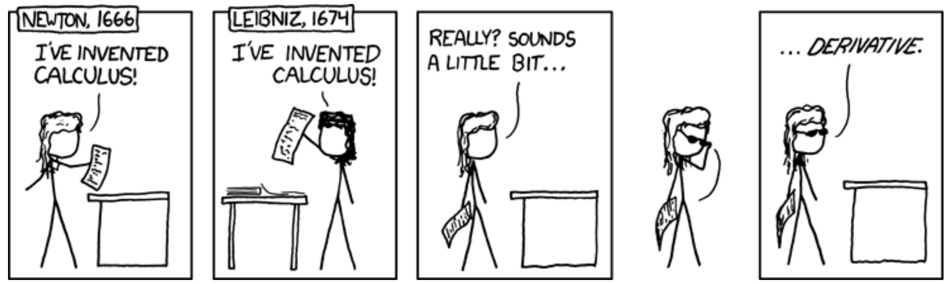
Worksheet 11: $\frac{d}{dx}$ + Review

Russell Buehler
b.r@berkeley.edu



www.xkcd.com

1. Write the general form for:

   (a) The Product Rule
   $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + \frac{d}{dx}[f(x)]g(x)$

   (b) The Quotient Rule
   $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{g(x)^2}$

2. Find the derivative.

   (a) $f(x) = e^x + 14\pi^2 e^4 + x^\pi + 4x^e + \frac{x+2}{\sqrt{x}}$
   $f'(x) = e^x + \pi x^{\pi-1} + 4ex^{e-1} + \frac{1}{2}x^{-\frac{1}{2}} - x^{-\frac{3}{2}}$

   (b) $y = \frac{x^3}{1-x^2}$
   $y' = \frac{(1-x^2)(3x^2) - x^3(-2x)}{(1-x^2)^2} = \frac{3x^2 - 3x^4 + 2x^4}{1-2x^2+x^4} = \frac{3x^2 - x^4}{1-2x^2+x^4}$

   (c) $g(x) = \sqrt{x}e^x$
   $g'(x) = (\frac{1}{2}x^{-\frac{1}{2}})(e^x) + e^x(\sqrt{x}) = \frac{e^x}{2\sqrt{x}} + \sqrt{x}e^x$

   (d) $z = \frac{t}{(2t-1)^2} = \frac{t}{4t^2-4t+1}$
   $z' = \frac{(4t^2-4t+1)(1) - t(8t-4)}{(4t^2-4t+1)^2} = \frac{4t^2-4t+1-8t^2+4t}{(4t^2-4t+1)^2} = \frac{-4t^2+1}{(4t^2-4t+1)^2} = \frac{-(2t-1)(2t+1)}{(2t-1)^4} = \frac{-(2t+1)}{(2t-1)^3}$

   (e) $v(t) = \frac{t-\sqrt{t}}{t^{\frac{1}{3}}}$
   $v'(t) = \frac{(t^{\frac{1}{3}})(1-\frac{1}{2}t^{-\frac{1}{2}}) - (t-\sqrt{t})(\frac{1}{3}t^{-\frac{2}{3}})}{t^{\frac{2}{3}}} = \frac{t^{\frac{1}{3}} - \frac{1}{2}t^{-\frac{1}{6}} - \frac{1}{3}t^{\frac{1}{3}} + \frac{1}{3}t^{-\frac{1}{6}}}{t^{\frac{2}{3}}} = \frac{\frac{2}{3}t^{\frac{1}{3}} - \frac{1}{6}t^{-\frac{1}{6}}}{t^{\frac{2}{3}}} = \frac{2}{3t^{\frac{1}{3}}} - \frac{1}{6t^{\frac{5}{6}}}$
   or
   $v(t) = t^{\frac{2}{3}} - t^{\frac{1}{6}}$, and so $v'(t) = \frac{2}{3}t^{-\frac{1}{3}} - \frac{1}{6}t^{-\frac{5}{6}}$

   (f) $K(y) = \frac{y(1-y^{\frac{4}{5}})}{y(e^y)} = \frac{1-y^{\frac{4}{5}}}{e^y}$
   $\frac{d}{dy}[K(y)] = \frac{e^y(\frac{4}{5}y^{-\frac{1}{5}}) - (1-y^{\frac{4}{5}})(e^y)}{e^{2y}} = \frac{\frac{4}{5}y^{-\frac{1}{5}} - 1 - y^{\frac{4}{5}}}{e^y}$

3. True or False; if true, give an explanation as to why. If false, give a counterexample or correction.

   (a) If f,g are differentiable, then $\frac{d}{dx}[f(x)+g(x)] = f'(x) + g'(x)$.
   True; this is just the sum rule.

   (b) If f,g are differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.
   False; let $f(x) = g(x) = x$. Then, $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[x^2] = 2x$, but $f'(x)g'(x) = (1)(1) = 1$.

   (c) If $y = e^2$, then $y' = 2e$.
   False; $y' = 0$

   (d) The derivative of a polynomial is a polynomial.
   True; just apply the sum and power rules.

