Worksheet 16: Derivative Applications

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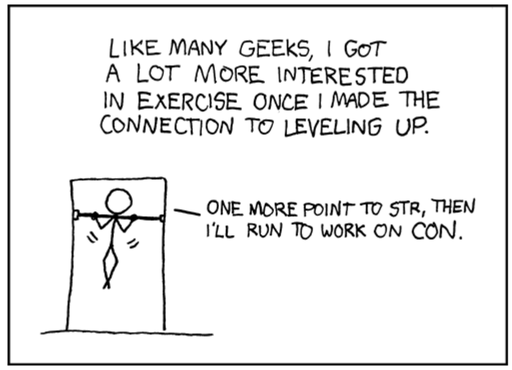

www.xkcd.com

1. Find the derivative by implicit differentiation: $x^3 + x^2y + 4y^2 = 6$.

$$\begin{aligned} 3x^2 + 2xy + x^2\frac{dy}{dx} + 8y\frac{dy}{dx} &= 0 \\ x^2\frac{dy}{dx} + 8y\frac{dy}{dx} &= 3x^2 + 2xy \\ \frac{dy}{dx}(x^2 + 8y) &= 3x^2 + 2xy \\ \frac{dy}{dx} &= \frac{3x^2 + 2xy}{x^2 + 8y} \end{aligned}$$

2. Find the formula for the nth derivative $f^{(n)}(x)$ if $f(x) = \frac{1}{3x^3} = \frac{1}{3}x^{-3}$

Note first that,

$$\begin{aligned} f'(x) &= x^{-4} \\ f''(x) &= -4x^{-5} \\ f'''(x) &= 20x^{-6} \\ f''''(x) &= -120x^{-7} \end{aligned}$$

To determine $f^{(n)}(x)$, it's best to proceed piece by piece rather than attempting to intuit the entire function. Looking at the derivatives above, it's obvious that the x term is given by x^{-n-3}. Note next that every odd derivative will be negative, every even positive; to achieve this, we may multiply our function by $(-1)^n$. Finally, we need to find an expression for the constants $1, 4, 20, 120, \ldots$. Noting that we may factor the constants as $1, 4, 4(5), 4(5)(6), \ldots$ and thinking about what happens each time we take a derivative, we can determine that the constant term is given by $\frac{(n+2)!}{3!}$. We therefore have that,

$$f^{(n)}(x) = (-1)^n\left(\frac{(n+2)!}{3!}\right)x^{-n-3}$$

3. Differentiate $f(x) = \ln(\ln(\ln(x)))$

$$f'(x) = \frac{1}{\ln(\ln(x))}\left(\frac{1}{\ln(x)}\right)\left(\frac{1}{x}\right)$$

4. A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.

   (a) What is the relative growth rate?

   Noting that constant relative growth rates have (according to the book) the form $y = ce^{kt}$,

$$\begin{aligned} 400 &= ce^{2k} \\ \ln(400) &= \ln(c) + \ln(e^{2k}) \\ \ln(400) &= \ln(c) + 2k \\ k &= \frac{1}{2}(\ln(400) - \ln(c)) \\ &= \ln(\frac{20}{\sqrt{c}}) \end{aligned}$$

And so,

$$\begin{aligned}
25,600 &= ce^{6\ln(\frac{20}{\sqrt{c}})} \\
&= ce^{\ln(\frac{20}{\sqrt{c}})^6} \\
&= c(\frac{20}{\sqrt{c}})^6 \\
&= c(\frac{20^6}{c^3}) \\
&= \frac{20^6}{c^2} \\
c^2 &= \frac{20^6}{2^8(100)} \\
&= \frac{20^6}{2^6(20)(20)} \\
&= \frac{(2(10))^4}{2^6} \\
&= \frac{10^4}{2^2} \\
c &= \sqrt{\frac{10^4}{2^2}} \\
&= \frac{10^2}{2} \\
&= 50
\end{aligned}$$

Plugging this value in,

$$\begin{aligned}
k &= \frac{1}{2}(\ln(400) - \ln(50)) \\
&= \frac{1}{2}(\ln(80))
\end{aligned}$$

Finally, noting that the k above is the constant relative growth rate, we have $\frac{1}{2}(\ln(80))$ as our solution.

(b) What was the initial size of the culture?
Plugging in $t = 0$ gives c as the initial size, and–by above–$c = 50$.

(c) Find an expression for the number of bacteria after t hours.

$$y = 50e^{\frac{1}{2}(\ln(80))t}$$

(d) Find the rate of growth after 4.5 hours

$$\frac{dy}{dx} = 50(\frac{1}{2}(\ln(80)))e^{\frac{1}{2}(\ln(80))t}$$

Setting $t = 4.5$,

$$\begin{aligned}
&= 50(\frac{1}{2}(\ln(80)))e^{\frac{1}{2}(\ln(80))4.5} \\
&= 25\ln(80)e^{\frac{9}{4}\ln(80)} \\
&= 25\ln(80)e^{\ln(80^{\frac{9}{4}})} \\
&= 25\ln(80)80^{\frac{9}{4}}
\end{aligned}$$

(e) When will the population reach 50,000?

$$\begin{aligned} 50000 &= 50e^{\frac{1}{2}(\ln(80))t} \\ 1000 &= e^{\frac{1}{2}(\ln(80))t} \\ \ln(1000) &= \frac{1}{2}(\ln(80))t \\ t &= \frac{2\ln(1000)}{\ln(80)} \end{aligned}$$

5. Strontium-90 has a half-life of 28 days.

(a) A sample has a mass of 50mg initially; find a formula for the mass remaining after t days.

