Worksheet 12: Review

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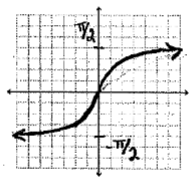
1. Sketch the following:

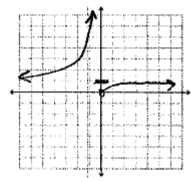
   (a) $f(x) = arctan(x)$
   What are the asymptotes?
   What is $f(1)$?

   Consulting the graph, the asymptotes are $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ (note that the since the tan graph is symmetric across the y-axis, the graph of its inverse has the same asymptotes only moved to the y-axis).

   Recalling that $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} = \cos(\frac{\pi}{4}), f(1) = arctan(1) = \frac{\pi}{4}$.

   (b) $f(x) = \frac{1}{4^{\frac{1}{x}}}$
   What are the asymptotes?

   Looking at the graph, the asymptotes are $y = 1$ and $x = 0$ (as a result of the left, not the right).

2. Evaluate the limit: $\lim_{x\to\infty} \sqrt{x^4 + 3x + 4}$.

   Analyzing this problem conceptually, we're taking the square root of $\infty + \infty + 4$ or ∞. Noting that square roots don't affect ∞, so the value of the limit is ∞.

3. Determine the infinitary limit: $\lim_{x\to\infty} \frac{x^2 - x - 2}{x + 1}$.

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2 - x - 2}{x+1} &= \lim_{x\to\infty} \frac{x^2 - x - 2}{x+1}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)\\ &= \lim_{x\to\infty} \frac{\frac{x^2-x-2}{x}}{\frac{x+1}{x}}\\ &= \lim_{x\to\infty} \frac{x - 1 - \frac{2}{x}}{1 + \frac{1}{x}}\end{aligned}$$

   Conceptually, this is:

$$\begin{aligned}&= \frac{\infty - 1 - 0}{1 + 0}\\ &= \infty\end{aligned}$$

4. Find the values at which f is discontinuous. Let f be defined by $f(x) = x+1$ if $x < 0$, $f(x) = e^x$ for $0 \leq x \leq 1$, and $f(x) = 2-x$ if $x > 1$.

   Note first that each of the individual functions making up f is defined over its entire domain and is either a function known to be continuous or a continuity-preserving combination of continuous functions. It follows that f is continuous at everywhere besides the junctures between functions, which remain to be checked. Consider, then, $x = 0$. At this point, one function approaches $0 + 1 = 1$ while the other approaches $e^0 = 1$. It follows that the limit will exist and equal $f(a)$ here, so the function is continuous at $x = 0$. Consider, then, $x = 1$. At this point, the left side is approaching $e^1 = e$ while the right approaches $2 - 1 = 1$. Since the left and right limits are going to different values, the limit to 1 doesn't exist and the function is not continuous at this point. It follows that f is only discontinuous at $x = 1$.

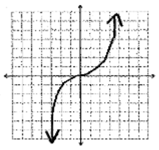
5. Differentiate $(x^3+1)e^x = x^3(e^x)+e^x$

   Using the product rule:

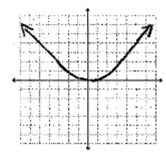
$$\frac{d}{dx}[(x^3)e^x + e^x] = (x^3)e^x + (3x^2)e^x + e^x$$

6. Sketch a function with following derivative:

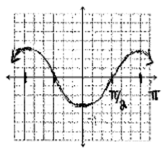
   (a) $f'(x) = x^2$



   (b) $f'(x) = arctan(x)$



   (c) $f'(x) = sin(x)$



7. Show that there exists a root of $arcsin(x) = 4x+1$ over the real numbers.

   Note first that this is an intermediate value theorem problem for $f(x) = 4x+1-arcsin(x)$. Since $f(x)$ is a continuity-preserving combination of continuous functions (with domain $\mathbb{R}$), it is continuous. It remains, then, to pick a closed interval $[a,b]$ where 0 is between $f(a)$ and $f(b)$. Note that, $f(0) = 1$, and so only a point with f-value below 0 needs to be found. Taking, for example, -1, we obtain $f(-1) = -4+1-(-\frac{\pi}{2}) = -3+\frac{\pi}{2}$, a negative number. By the intermediate value theorem, it therefore follows that there exists a point in $(-3+\frac{\pi}{2},1)$ where f is zero.

8. Find the equation of the tangent line to the curve $y = x^4 - 1$ at the point where $x = 1$.

   Taking the derivative, $y' = 4x^3$. It follows by the definition of the derivative that $4(1)^3 = 4$ is the slope of the tangent line at $x = 1$. Plugging in $x = 1$ to the original function, we see that the point requested is $(1,0)$. Thus, the equation of the tangent line at that point is $y = 4x+b$; solving for b,

$$0 = 4(1) + b$$

$$b = -4$$

   And so the final result is $y = 4x - 4$.