

Math 1A
Midterm 2
2006

1. Differentiate $\sin(\cos(\tan(x)))$.

Solution. If $f(x) = \sin(\cos(\tan(x)))$ then by the chain rule we have that

$$f'(x) = [\cos(\cos(\tan(x)))] \cdot [-\sin(\tan(x))] \cdot [\sec^2(x)]$$

The brackets are only there to emphasize the applications of the chain rule. □

2. Find an equation of the tangent line to the curve $y = \frac{1}{\sin(x) + \cos(x)}$ at the point $(0, 1)$.

Solution. $y' = \frac{-(\cos(x) - \sin(x))}{(\sin(x) + \cos(x))^2}$ so $y'(0) = \frac{-(1-0)}{(0+1)^2} = \frac{-1}{1} = -1$ and so the equation of the tangent line at the point $(0, 1)$ is

$$y - 1 = -x$$

□

3. Find $\frac{dy}{dx}$ by implicit differentiation if $1 + x = \sin(xy^2)$.

Solution. Differentiate both sides of $1 + x = \sin(xy^2)$ with respect to x :

$$\begin{aligned}\frac{d}{dx}(1 + x) &= \frac{d}{dx}(\sin(xy^2)) \\ 1 &= \cos(xy^2)(y^2 + 2xyy') \\ \sec(xy^2) &= y^2 + 2xyy' \\ \sec(xy^2) - y^2 &= 2xyy' \\ y' &= \frac{\sec(xy^2) - y^2}{2xy}\end{aligned}$$

□

4. Find a formula for the n th derivative of x^{-3} .

Solution. The strategy here is to find the derivative for a few values of n (e.g. $n = 1, 2, 3$) and recognize a pattern. To this end, let $f(x) = x^{-3}$. Then

$$\begin{aligned}f'(x) &= -3x^{-4} \\f''(x) &= 12x^{-5} \\f'''(x) &= -60x^{-6}\end{aligned}$$

We see that the exponent keeps coming down and the sign keeps alternating. Formally,

$$f^{(n)}(x) = (-1)^n \frac{(n+2)!}{2} x^{-3-n}$$

□

5. Differentiate $x^{\sin(x)}$.

Solution. Let $f(x) = x^{\sin(x)} = e^{\ln(x)\sin(x)}$. Then

$$f'(x) = e^{\ln(x)\sin(x)} \left(\frac{\sin(x)}{x} + \ln(x) \cos(x) \right) = x^{\sin(x)} \left(\frac{\sin(x)}{x} + \ln(x) \cos(x) \right)$$

□

6. Find the derivative of $\sinh(x) \tanh(x)$.

Solution. Let $f(x) = \sinh(x) \tanh(x)$. Then

$$\begin{aligned}f'(x) &= \cosh(x) \tanh(x) + \sinh(x) \operatorname{sech}^2(x) \\&= \sinh(x) + \tanh(x) \operatorname{sech}(x)\end{aligned}$$

□

7. Use differentials or a linear approximation to estimate $\ln(.97)$.

Solution. Let $f(x) = \ln(x)$. Then

$$\begin{aligned}f'(x) &= \frac{1}{x} \\f'(1) &= 1\end{aligned}$$

and so the tangent line to $f(x)$ at the point $(1, 0)$ is

$$\begin{aligned}y - 0 &= 1(x - 1) \\y &= x - 1\end{aligned}$$

meaning our approximation is this line evaluated at $x = .97$ so

$$\ln(.97) \approx .97 - 1 = -.03$$

□

8. Find the absolute maximum and absolute minimum values $f(x) = x^3 - 3x - 1$ on the interval $[-3, 3]$.

Solution.

$$f'(x) = 3x^2 - 3$$

so

$$\begin{aligned} f'(x) = 0 &\iff 3x^2 - 3 = 0 \\ &\iff 3x^2 = 3 \\ &\iff x^2 = 1 \\ &\iff x = \pm 1 \end{aligned}$$

