Worksheet 14: Implicit Differentiation

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1. Using Leibniz notation, find the derivative of $x^2 + y^2 = 1$ without solving for y. Why is Leibniz notation good for implicit differentiation?

$$\begin{aligned} x^2 + y^2 &= 1 \\ 2x\frac{dx}{dx} + 2y\frac{dy}{dx} &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2x &= -2y\frac{dy}{dx} \\ \frac{2x}{-2y} &= \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Leibniz notation is good for implicit differentiation because it shows that we aren't actually treating x and y differently–we're simply taking the derivative with respect to x, so the $\frac{dx}{dx}$ term cancels to 1.

2. Write out the derivatives of the inverse trigonometric functions; what patterns do you see?

$$\begin{array}{lcl|lcl} \frac{d}{dx}[\sin^{-1}(x)] &=& \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}[\csc^{-1}(x)] &=& -\frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}[\cos^{-1}(x)] &=& -\frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}[\sec^{-1}(x)] &=& \frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}[\tan^{-1}(x)] &=& \frac{1}{1+x^2} & \frac{d}{dx}[\cot^{-1}(x)] &=& -\frac{1}{x^2+1} \end{array}$$

Note that sin and cos are only different by a negative sign and that csc and sec mirror them. Furthermore, reading left to right, the sign always flips. Finally, tan and cot are almost exactly the same.

3. Find the derivative:

(a) $x^3 + y^3 = 6xy$

$$\begin{aligned} x^3 + y^3 &= 6xy \\ 3x^2\frac{dx}{dx} + 3y^2\frac{dy}{dx} &= 6y\frac{dx}{dx} + 6x\frac{dy}{dx} \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\ -6x\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx}(-6x + 3y^2) &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{-6x + 3y^2} \end{aligned}$$

(b) $f(x) = \arcsin(x^3 + 1)$

$$f'(x) = \frac{1}{\sqrt{1 - (x^3+1)^2}}(3x^2)$$

(c) $xe^y = x - y$

$$\begin{aligned}
\frac{dx}{dx}e^y + xe^y\frac{dy}{dx} &= \frac{dx}{dx} - \frac{dy}{dx} \\
e^y + xe^y\frac{dy}{dx} &= 1 - \frac{dy}{dx} \\
\frac{dy}{dx} + xe^y\frac{dy}{dx} &= 1 - e^y \\
\frac{dy}{dx}(1 + xe^y) &= 1 - e^y \\
\frac{dy}{dx} &= \frac{1 - e^y}{1 + xe^y}
\end{aligned}$$

(d) $y = e^{cot^{-1}(x^2)} + x^3$
See next worksheet solution.

(e) $\tan(x - y) = \frac{y}{1+x^2}$
See next worksheet solution.

4. Use implicit differentiation to find an equation of the tangent line to the curve $\sin(x + y) = 2x - 2y$ at the point (π, π).
See next worksheet solution.