Worksheet 13: Trigonometric Derivatives & Chain Rule

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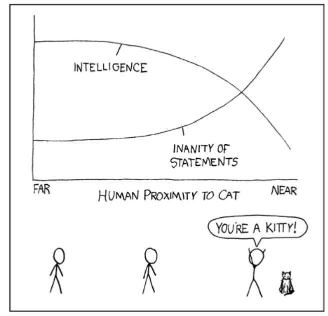

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1. Find the derivative:

   (a) $y = 2\sec(x) - \csc(x)$
   $y' = 2\sec(x)\tan(x) - (-\csc(x)\cot(x))$
   $y' = 2\sec(x)\tan(x) + \csc(x)\cot(x)$

   (b) $f(\theta) = \sin(\theta)\cos(\theta)$
   $f'(\theta) = -\sin(\theta)\sin(\theta) + \cos(\theta)\cos(\theta) = -\sin^2(\theta) + \cos^2(\theta)$

   (c) $f(\theta) = \sin(\theta)\csc(\theta) = \sin(\theta)\frac{1}{\sin(\theta)} = 1$
   $f'(\theta) = 0$

   (d) $y = \frac{1-\sec(x)}{\tan(x)}$
   It's easier to divide the fraction out (obtaining $\cot(x) - \csc(x)$), but the long way is:

$$\begin{aligned} y' &= \frac{\tan(x)(-\sec(x)\tan(x)) - (1-\sec(x))(sec^2(x))}{\tan^2(x)} \\ &= \frac{-\sec(x)\tan^2(x) - \sec^2(x) + \sec^3(x)}{\tan^2(x)} \end{aligned}$$

   And to obtain the same result as the easier way,

$$\begin{aligned} &= \frac{-\frac{\sin^2(x)}{\cos^3(x)} - \frac{1}{\cos^2(x)} + \frac{1}{\cos^3(x)}}{\frac{\sin^2(x)}{\cos^2(x)}} \\ &= -\sec(x) - \csc^2(x) + \csc^2(x)\sec(x) \\ &= -\csc^2(x) + \sec(x)(\csc^2(x) - 1) \\ &= -\csc^2(x) + \sec(x)(\cot^2(x)) \\ &= -\csc^2(x) + \csc(x)\cot(x) \end{aligned}$$

2. Evaluate: $\lim_{x\to 0} \frac{\sin 4x}{\sin 6x}$

$$\begin{aligned} \lim_{x\to 0} \frac{\sin 4x}{\sin 6x} &= \lim_{x\to 0} \frac{\sin 4x}{\sin 6x}\left(\frac{\frac{1}{24x}}{\frac{1}{24x}}\right) \\ &= \lim_{x\to 0} \frac{\frac{\sin 4x}{24x}}{\frac{\sin 6x}{24x}} \\ &= \lim_{x\to 0} \frac{\left(\frac{1}{6}\right)\frac{\sin 4x}{4x}}{\left(\frac{1}{4}\right)\frac{\sin 6x}{6x}} \\ &= \left(\frac{4}{6}\right)\lim_{x\to 0} \frac{\frac{\sin 4x}{4x}}{\frac{\sin 6x}{6x}} \\ &= \left(\frac{4}{6}\right)\frac{\lim_{x\to 0} \frac{\sin 4x}{4x}}{\lim_{x\to 0} \frac{\sin 6x}{6x}} \\ &= \left(\frac{4}{6}\right)\frac{1}{1} \\ &= \frac{4}{6} \end{aligned}$$

3. Evaluate: $\lim_{x\to 0} \dfrac{\sin x^2}{x}$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x^2}{x} &= \lim_{x\to 0} \frac{\sin x^2}{x}\left(\frac{x}{x}\right)\\ &= \lim_{x\to 0} \frac{\sin x^2}{x^2}\left(x\right)\\ &= \left[\lim_{x\to 0} \frac{\sin x^2}{x^2}\right]\left[\lim_{x\to 0} x\right]\\ &= 1(0)\\ &= 0\end{aligned}$$

4. Write the chain rule in both Leibniz and Newtonian notation.

   Assume $F(x) = f(g(x))$. Then,

$$F'(x) = f'(g(x))g'(x)$$

$$\frac{d}{dx}F(x) = \frac{dF}{dg}\left(\frac{dg}{dx}\right)$$

5. Find the derivative:

   (a) $y = (x+1)^{10}$
   $y' = 10(x+1)^9(1) = 10(x+1)^9$

   (b) $f(x) = e^{-x}$
   $f'(x) = e^{-x}(-1) = -e^{-x}$

   (c) $y = (x^2 + e^{2x-1})^2$
   $y' = 2(x^2 + e^{2x-1})^1(2x + e^{2x-1}(2)) = 4(x^2 + e^{2x-1})(x + e^{2x-1})$

   (d) $f(\theta) = \sin(\cos(\tan(\theta)))$
   $f'(\theta) = \cos(\cos(\tan(\theta)))(-\sin(\tan(\theta))(\sec^2(\theta))) = -\sec^2(\theta)\sin(\tan(\theta))\cos(\cos(\tan(\theta)))$

   (e) $y = 2^x$
   $y' = 2^x \ln(2)$

   (f) $f(x) = ((3x^5 + e^{2x} + x^4)^{12} + 2x)^3$
   $f'(x) = 3((3x^5 + e^{2x} + x^4)^{12} + 2x)^2(12(3x^5 + e^{2x} + x^4)^{11}(15x^4 + 2e^2x + 4x^3) + 2)$

6. You've learned about composing two functions, e.g., if $f(x) = x^2$ and $g(x) = x + 1$, then $f \circ g(x) = (x+1)^2$. Decomposing a function reverses this process. For instance, if $h(x) = (x+1)^2$ then $f \circ g(x)$, where f and g are as above, is called a decomposition of h.

   (a) $h(x) = (x+1)^2$ can be decomposed in several other ways. Find one.

   (b) Can a function *always* be decomposed in more than one way? For instance, can $h(x) = x$ be decomposed in more than one way? Give an example of a function that can't be decomposed in two ways, or explain why you think that all functions can be decomposed in more than one way.

   (a) Consider $f(x) = x$ and $g(x) = (x+1)^2$; then $f(g(x))$ and $g(f(x))$ are both decompositions.

   (b) Every function can be decomposed in multiple ways; for a function $f(x)$, both decompositions given by $h(g(x))$ and $g(h(x)$ where $g(x) = x$ and $h(x) = f(x)$ are always viable. Less trivially, $g(h(x))$ where $g(x) = x^3$ and $h(x) = f(x)^{\frac{1}{3}}$ is also always available.

7. Before the midterm, you found the derivative of $f(x) = |x|$ by cases; find the derivative of $f(x)$ with the chain rule instead.

   Notice that $f(x) = |x| = \sqrt{x^2}$. Thus, $f'(x) = \frac{1}{2}(x^2)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2}}$. Note, however, that the domain of the derivative above is unknown; being able to solve for the derivative of a function algebraically doesn't make it differentiable over its domain!

8. Using Leibniz notation, find the derivative of $x^2 + y^2 = 1$ without solving for y. Why is Leibniz notation good for implicit differentiation?

   See the solutions to the next worksheet.