Now

$$\begin{aligned} f(1) &= 1 - 3 - 1 = -3 \\ f(-1) &= -1 + 3 - 1 = 1 \\ f(3) &= 27 - 9 - 1 = 17 \\ f(-3) &= -27 + 9 - 1 = -19 \end{aligned}$$

so -19 is the absolute minimum and 17 is the absolute maximum on the interval $[-3, 3]$. \square

9. Find all critical numbers of the function $f(x) = 5x^{\frac{2}{3}} + x^{\frac{5}{3}}$.

Solution.

$$f'(x) = \frac{10}{3x^{\frac{1}{3}}} + \frac{5x^{\frac{2}{3}}}{3}$$

so immediately we see that $x = 0$ is a critical point. Now we want to solve $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\iff \frac{10}{3x^{\frac{1}{3}}} + \frac{5x^{\frac{2}{3}}}{3} = 0 \\ &\iff \frac{10}{3x^{\frac{1}{3}}} = -\frac{5x^{\frac{2}{3}}}{3} \\ &\iff -2 = x \end{aligned}$$

Thus the critical points of $f(x)$ are

$$x = 0, -2$$

\square

10. Show that the equation $2x - 1 - \sin(x) = 0$. has exactly one real root.

Solution. Let $f(x) = 2x - 1 - \sin(x)$. First we show that $f(x)$ has at least one real root. Observe that $f(x)$ is a continuous function. Now

$$\begin{aligned} f(0) &= -1 < 0 \\ f(\pi) &= 2\pi - 1 > 0 \end{aligned}$$

so by the intermediate value theorem

$$\exists c \in (0, \pi) : f(c) = 0.$$

Now

$$f'(x) = 2 - \cos(x) \geq 1$$

so $f'(x)$ has no real roots. Suppose, for a contradiction, $f(x)$ has two or more real roots. By Rolle's theorem, $f'(x)$ would have a real root which contradicts the fact that $f'(x) \geq 1$ has no real roots. Thus it must be the case that $f(x)$ has at most one real root. Since we know that $f(x)$ has at least one real root, we now have that $f(x)$ has exactly one real root. \square

11. Find the intervals on which f is increasing or decreasing and all local maximum and minimum values of $f(x) = 3x^{\frac{2}{3}} - x$.

Solution.

$$f'(x) = \frac{2}{x^{\frac{1}{3}}} - 1$$

so

$$\begin{aligned} f'(x) = 0 &\iff \frac{2}{x^{\frac{1}{3}}} - 1 = 0 \\ &\iff \frac{2}{x^{\frac{1}{3}}} = 1 \\ &\iff 2 = x^{\frac{1}{3}} \\ &\iff 8 = x \end{aligned}$$

and so the critical points of $f(x)$ are $x = 0, 8$. Now we solve for when $f'(x) < 0$:

$$\begin{aligned} f'(x) < 0 &\iff \frac{2}{x^{\frac{1}{3}}} - 1 < 0 \\ &\iff \frac{2}{x^{\frac{1}{3}}} < 1 \\ &\iff 2 < x^{\frac{1}{3}} \text{ or } x < 0 \\ &\iff 8 < x \text{ or } x < 0 \end{aligned}$$

and so

$$f'(x) < 0 \iff x \in (-\infty, 0) \cup (8, \infty).$$

Then the only remaining possibility is that

$$f'(x) > 0 \iff x \in (0, 8)$$

Thus f is increasing on $(0, 8)$, decreasing on $(-\infty, 0) \cup (8, \infty)$, $f(8) = 4$ is a local maximum, and $f(0) = 0$ is a local minimum. \square

12. Find the limit $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x}$.

Solution. We do not need to apply L'Hôpital's rule. The numerator tends to $-\infty$. The bottom tends to 0 but stays positive. Thus

$$\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x} = -\infty.$$

□

13. Find the limit $\lim_{x \rightarrow 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{x^4}$

Solution. The numerator and denominator both tend to 0 so we may apply L'Hôpital's rule:

