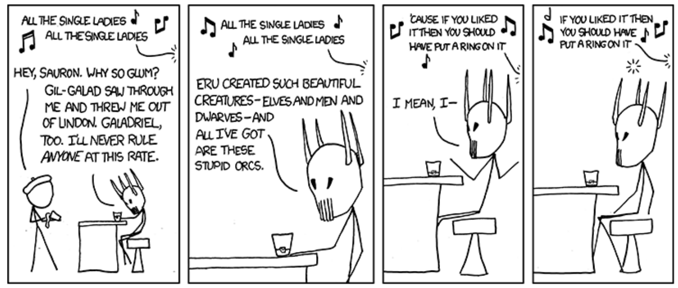
# Worksheet 22: Concavity

Russell Buehler
b.r@berkeley.edu

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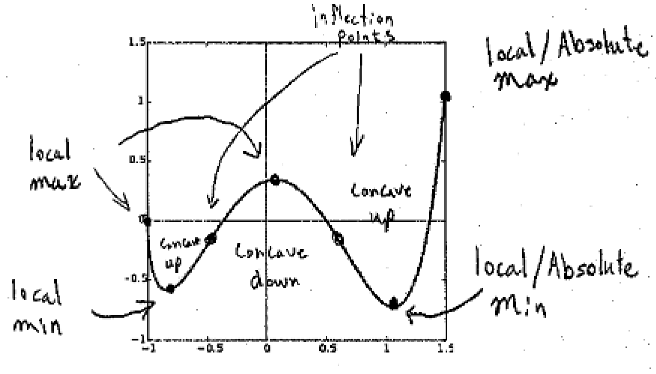
1. If you haven't already, label the local maxima/minima, absolute maximum/minimum, inflection points, and where the graph is concave up or concave down.




2. What is the first derivative test? What is the second derivative test?

   The first derivative test states that if $c$ is a critical number of a continuous function $f$:

   (a) If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$
   (b) If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$
   (c) If $f'$ does not change sign at $c$, then $f$ has no local max or min at $c$

   The second derivative test states that if $f''$ is continuous near $c$,

   (a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.
   (b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

3. If you haven't already, for $f(x) = 4x^3 + 3x^2 - 6x + 1$, find the intervals on which $f$ is increasing or decreasing, the local maxima and minima, and the inflection points.

   We begin by taking the derivative of $f$:

$$f'(x) = 12x^2 + 6x - 6$$

   Setting the derivative equal to 0 to find points where the function might switch increasing/decreasing:

$$\begin{aligned} 12x^2 + 6x - 6 &= 0 \\ 6(2x^2 + x - 1) &= \\ 2x^2 + x - 1 &= 0 \\ (2x-1)(x+1) &= 0 \\ x &= \frac{1}{2}, -1 \end{aligned}$$

   Plugging in values to determine increasing or decreasing, we obtain $f'(-2) = 30$ (increasing), $f'(0) = -6$ (decreasing), and $f(1) = 12$ (increasing). Solving for possible inflection points (taking the second derivative and setting equal to zero/looking for points where $f''(x)$ is undefined):

$$\begin{aligned} 24x + 6 &= 0 \\ x &= -\frac{1}{4} \end{aligned}$$

Checking if the point above is actually an inflection point, $f''(0) = 6$ (concave up) while $f''(-1) = -18$ (concave down) shows it is.

Thus, in sum, $f$ is increasing over $(-\infty, -1)$ and $(\frac{1}{2}, \infty)$, decreasing over $(-1, \frac{1}{2})$, has a local max at -1 and local min at $\frac{1}{2}$ (by first derivative test; note that no other critical points exist), and finally an inflection point at $-\frac{1}{4}$.

4. Show that the equation $x^3 - 15x + c$ has at most one root in the interval $[-2, 2]$.

Assume that $x^3 - 15x + c$ has more than one root in the interval $[-2, 2]$. Then, by Rolle's theorem with these two points, there is a point in $[-2, 2]$ where $3x^2 - 15 = 0$. Note, however,

$$3x^2 - 15 = 0$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

–both outside of $[-2, 2]$, a contradiction. It follows, then, that the equation $x^3 - 15x + c$ has at most one root in the interval $[-2, 2]$.

5. Find the intervals on which $f$ is increasing/decreasing and the local maxima and minima values for $f(x) = x - 2\sin(x)$, $0 < x < 3\pi$.

We begin by taking the derivative of $f$:

$$f'(x) = 1 - 2\cos(x)$$

Setting the derivative equal to 0 to find points where the function might switch increasing/decreasing:

$$1 - 2\cos(x) = 0$$
$$\cos(x) = \frac{1}{2}$$

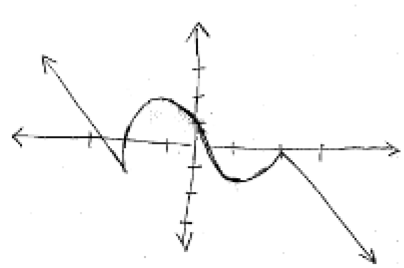
Recalling that the relevant interval is $0 < x < 3\pi$:

$$x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

Noting that there are no points at which the derivative is undefined, we plug in values to determine increasing or decreasing, we obtain $f'(0) = -1$ (decreasing), $f'(\frac{\pi}{2}) = 1$ (increasing), $f(2\pi) = -1$ (decreasing), $f(2\pi + \frac{\pi}{2}) = 1$ (increasing).

