Worksheet 15: Implicit & Logarithmic Differentiation

Russell Buehler
b.r@berkeley.edu

WHILE IT'S TECHNICALLY TRUE, I WISH SHE'D STOP PREFACING EVERY SENTENCE WITH THAT.

www.xkcd.com

1. If you haven't already, find the following derivatives:

(a) $y = e^{cot^{-1}(x^2)} + x^3$

$$y' = e^{cot^{-1}(x^2)}\left[-\frac{1}{(x^2)^2+1}(2x)\right] + 3x^2$$

(b) $\tan(x-y) = \frac{y}{1+x^2}$

$$\begin{aligned}
\tan(x-y)[1-\frac{dy}{dx}] &= \frac{(1+x^2)\frac{dy}{dx} - y(2x)}{(1+x^2)^2} \\
(1+x^2)^2\tan(x-y)[1-\frac{dy}{dx}] &= (1+x^2)\frac{dy}{dx} - y(2x) \\
(1+x^2)^2\tan(x-y) - (1+x^2)^2\tan(x-y)\frac{dy}{dx} &= (1+x^2)\frac{dy}{dx} - y(2x) \\
-(1+x^2)\frac{dy}{dx} - (1+x^2)^2\tan(x-y)\frac{dy}{dx} &= -y(2x) - (1+x^2)^2\tan(x-y) \\
\frac{dy}{dx}\left[-(1+x^2) - (1+x^2)^2\tan(x-y)\right] &= -y(2x) - (1+x^2)^2\tan(x-y) \\
\frac{dy}{dx} &= \frac{-y(2x) - (1+x^2)^2\tan(x-y)}{-(1+x^2) - (1+x^2)^2\tan(x-y)}
\end{aligned}$$

(c) $f(x) = \log_5(xe^x)$

$$\begin{aligned}
f'(x) &= \frac{1}{xe^x\ln(5)}(xe^x + e^x) \\
&= \frac{x+1}{x\ln(5)}
\end{aligned}$$

(d) $x\ln(|x|) - x$

$$\begin{aligned}
\frac{d}{dx}[x\ln(|x|) - x] &= (x)\frac{1}{x} + \ln(|x|) - 1 \\
&= 1 + \ln(|x|) - 1 \\
&= \ln(|x|)
\end{aligned}$$

(e) $H(y) = \ln\left(\frac{(5y^3+y^2)^2}{\sqrt{y^3+3}}\right)$

$$\begin{aligned}
H(y) &= \ln\left(\frac{(5y^3+y^2)^2}{\sqrt{y^3+3}}\right) \\
&= \ln((5y^3+y^2)^2) - \ln(\sqrt{y^3+3}) \\
&= 2\ln(5y^3+y^2) - \frac{1}{2}\ln(y^3+3) \\
H'(y) &= \frac{2}{5y^3+y^2}(15y^2+2y) - \frac{1}{2(y^3+3)}(3y^2) \\
&= \frac{30y+4}{5y^2+y} - \frac{3y^2}{2y^3+6}
\end{aligned}$$

2. Use implicit differentiation to find an equation of the tangent line to the curve $\sin(x+y) = 2x - 2y$ at the point (π, π).

$$\begin{aligned}
\frac{d}{dx}[\sin(x+y)] &= \frac{d}{dx}[2x - 2y] \\
\cos(x+y)[1 + \frac{dy}{dx}] &= 2 - 2\frac{dy}{dx} \\
\cos(x+y) + \cos(x+y)\frac{dy}{dx} &= \\
\cos(x+y)\frac{dy}{dx} + 2\frac{dy}{dx} &= -\cos(x+y) + 2 \\
\frac{dy}{dx}[\cos(x+y) + 2] &= -\cos(x+y) + 2 \\
\frac{dy}{dx} &= \frac{-\cos(x+y) + 2}{\cos(x+y) + 2}
\end{aligned}$$

Plugging in the point (π, π),

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-\cos(2\pi) + 2}{\cos(2\pi) + 2} \\
&= \frac{-1 + 2}{1 + 2} \\
&= \frac{1}{3}
\end{aligned}$$

Solving for b,

$$\begin{aligned}
y &= mx + b \\
\pi &= \frac{1}{3}(\pi) + b \\
\pi - \frac{1}{3}(\pi) &= b \\
b &= \frac{2}{3}\pi
\end{aligned}$$

