Worksheet 19: Review

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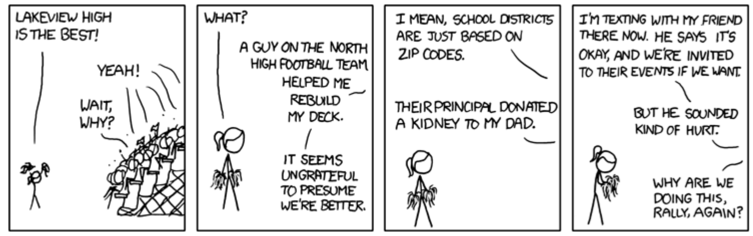

www.xkcd.com

1. Use linear approximation or differentials to approximate 2.001^3.

Note that the function is $f(x) = x^3$, so taking the derivative: $f'(x) = 3x^2$. Solving by differentials, $dy = 3x^2 dx$. Plugging in $x = 2$ and $dx = .001$,

$$\begin{aligned} dy &= 3(2)^2(.001) \\ &= (12)(\frac{1}{1000}) \\ &= \frac{3}{250} \end{aligned}$$

The final approximation is therefore $f(2) + \frac{3}{250} = 8 + \frac{3}{250}$.

2. Find all the critical numbers of the function $f(x) = x\ln(x)$.

Taking the derivative, $f'(x) = (x)\frac{1}{x} + \ln(x) = \ln(x) + 1$. Note that the derivative is undefined for $x \leq 0$, but so is the original function, and so we obtain no critical points. The only remaining possibilities, then, are points where the derivative is 0. Thus,

$$\begin{aligned} \ln(x) + 1 &= 0 \\ \ln(x) &= -1 \\ x &= e^{-1} \end{aligned}$$

The only critical number is therefore e^{-1}.

3. Find the derivative of $t(\theta) = \tanh^{-1}(5\theta^6)$

$$\begin{aligned} t'(\theta) &= \frac{1}{1-(5\theta^6)^2}(30\theta^5) \\ &= \frac{30\theta^5}{1-25\theta^{12}} \end{aligned}$$

4. Find the absolute minimum and maximum of $g(q) = 2^q$ in the interval $(0, 5)$.

Since we don't have endpoints, we can't check them. Moving on to critical numbers, $g'(q) = 2^q \ln(2)$ which is always defined and never 0. It follows immediately that there is no absolute minimum or maximum.

5. Find $\frac{dy}{dx}$ by implicit differentiation: $e^{\frac{x}{y}} = x - y$

Using some of our log rules,

$$\begin{aligned} \ln(e^{\frac{x}{y}}) &= \ln(x-y) \\ \frac{x}{y} &= \ln(x-y) \\ xy^{-1} &= \ln(x-y) \end{aligned}$$

Taking the derivative,

$$\begin{aligned} y^{-1} + x(-y^{-2})\frac{dy}{dx} &= \frac{1}{x-y}(1-\frac{dy}{dx}) \\ &= \frac{1}{x-y} - \frac{1}{x-y}\frac{dy}{dx} \\ \frac{1}{x-y}\frac{dy}{dx} + x(-y^{-2})\frac{dy}{dx} &= \frac{1}{x-y} - y^{-1} \\ \frac{dy}{dx} &= \frac{\frac{1}{x-y} - y^{-1}}{\frac{1}{x-y} - xy^{-2}} \end{aligned}$$

6. Find the derivative of $t(\theta) = \cosh(3x^2)$

$$\begin{aligned} t'(\theta) &= \sinh(3x^2)(6x) \\ &= 6x\sinh(3x^2) \end{aligned}$$

7. Find all the critical numbers of the function $f(x) = 5$.

Taking the derivative,

$$f'(x) = 0$$

Since any x-value with a zero derivative is a critical number, all real numbers are critical numbers

8. Find the absolute minimum and maximum of $f(x) = x^2 - 2x$ in the interval $[0, 3]$.

Taking the derivative,

$$f'(x) = 2x - 2$$

Since the derivative is defined for all reals, we move to finding those points with a zero derivative:

$$\begin{aligned} 2x - 2 &= 0 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

Checking both this x-value and the endpoints,

$$\begin{aligned} f(1) &= -1 \\ f(0) &= 0 \\ f(3) &= 3 \end{aligned}$$

we have that $x = 1$ is the absolute minimum and $x = 3$ is the absolute maximum.

9. ($\star$) (True or False) and why.

(a) If $f(x) = (x^6 - x^4)^5$, then $f^{(31)}(x) = 0$

True; note that when $f(x)$ is expanded, the degree of the result is $5(6) = 30$. Thus, the 31st derivative will be the derivative of a constant, and therefore 0.

(b) The derivative of a rational function is a rational function.

True; we're taking the derivative of something with the form $\frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are polynomials. By the quotient rule, this means that the result is $\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$; noting that polynomials are closed under multiplication, derivatives, and subtraction, the result is still a rational function.

(c) If $g(x) = x^5$, then $\lim_{x \to 2} \dfrac{g(x) - g(2)}{x - 2} = 80$

True; the crucial step is to note that $\lim_{x \to 2} \dfrac{g(x) - g(2)}{x - 2} = g'(2)$. So, $g'(x) = 5x^4$, and $g'(2) = 5(2)^4 = 5(16) = 80$.