Worksheet 23: Bernoulli's Rule

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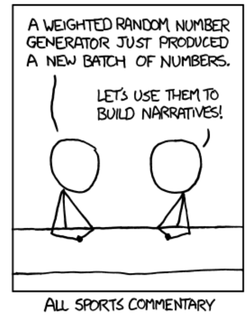

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1. State the assumptions of Bernoulli's (l'Hospital's) Rule.

   For $\lim_{x\to a} \frac{f(x)}{g(x)}$, f, g must be differentiable and $g'(x) \neq 0$ on an open interval I that contains a, except possibly at a.

2. Find the value of the following limits:

   (a) $\lim_{x\to\infty} \frac{e^x}{x^2}$

   Noting that, as written, the limit has indeterminate form $\frac{\infty}{\infty}$ and the other conditions of l'Hospital's rule are met (top and bottom differentiable, open interval around limit non-zero),

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

   Noting that, as written, the limit has indeterminate form $\frac{\infty}{\infty}$ and the other conditions of l'Hospital's rule are met (top and bottom differentiable, open interval around limit non-zero),

$$= \lim_{x\to\infty} \frac{e^x}{2}$$
$$= \infty$$

   (b) $\lim_{x\to 0^+} x\ln(x)$

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{1}{\frac{1}{x}}(\ln(x))$$
$$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$
$$= -\lim_{x\to 0^+} \frac{-\ln(x)}{\frac{1}{x}}$$

   Noting that, as written, the limit has indeterminate form $\frac{\infty}{\infty}$ and the other conditions of l'Hospital's rule are met (top and bottom differentiable, open interval around limit non-zero),

$$-\lim_{x\to 0^+} \frac{-\ln(x)}{\frac{1}{x}} = -\lim_{x\to 0^+} \frac{-\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= -\lim_{x\to 0^+} x$$
$$= -0$$
$$= 0$$

   (c) $\lim_{x\to \frac{\pi}{2}^-} \sec(x) - \tan(x)$

$$\lim_{x\to \frac{\pi}{2}^-} \sec(x) - \tan(x) = \lim_{x\to \frac{\pi}{2}^-} \frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}$$
$$= \lim_{x\to \frac{\pi}{2}^-} \frac{1-\sin(x)}{\cos(x)}$$

Noting that, as written, the limit has indeterminate form $\frac{0}{0}$ and the other conditions of l'Hospital's rule are met (top and bottom differentiable, open interval around limit non-zero),

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin(x)}{\cos(x)} &= \lim_{x\to\frac{\pi}{2}^-}\frac{-\cos(x)}{-\sin(x)}\\ &= \lim_{x\to\frac{\pi}{2}^-}\frac{\cos(x)}{\sin(x)}\\ &= \frac{0}{1}\\ &= 0\end{aligned}$$

(d) $\lim_{x\to 0^+} x^x$

Note that,

$$\begin{aligned}y &= x^x\\ \ln(y) &= x\ln(x)\\ e^{\ln(y)} &= e^{x\ln(x)}\\ y &= \end{aligned}$$

And so,

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)}$$

Note further than in the rightmost limit e is constant, and the only value affected by the limit is $x\ln(x)$. Moreover, the value of $\lim_{x\to 0^+} x\ln(x) = 0$ by above. It follows immediately that,

$$\begin{aligned}\lim_{x\to 0^+} x^x &= e^0\\ &= 1\end{aligned}$$

(e) $\lim_{x\to\infty}\sqrt{x}e^{-\frac{x}{2}}$

$$\lim_{x\to\infty}\sqrt{x}e^{-\frac{x}{2}} = \lim_{x\to\infty}\frac{\sqrt{x}}{e^{\frac{x}{2}}}$$

Noting that, as written, the limit has indeterminate form $\frac{\infty}{\infty}$ and the other conditions of l'Hospital's rule are met (top and bottom differentiable, open interval around limit non-zero),

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x}}{e^{\frac{x}{2}}} &= \lim_{x\to\infty}\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}e^{\frac{x}{2}}}\\ &= \lim_{x\to\infty}\frac{x^{-\frac{1}{2}}}{e^{\frac{x}{2}}}\\ &= \lim_{x\to\infty}\frac{1}{x^{\frac{1}{2}}e^{\frac{x}{2}}}\\ &= \frac{1}{\infty(\infty)}\\ &= 0\end{aligned}$$

3. Show that

$$\lim_{x\to\infty}\frac{\ln(x)}{x^p} = 0$$

for any number $p > 0$. This proves that the logarithmic function approaches ∞ more slowly than any positive power of x.