The tangent line at the point (π, π) is thus,

$$y = \frac{1}{3}x + \frac{2}{3}\pi$$

3. Use logarithmic differentiation to find the derivative of $y = \frac{x^{\frac{5}{3}}\sqrt{x^4+2}}{(4x+1)^5}$

$$\begin{aligned}
y &= \frac{x^{\frac{5}{3}}\sqrt{x^4+2}}{(4x+1)^5} \\
\ln(y) &= \ln\left(\frac{x^{\frac{5}{3}}\sqrt{x^4+2}}{(4x+1)^5}\right) \\
&= \ln(x^{\frac{5}{3}}\sqrt{x^4+2})) - \ln((4x+1)^5) \\
&= \ln(x^{\frac{5}{3}}) + \ln(\sqrt{x^4+2})) - \ln((4x+1)^5) \\
&= \frac{5}{3}\ln(x) + \frac{1}{2}\ln(x^4+2) - 5\ln(4x+1)
\end{aligned}$$

Taking the derivative,

$$\begin{aligned}
\frac{dy}{dx}\left(\frac{1}{y}\right) &= \frac{5}{3x} + \frac{4x^3}{2(x^4+2)} - \frac{20}{4x+1} \\
\frac{dy}{dx} &= y\left[\frac{5}{3x} + \frac{4x^3}{2(x^4+2)} - \frac{20}{4x+1}\right] \\
\frac{dy}{dx} &= \frac{x^{\frac{5}{3}}\sqrt{x^4+2}}{(4x+1)^5}\left[\frac{5}{3x} + \frac{4x^3}{2(x^4+2)} - \frac{20}{4x+1}\right]
\end{aligned}$$

4. Find the derivative of $y = \sqrt{x}^{\;x}$.

$$\begin{aligned} y &= \sqrt{x}^{\;x} \\ &= x^{\frac{1}{2}x} \\ \ln(y) &= \ln(x^{\frac{1}{2}x}) \\ &= \frac{1}{2}x\ln(x) \end{aligned}$$

Taking the derivative,

$$\begin{aligned} \left(\frac{1}{y}\right)\frac{dy}{dx} &= \frac{1}{2}\ln(x) + \frac{1}{2}\left(\frac{x}{x}\right) \\ \frac{dy}{dx} &= y\left[\frac{1}{2}\ln(x) + \frac{1}{2}\right] \\ \frac{dy}{dx} &= \sqrt{x}^{\;x}\left[\frac{1}{2}\ln(x) + \frac{1}{2}\right] \end{aligned}$$

5. Find the derivative of $y = \sin(x)^{\ln(x)}$.

$$\begin{aligned} y &= \sin(x)^{\ln(x)} \\ \ln(y) &= \ln(\sin(x)^{\ln(x)}) \\ \ln(y) &= \ln(x)\ln(\sin(x)) \end{aligned}$$

Taking the derivative,

$$\begin{aligned} \left(\frac{1}{y}\right)\frac{dy}{dx} &= \frac{1}{x}(\ln(\sin(x))) + \ln(x)(\frac{\cos(x)}{\sin(x)}) \\ \frac{dy}{dx} &= y[\frac{\ln(\sin(x))}{x} + \frac{\ln(x)\cos(x)}{\sin(x)}] \\ \frac{dy}{dx} &= \sin(x)^{\ln(x)}[\frac{\ln(\sin(x))}{x} + \frac{\ln(x)\cos(x)}{\sin(x)}] \end{aligned}$$

6. ($\star$) Give an example of y as a function of x. Now give a function which has an implicit relationship between x and y. Can you solve your second function for y (i.e. make it explicit)? Is it always possible to do so?

$y = x^2$ is an example of an explicit function while $x^2 + y^2 = 1$ is an example of a function with an implicit relationship between x and y. Note that you can solve the given implicit function, but–in general–it is not always possible to do so: $\sin(x+y) = x+y$.

7. True or False; if false, provide a counterexample. If true, provide a true generalization.

If P is a degree 6 polynomial, then $\frac{d^7}{dx^7}[P] = 0$.

True; in general, if P is a polynomial, then the $\deg(P) + 1$ derivative of P is 0.