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{x^4} = \lim_{x \rightarrow 0} \frac{-\sin(x) + x}{4x^3}$$

Again, the numerator and denominator both tend to 0 so we may apply L'Hôpital's rule another time:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{-\sin(x) + x}{4x^3} &= \lim_{x \rightarrow 0} \frac{-\cos(x) + 1}{12x^2} \\ &= \lim_{x \rightarrow 0} \frac{\sin(x)}{24x} \quad (\text{again we have used L'Hôpital's rule}) \\ &= \lim_{x \rightarrow 0} \frac{\cos(x)}{24} \quad (\text{again we have used L'Hôpital's rule}) \\ &= \frac{1}{24}. \end{aligned}$$

□

14. Sketch the curve $y = \sqrt[3]{x^2 - 1}$.

Solution. The domain of $y = \sqrt[3]{x^2 - 1}$ is \mathbb{R} , the entire real line. Now

$$y' = \frac{2x}{3(x^2 - 1)^{\frac{2}{3}}}$$

so immediately we see that the critical points are $x = 0, \pm 1$. Notice that the denominator is always positive since

$$(x^2 - 1)^{\frac{2}{3}} = (\sqrt[3]{x^2 - 1})^2$$

meaning the sign of y' is determined by the numerator, $2x$. Thus

$$\begin{aligned} y' &> 0 \iff 0 < x < 1 \text{ and } 1 < x \\ y' &< 0 \iff x < -1 \text{ and } -1 < x < 0. \end{aligned}$$

Thus $\sqrt[3]{0^2 - 1} = -1$ is a local minimum. We also see that y is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. y clearly has zeroes at $x = \pm 1$. For large $|x|$, we have that

$$x^2 - 1 \sim x^2 \text{ so } \sqrt[3]{x^2 - 1} \sim \sqrt[3]{x^2} = x^{\frac{2}{3}}.$$

The function obtains a local minimum when $x = 0$ as we observed earlier. The function is not differentiable at $x = \pm 1$. In fact, from our formula for the derivative, it is easy to see that

$$\begin{aligned}\lim_{x \rightarrow -1} y' &= \lim_{x \rightarrow -1} \frac{2x}{3(x^2 - 1)^{\frac{2}{3}}} = -\infty \\ \lim_{x \rightarrow 1} y' &= \lim_{x \rightarrow 1} \frac{2x}{3(x^2 - 1)^{\frac{2}{3}}} = \infty.\end{aligned}$$

Click here to see the graph. Zoom in/out as necessary. □

15. Sketch the curve $y = \frac{\ln(x)}{x}$ for $x > 0$.

Solution. The domain of y is $\mathbb{R}_{>0}$, the positive real axis. Now

$$y' = \frac{x \cdot \frac{1}{x} - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}.$$

We proceed to solve $y' = 0$:

$$\begin{aligned}y' = 0 &\iff \frac{1 - \ln(x)}{x^2} = 0 \\ &\iff 1 - \ln(x) = 0 \\ &\iff 1 = \ln(x) \\ &\iff e = x\end{aligned}$$

so the only critical point is $x = e$. Again, notice that the denominator of the derivative is always positive so the sign of the derivative is determined by the sign of the numerator. Thus

$$\begin{aligned}y' > 0 &\iff 1 - \ln(x) > 0 \iff 1 > \ln(x) \iff x < e \\ y' < 0 &\iff 1 - \ln(x) < 0 \iff 1 < \ln(x) \iff e < x\end{aligned}$$

whence y is decreasing on (e, ∞) , increasing on $(0, e)$. Moreover, $\frac{\ln(e)}{e} = \frac{1}{e}$ is a local maximum. The function has a zero at $x = 1$. Now we want to compute $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x}$:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} \quad (\text{by L'Hôpital's rule}) \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} \\ &= 0.\end{aligned}$$

Next, we compute $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x} = -\infty$ (this was problem #12). The function is differentiable for every $x \in \mathbb{R}_{>0}$. Click here to see the graph. Zoom in/out as necessary. □