4. Determine for what values of x the function $f(x) = x|x|$ is differentiable and find a formula for f'.

   We work by cases. Case 1: Assume $x \geq 0$. Then $f(x) = x^2$ and $f'(x) = 2x$. Case 2: Assume $x < 0$, then $f(x) = -x^2$ and $f'(x) = -2x$. I could write the derivative as a piecewise function, but notice that, in both cases, $f'(x) = 2|x|$ (note that $x < 0$ for the second case). It follows that the derivative of $f(x) = x|x|$ is $2|x|$. It remains, however, to determine what the domain of the derivative is (note that the process above also applies to $|x|$ which is not differentiable at 0). In particular, note that by the product rule the derivative exists at every point except 0. It remains, then, to check this point. By definition, the derivative is:

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)|x+h|-(x)|x|}{h}\\
&= \lim_{h\to 0}\frac{(0+h)|0+h|-(0)|0|}{h}\\
&= \lim_{h\to 0}\frac{(h)|h|}{h}\\
&= \lim_{h\to 0}|h|\\
&= 0
\end{aligned}
$$

and so the derivative is defined at $x = 0$ as well. It follows that the derivative is defined over all reals.

5. Find all points on the curve $y = \frac{7}{3}x^3 + \frac{1}{4}x^4 + 6x^2$ where the tangent is horizontal.

   Note first that if the tangent is horizontal, its slope is 0. To find the points with slope 0, we simply take the derivative:

$$y' = 7x^2 + x^3 + 12x$$

   set it equal to 0,

$$7x^2 + x^3 + 12x = 0$$

   and solve for x.

$$x(x+3)(x+4) = 0$$

   The x-values for the points with horizontal tangent lines are therefore $0, -3$, and -4. Plugging these values into the original function, we obtain the points $(0,0), (-3, \frac{45}{4})$, and $(-4, \frac{20}{3})$ as our solution.

6. Solve.

   (a) $\lim\limits_{x\to\infty}\dfrac{11x^3-5}{x^2-5x+11}$

$$
\begin{aligned}
\lim_{x\to\infty}\frac{11x^3-5}{x^2-5x+11} &= \lim_{x\to\infty}\frac{11x^3-5}{x^2-5x+11}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)\\
&= \lim_{x\to\infty}\frac{\frac{11x^3-5}{x^2}}{\frac{x^2-5x+11}{x^2}}\\
&= \lim_{x\to\infty}\frac{11x-\frac{5}{x^2}}{1-\frac{5}{x^2}+\frac{11}{x^2}}\\
&= \infty
\end{aligned}
$$

   (b) $\lim\limits_{x\to-\infty}\dfrac{(2x+1)(x+17)}{(x+4)(x+7)}$

$$
\begin{aligned}
\lim_{x\to-\infty}\frac{(2x+1)(x+17)}{(x+4)(x+7)} &= \lim_{x\to-\infty}\frac{2x^2+35x+17}{x^2+11x+28}\\
&= \lim_{x\to-\infty}\frac{2x^2+35x+17}{x^2+11x+28}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)\\
&= \lim_{x\to-\infty}\frac{\frac{2x^2+35x+17}{x^2}}{\frac{x^2+11x+28}{x^2}}\\
&= \lim_{x\to-\infty}\frac{2+\frac{35}{x}+\frac{17}{x^2}}{1+\frac{11}{x}+\frac{28}{x^2}}\\
&= 2
\end{aligned}
$$

(c) $\lim\limits_{x \to 4} \dfrac{2-\sqrt{x}}{4x-x^2}$

$$\begin{aligned}
\lim_{x \to 4} \frac{2-\sqrt{x}}{4x-x^2} &= \lim_{x \to 4} \frac{2-\sqrt{x}}{x(4-x)} \\
&= \lim_{x \to 4} \frac{2-\sqrt{x}}{x(2-\sqrt{x})(2+\sqrt{x})} \\
&= \lim_{x \to 4} \frac{1}{x(2+\sqrt{x})} \\
&= \frac{1}{4(2+2)} \\
&= \frac{1}{16}
\end{aligned}$$

7. Let $f(x)$ be a curve.

- Give an expression for the secant line through points $(y, f(y))$ and $(z, f(z))$
  By definition, the slope of the secant line is $m = \frac{f(y)-f(z)}{y-z}$; solving for b,

$$\begin{aligned}
f(y) &= \frac{f(y)-f(z)}{y-z}y + b \\
f(y) - \frac{f(y)-f(z)}{y-z}y &= b \\
\frac{f(y)(y-z)}{y-z} - \frac{yf(y)-yf(z)}{y-z} &= b \\
\frac{f(y)y - f(y)z - yf(y) + yf(z)}{y-z} &= b \\
\frac{-f(y)z + yf(z)}{y-z} &= b
\end{aligned}$$

  And thus, the equation of the secant line (replacing the standard y with y'),

$$y' = \frac{f(y)-f(z)}{y-z}x + \frac{-f(y)z + yf(z)}{y-z}$$

- Give an expression for the tangent line through the point $x = w$
  Note that we can find the slope at $x = w$ using the definition of the derivative; the derivative is defined as

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

  so the slope or derivative at w is

$$\lim_{h \to 0} \frac{f(w+h)-f(w)}{h}$$

  An equation for the tangent line at $(w, f(w))$ is thus,

$$y - f(w) = \left(\lim_{h \to 0} \left[\frac{f(w+h)-f(w)}{h} \right] \right)(x-w)$$