Worksheet 18: Maxima, Minima, & Hyperbolic Trig

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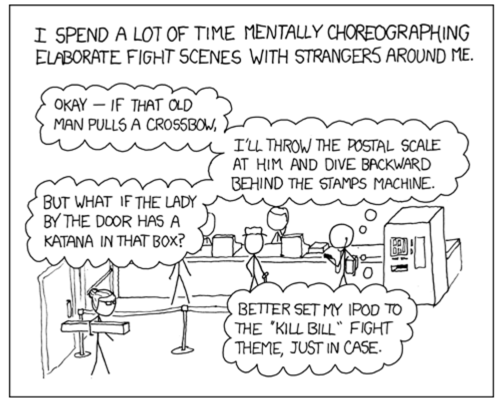


www.xkcd.com

1. Find the derivative:

   (a) $f(x) = \tanh(1 + e^{2x})$

   $$\begin{aligned} f'(x) &= \text{sech}^2(1 + e^{2x})(e^{2x})(2) \\ &= 2e^{2x}\text{sech}^2(1 + e^{2x}) \end{aligned}$$

   (b) $g(x) = \sinh(\cosh(x))$

   $$\begin{aligned} f'(x) &= \cosh(\cosh(x))(\sinh(x)) \\ &= \cosh(\cosh(x))\sinh(x) \end{aligned}$$

   (c) $y = \text{sech}^{-1}(e^{-x})$

   $$\begin{aligned} f'(x) &= -\frac{1}{e^{-x}\sqrt{1-(e^{-2x})}}(-e^{-x}) \\ &= \frac{1}{\sqrt{1-(e^{-2x})}} \end{aligned}$$

2. What does the extreme value theorem say?

   For a continuous function over a closed interval, the function has an absolute maximum and an absolute minimum in the interval.

3. What does Fermat's theroem say?

   If c is a local maxima or minima and $f'(c)$ exists, then $f'(c) = 0$.

4. ($\star$) Does the converse (reverse the 'if' and 'then' portions) of Fermat's theorem hold? If not, provide a counterexample. If so, explain why.

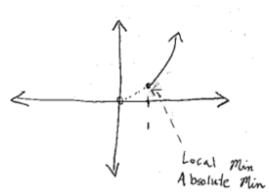
   No; $f(x) = x^3$ has a point with derivative 0 ($x = 0$) that is not a local maxima or minima.

5. Based on the last three problems, what kinds of points do we need to check in order to find an absolute minimum or maximum?

   Endpoints, points where the derivative doesn't exist, and points where the derivative is 0.

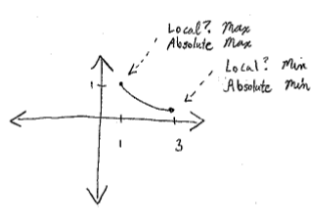
6. Sketch the following functions by hand and label the local minima, local maxima, and absolute minimum and maximum:

   (a) $f(x) = x^2$, $x \geq 1$

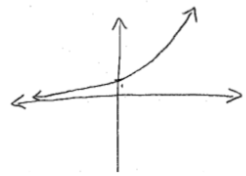
   According to the definition your book initially gives, the endpoints are a local maxima/minima; it's not common to think of them as such.

   (b) $g(\theta) = \frac{1}{\theta}$, $1 \leq x \leq 3$

According to the definition your book initially gives, the endpoints are a local maxima/minima; it's not common to think of them as such.

(c) $y = e^x$



7. List the values you would need to check in order to find all the local minima and maxima:

(a) $f(t) = 2t^3 + t^2 + 2t$

$$\begin{aligned} f'(t) &= 6t^2 + 2t + 2 \\ &= 2(3t^2 + t + 1) \end{aligned}$$

Note now that $3t^2+t+1$ never reaches 0, and so we don't need to check any x-values where the derivative is 0. Similarly, the derivative is defined for all t, and so it exists everywhere. Thus, there are no local minima or maxima (if you remembered the graph of x^3, you could have skipped this work).

(b) $h(p) = \frac{p-1}{p^2+4}$

$$\begin{aligned} h'(p) &= \frac{p^2 + 4 - (p-1)(2p)}{(p^2+4)^2} \\ &= \frac{p^2 + 4 - 2p^2 + 2p}{(p^2+4)^2} \\ &= \frac{-p^2 + 2p + 4}{(p^2+4)^2} \end{aligned}$$

We now need to find those x-values for which the derivative is either undefined or 0. Starting with the former, note that the derivative is defined when $x = \sqrt{-4}$. Since we're working in the reals, however, we ignore this. Next, note that the derivative is 0 when $-p^2 + 2p + 4$ is 0. So,

$$-p^2 + 2p + 4 = 0$$

This doesn't actually factor nicely, so the option of last resort is the quadratic formula:

$$\frac{-2 \pm \sqrt{4 - 4(-1)(4)}}{-2} = \frac{-2 \pm \sqrt{20}}{-2}$$

The critical points, then, are $x = \frac{-2+\sqrt{20}}{-2}$ and $x = \frac{-2-\sqrt{20}}{-2}$ (ugly, I know).

8. Find the absolute maximum and minimum of the function on the given interval.

(a) $f(x) = 5 + 54x - 2x^3$, $[0, 4]$

Taking the derivative, $f'(x) = -6x^2 + 54$ which is defined for all numbers. Solving for points with derivative 0,

$$\begin{aligned} -6x^2 + 54 &= 0 \\ 6x^2 &= 54 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

Checking the end and critical points:

$$\begin{aligned} f(-3) &= -103 \\ f(3) &= 113 \\ f(0) &= 5 \\ f(4) &= 93 \end{aligned}$$

The absolute maximum is therefore at $x = 3$ and the absolute minimum is $x = -3$

(b) $h(y) = (y^2 - 1)^3$, $[-1, 2]$
Taking the derivative, $f'(x) = 3(y^2 - 1)^2(2y) = 6y(y^2 - 1)$ which is defined for all numbers. Solving for points with derivative 0,

$$\begin{aligned} 6y(y^2 - 1) &= 0 \\ y &= 0 \end{aligned}$$

or

$$\begin{aligned} y^2 - 1 &= 0 \\ y^2 &= 1 \\ y &= 1 \end{aligned}$$

Checking the end and critical points:

$$\begin{aligned} h(1) &= 0 \\ h(0) &= -1 \\ h(-1) &= 0 \\ h(2) &= 27 \end{aligned}$$

The absolute maximum is therefore at $y = 2$ and the absolute minimum is $y = 0$