Noting that half-life decay is represented by a formula of the form $y = ce^{kt}$,

$$\begin{aligned} 25 &= 50e^{28k} \\ \frac{1}{2} &= e^{28k} \\ \ln\left(\frac{1}{2}\right) &= 28k \\ k &= \frac{1}{28}\ln\left(\frac{1}{2}\right) \end{aligned}$$

And so, the requested formula is:

$$y = 50e^{\frac{1}{28}\ln\left(\frac{1}{2}\right)t}$$

(b) How long does it take the sample to decay to a mass of 2mg?

$$\begin{aligned} 2 &= 50e^{\frac{1}{28}\ln\left(\frac{1}{2}\right)t} \\ \ln\left(\frac{1}{25}\right) &= \frac{1}{28}\ln\left(\frac{1}{2}\right)t \\ t &= \frac{28\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{1}{2}\right)} \end{aligned}$$

6. (a) If A is the area of a circle with radius r and the circle expands as time passes, find $\frac{dA}{dt}$ in terms of $\frac{dr}{dt}$.

Note that the formula for the area of a circle is: $A = \pi r^2$. Thus, taking the derivative,

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of $\frac{1m}{s}$, how fast is the area of the spill increasing when the radius is 30m?

Taking the formula above and plugging in the given values,

$$\begin{aligned} \frac{dA}{dt} &= 2\pi(30)(1) \\ &= 60\pi \end{aligned}$$

7. Suppose $4x^2 + 9y^2 = 36$ where x and y are functions of t.

(a) If $\frac{dy}{dt} = \frac{1}{3}$, find $\frac{dx}{dt}$ when $x = 2$ and $y = \frac{2}{3}\sqrt{5}$.

$$8x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0$$

Plugging in values,

$$\begin{aligned} 8(2)\frac{dx}{dt} + 18(\frac{2}{3}\sqrt{5})(\frac{1}{3}) &= 0 \\ 16\frac{dx}{dt} + 4\sqrt{5} &= 0 \\ \frac{dx}{dt} &= -\frac{1}{4}\sqrt{5} \end{aligned}$$

(b) If $\frac{dx}{dt} = 3$, find $\frac{dy}{dt}$ when $x = -2$ and $y = \frac{2}{3}\sqrt{5}$.

$$8x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0$$

Plugging in values,

$$\begin{aligned} 8(-2)(3) + 18(\frac{2}{3}\sqrt{5})(\frac{dy}{dt}) &= 0 \\ -48 + 12\sqrt{5}(\frac{dy}{dt}) &= 0 \\ \frac{dy}{dt} &= \frac{4}{\sqrt{5}} \end{aligned}$$

8. Find the line tangent to the curve $f(x) = (1 + 3x)^{10}$ at $(0, 1)$.

$$\begin{aligned} f'(x) &= 10(1 + 3x)^9(3) \\ f'(0) &= 10(1)(3) \\ &= 30 \end{aligned}$$

Solving for b,

$$\begin{aligned} y &= mx + b \\ 1 &= 30(0) + b \\ b &= 1 \end{aligned}$$

The tangent line requested is thus,

$$y = 30x + 1$$

9. ($\star$) Find the third degree polynomial Q such that $Q(1) = 1$, $Q'(1) = 1$, $Q''(1) = 6$, and $Q'''(1) = 12$.

Note first that starting with the higher derivatives is more informative since most of the terms will have gone to 0 already. In particular, observe that

$$Q'''(x) = 12$$

And so,

$$\begin{aligned} Q''(x) &= 12x + c \\ Q''(1) &= 12(1) + c \\ 6 &= 12 + c \\ c &= -6 \\ Q''(x) &= 12x - 6 \end{aligned}$$

Similarly,

$$\begin{aligned}
Q'(x) &= 6x^2 - 6x + c \\
Q'(1) &= 6(1^2) - 6(1) + c \\
1 &= 6 - 6 + c \\
c &= 1 \\
Q'(x) &= 6x^2 - 6x + 1
\end{aligned}$$

And finally,

$$\begin{aligned}
Q(x) &= 2x^3 - 3x^2 + x + c \\
Q(1) &= 2(1)^3 - 3(1)^2 + 1 + c \\
1 &= 2 - 3 + 1 + c \\
c &= 1
\end{aligned}$$

And thus,

$$Q(x) = 2x^3 - 3x^2 + x + 1$$