Thus, we have that $f$ is decreasing over $(0, \frac{\pi}{3})$ and $(\frac{5\pi}{3} \frac{7\pi}{3})$, increasing over $(\frac{\pi}{3}, \frac{5\pi}{3})$ and $(\frac{5\pi}{3}, 3\pi)$, has a local max at $\frac{5\pi}{3}$ and local mins at $\frac{\pi}{3}$, $\frac{7\pi}{3}$ (by first derivative test; note that no other critical points exist).

6. Sketch the graph of a function that satisfies all of the following: $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$, $f''(x) < 0$ if $-2 < x < 0$, inflection point at $(0, 1)$.



7. Find all critical numbers of $5x^{\frac{2}{3}} + x^{\frac{5}{3}}$.

Taking the derivative and setting it equal to zero,

$$\begin{aligned}\frac{10}{3}x^{-\frac{1}{3}} + \frac{5}{3}x^{\frac{2}{3}} &= 0\\ \frac{10}{3x^{\frac{1}{3}}} + \frac{5x}{3x^{\frac{1}{3}}} &=\\ \frac{10+5x}{3x^{\frac{1}{3}}} &=\end{aligned}$$

And so we need

$$\begin{aligned}10 + 5x &= 0\\ x &= -2\end{aligned}$$

Noting further that the derivative is only undefined when $x = 0$, we have that the critical numbers are -2 and 0.

8. Find the intervals on which $f$ is increasing/decreasing and the local maxima and minima values for $f(x) = xe^x$

We begin by taking the derivative of $f$:

$$f'(x) = (1)e^x + xe^x$$

Setting the derivative equal to 0 to find points where the function might switch increasing/decreasing:

$$\begin{aligned}e^x + xe^x &= 0\\ e^x(1+x) &= 0\end{aligned}$$

Since $e^x$ is never 0, we need only consider the case below:

$$\begin{aligned}1 + x &= 0\\ x &= -1\end{aligned}$$

Noting that there are no points at which the derivative is undefined, we plug in values to determine increasing or decreasing, we obtain $f'(-2) = -e^{-2}$ (decreasing) and $f'(0) = 1$ (increasing).

Thus, we have that $f$ is decreasing over $(-\infty, -1)$ and increasing over $(-1, \infty)$. It therefore has a local min at $-1$ (by first derivative test; note that no other critical points exist).

9. Sketch $f(x) = \sqrt[3]{x^3 - x}$ showing: increasing, decreasing, zeroes, behavior for $|x|$ large, behavior for $|x|$ small, and points where the function is not differentiable. You need not show convexity or points of inflection.

We begin by taking the derivative of $f$:

$$\begin{aligned}f'(x) &= \frac{1}{3}(x^3 - x)^{-\frac{2}{3}}(3x^2 - 1)\\ &= \frac{3x^2 - 1}{3(x^3 - x)^{\frac{2}{3}}}\end{aligned}$$

Setting the derivative equal to 0 to find points where the function might switch increasing/decreasing and noting that only the top need be considered:

$$\begin{aligned}3x^2 - 1 &= 0\\ x^2 &= \frac{1}{3}\\ x &= \pm\sqrt{\frac{1}{3}}\end{aligned}$$

Noting that the derivative is undefined at $0, \pm 1$, we plug in values to determine increasing or decreasing, we obtain $f'(-2)$ as positive (increasing), f(-.75) as positive (increasing), f(-.5) as negative (decreasing), f(.5) as negative (decreasing), f(.75) as positive (increasing), and $f'(2)$ as positive (increasing).

Thus, we have that $f$ is increasing over $(-\infty, -\sqrt{\frac{1}{3}})$ and $(\sqrt{\frac{1}{3}}, \infty)$, decreasing over $(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$. It therefore has a local max at $-\sqrt{\frac{1}{3}}$ and local min at $\sqrt{\frac{1}{3}}$ (by first derivative test; note that no other critical points exist).

Consider next the roots of the function:

$$\sqrt[3]{x^3 - x} = 0$$
$$\sqrt[3]{x(x^2 - 1)} = 0$$
$$\sqrt[3]{x(x-1)(x+1)} = 0$$

The function has zeroes, then, at $0, \pm 1$.

Finally, we consider the behavior of the function for $|x|$ large and for $|x|$ small. If $x$ is approaching $\infty$, then the function approaches infinity. If $x$ is approaching $-\infty$, then the function is approaching $-\infty$. If $x$ is approaching 0 from the left, then the function approaches zero from the right. If $x$ is approaching zero from the right, then the function approaches zero from the left. Putting all of the above together,