Worksheet 21: The Mean Value Theorem

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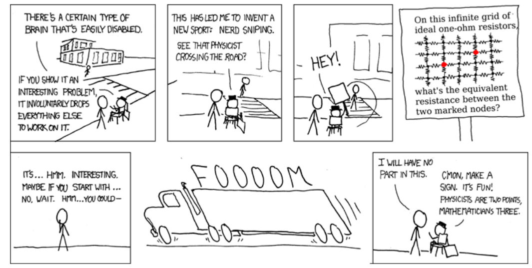

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1. Verify that $f(x) = x^3 - x^2 - 6x + 2$ satisfies the hypotheses of Rolle's theorem for the interval $[0, 3]$, then find all c that satisfy the conclusion.

   Note that f is continuous over $[0, 3]$ since polynomials are continuous over their domains, f is differentiable over $(0, 3)$ since polynomials are differentiable over the reals, and $f(0) = 2 = f(3)$. It follows, then, that Rolle's theorem applies. To find all c satisfying Rolle's theorem, we take the derivative of f and set it equal to 0:

   $$3x^2 - 2x - 6 = 0$$

   A few attempts to factor the polynomial above show it's not factorable, and so we resort to the quadratic formula:

   $$\begin{aligned} \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} &= \frac{2 \pm \sqrt{4 + 72}}{6} \\ &= \frac{2 \pm \sqrt{76}}{6} \\ &= \frac{1}{3} \pm \sqrt{\frac{76}{36}} \end{aligned}$$

2. Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$, but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Is this a counterexample to Rolle's theorem? Why or why not?

   Note that $f(0) = 0 = f(\pi)$, but yet taking the derivative,

   $$\begin{aligned} \sec^2(x) &= 0 \\ \frac{1}{\cos^2(x)} &= \end{aligned}$$

   which isn't possible. This isn't a counterexample, however, because $\tan(x)$ is not continuous over $[0, \pi]$ (look at $\frac{\pi}{2}$ in particular).

3. Verify that $f(x) = x^3 - 3x + 2$ satisfies the hypotheses of the mean value theorem on $[-2, 2]$, then find all c that satisfy the conclusion.

   Note that f is continuous over $[-2, 2]$ since polynomials are continuous over their domains, f is differentiable over $(-2, 2)$ since polynomials are differentiable over the reals. It follows, then, that that mean value theorem applies. To find all c satisfying the mean value theorem, we take the derivative of f and set it equal to the slope of the secant line between $(-2, f(-2))$ and $(2, f(2))$:

   $$\begin{aligned} 3x^2 - 3 &= \frac{f(2) - f(-2)}{2 - (-2)} \\ &= \frac{4 - 0}{2 - (-2)} \\ &= 1 \\ 3x^2 &= 4 \\ x &= \pm\sqrt{\frac{4}{3}} \end{aligned}$$

4. Let $f(x) = \frac{x^3-x^2}{x-1}$ on $[0, 2]$. Show that there is no value of c such that $f'(c) = \frac{f(2)-f(0)}{2-0}$. Is this a counterexample to the mean value theorem? Why or why not?

Note first that,

$$\begin{aligned} \frac{x^3 - x^2}{x-1} &= \frac{x^2(x-1)}{x-1} \\ &= x^2 \end{aligned}$$

Taking the derivative,

$$f'(x) = 2x$$

Setting it equal to the slope of the secant line through the given points,

$$\begin{aligned} 2x &= \frac{f(0) - f(2)}{0-2} \\ &= \frac{0-4}{0-2} \\ &= 2 \\ x &= 1 \end{aligned}$$

Note, however, that 1 is not in the domain of the given function. This is not a counterexample since there is a point discontinuity at $x = 1$.

5. ($\star$) If for two functions $f(x)$ and $g(x)$, we know that $f'(x) = g'(x)$ for every x in an interval (a, b), it must be the case that $f - g$ is constant on (a, b). Why? What can we say about $f(x)$ in terms of $g(x)$?

   Because the functions always rise, fall, or stay constant together over the interval (a, b), they are parallel over this interval–and thus $f - g$ is constant on (a, b). In terms of g, $f(x) = g(x) + c$ for some constant c.

6. ($\star$) Using the mean value theorem and Rolle's theorem, show that $x^3 + x - 1 = 0$ has exactly one real root.

   Noting that polynomials are continuous over the reals and $f(0) = -1$ while $f(1) = 1$, by the intermediate value theorem we have that $x^3 + x - 1 = 0$ has at least one real root.

   We show, then, that $x^3 + x - 1 = 0$ cannot have more than one real root. Assume it does. Then, noting that polynomials are differentiable over the reals, we have by Rolle's theorem–using these two roots–that there is a point where $3x^2 + 1 = 0$ (the derivative of the original function).

$$\begin{aligned} 3x^2 + 1 &= 0 \\ x^2 &= -\frac{1}{3} \\ x &= \sqrt{-\frac{1}{3}} \end{aligned}$$

   a contradiction. We have, then, that $x^3 + x - 1 = 0$ cannot have more than one real root which, combined with our earlier result gives that $x^3 + x - 1 = 0$ has exactly one real root as requested.

7. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

   Assume that the equation $x^4 + 4x + c = 0$ has more than two real roots. Then, noting that polynomials are differentiable and continuous over the reals, we have by Rolle's theorem–using first the first two roots, then the second and third roots–that there are two distinct points where $4x^3 + 4 = 0$ (the derivative of the original function).

$$\begin{aligned} 4x^3 + 4 &= 0 \\ 4x^3 &= -4 \\ x &= \sqrt[3]{-1} \\ &= -1 \end{aligned}$$

a contradiction (we were promised two!). We have, then, that $x^4 + 4x + c = 0$ has at most two real roots, as requested.

8. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.

    Noting that differentiability over the reals implies continuity over the reals, we have by Rolle's theorem–using the two roots–that there is a point where $f'(x) = 0$, as requested.

    (b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.

    Noting that differentiability over the reals implies continuity over the reals, we have by Rolle's theorem–using first the first and second roots, then the second and third roots–that there are two distinct points where $f'(x) = 0$. Thus, reasoning exactly as earlier, we have by Rolle's theorem–using the two roots of $f'(x)$–that there is a point where $f''(x) = 0$, as requested.

    (c) Can you generalize parts (a) and (b)?
    Yes; if f is n differentiable and has $n + 1$ roots, then $f^{(n)}(x)$ has at least one real root.