Noting that, as written, the limit has indeterminate form $\frac{\infty}{\infty}$ and the other conditions of l'Hospital's rule are met (top and bottom differentiable, open interval around limit non-zero),

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln(x)}{x^p} &= \lim_{x\to\infty}\frac{\frac{1}{x}}{px^{p-1}}\\ &= \lim_{x\to\infty}\frac{1}{(x)px^{p-1}}\\ &= \lim_{x\to\infty}\frac{1}{px^p}\end{aligned}$$

Since, by assumption, $p > 0$,

$$\begin{aligned}&= \frac{1}{\infty}\\ &= 0\end{aligned}$$

4. ($\star$) True or False; justify your answer

   (a) If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.
   True; noting that differentiability implies continuity, this is just Rolle's theorem.

   (b) If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of $f(x)$.
   False; consider the graph $y = (x-2)^4$

   (c) There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.
   True; consider $f(x) = \frac{1}{x}$ over $(0, \infty)$ and stretch it to cover the entire real line.

   (d) There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.
   True; place points so that $f(1) = -2$ and $f(3) = 0$, draw a line covering the entire real line with slope greater that 1 and holes at these points. This is, however, false if the function is continuous $[1, 3]$ and differentiable $(1, 3)$.

   (e) If f,g are increasing on an interval I, $f + g$ is increasing on I.
   True; $\frac{d}{dx}[f + g] = f' + g'$ and by assumption $f'(x) > 0$ and $g'(x) > 0$, so $f'(x) + g'(x) > 0$.

5. Sketch $f(x) = \sqrt[3]{x^3 - x}$ showing: increasing, decreasing, zeroes, behavior for $|x|$ large, behavior for $|x|$ small, and points where the function is not differentiable. You need not show convexity or points of inflection.

We begin by taking the derivative of f:

$$\begin{aligned}f'(x) &= \frac{1}{3}(x^3 - x)^{-\frac{2}{3}}(3x^2 - 1)\\ &= \frac{3x^2 - 1}{3(x^3 - x)^{\frac{2}{3}}}\end{aligned}$$

Setting the derivative equal to 0 to find points where the function might switch increasing/decreasing and noting that only the top need be considered:

$$\begin{aligned}3x^2 - 1 &= 0\\ x^2 &= \frac{1}{3}\\ x &= \pm\sqrt{\frac{1}{3}}\end{aligned}$$

Noting that the derivative is undefined at $0, \pm 1$, we plug in values to determine increasing or decreasing, we obtain $f'(-2)$ as positive (increasing), f(-.75) as positive (increasing), f(-.5) as negative (decreasing), f(.5) as negative (decreasing), f(.75) as positive (increasing), and $f'(2)$ as positive (increasing).

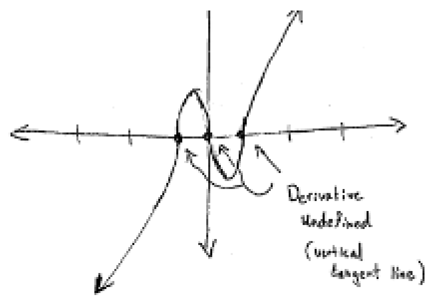
Thus, we have that f is increasing over $(-\infty, -\sqrt{\frac{1}{3}})$ and $(\sqrt{\frac{1}{3}}, \infty)$, decreasing over $(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$. It therefore has a local max at $-\sqrt{\frac{1}{3}}$ and local min at $\sqrt{\frac{1}{3}}$ (by first derivative test; note that no other critical points exist).

Consider next the roots of the function:

$$\sqrt[3]{x^3 - x} = 0$$
$$\sqrt[3]{x(x^2 - 1)} = 0$$
$$\sqrt[3]{x(x-1)(x+1)} = 0$$

The function has zeroes, then, at $0, \pm 1$.

Finally, we consider the behavior of the function for $|x|$ large and for $|x|$ small. If x is approaching ∞, then the function approaches infinity. If x is approaching $-\infty$, then the function is approaching $-\infty$. If x is approaching 0 from the left, then the function approaches zero from the right. If x is approaching zero from the right, then the function approaches zero from the left. Putting all of the above together,



6. In section 4.5, Stewart gives a list of seven main attributes of functions which should be taken into account when sketching a curve; list them.

   (a) Domain
   (b) Intercepts
   (c) Symmetry (even, odd, period)
   (d) Asymptotes (horizontal, vertical, slant)
   (e) Increasing/Decreasing
   (f) Maximums and Minimums (local, absolute)
   (g) Concavity, Points of Inflection