Computing modular forms for the Weil representation

by

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Abstract

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We describe an algorithm to compute bases of modular forms with rational coefficients for the Weil representation associated to an even lattice. In large enough weights the forms we construct are zero-values of Jacobi forms of rational index, while in smaller weights our construction uses the theory of mock modular forms. The main application is in computing automorphic products.

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Contents

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Symbols

(f,g)	The Petersson scalar product of two cusp forms f,g
$\langle v,w\rangle$	The scalar product on $\mathbb{C}[A]$
A	Discriminant group of Λ
a,b,c,d	Entries of a matrix in $SL_2(\mathbb{Z})$
(b^+,b^-)	The signature of Λ
β,γ,δ	Elements of the discriminant group of Λ
Δ	The normalized cusp form of weight 12
E_k	The Eisenstein series of weight k with constant term $\mathfrak{e}_0$
E_k^*	The real-analytic deformation of E_k
$E_{k,m,\beta}$	The Jacobi Eisenstein series of weight k and index (m,β) with constant term $\mathfrak{e}_0$
e	The dimension of Λ, or 2.71828...
$\mathbf{e}$	$\mathbf{e}(\tau)=e^{2\pi i\tau}$
$\mathfrak{e}_\gamma$	A basis element of $\mathbb{C}[A]$
Γ	The Gamma function $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,\mathrm{d}x$
$\tilde{\Gamma}$	The metaplectic group
Γ_∞	The subgroup of $\tilde{\Gamma}$ that fixes $\mathfrak{e}_0$
$H(n)$	The Hurwitz class number
$\mathcal{H}$	Heisenberg group
$I(k,y,n,s)$	An integral defined in section 4.2
$\mathcal{J}$	Jacobi group
$\mathcal{J}_\infty$	The subgroup of $\mathcal{J}$ that fixes $\mathfrak{e}_0$
K_c	A Kloosterman sum

k	The weight of modular forms or Jacobi forms
L_p	A local L-function
Λ	An even lattice
Λ'	The dual lattice of Λ
$P_{k,m,\beta}$	The Poincaré series of weight k and index (m,β)
Q	A quadratic form
$Q_{k,m,\beta}$	The Poincaré square series of weight k and index (m,β)
q	$q = e^{2\pi i\tau}$
$R_{k,m,\beta}$	A rational Poincaré series of weight k and index (m,β)
ρ	Weil representation
ρ_β	The semidirect product of ρ and σ_β
S	The element $(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau})$ of $Mp_2(\mathbb{Z})$
$\mathbf{S}$	A Gram matrix of the quadratic form Q
σ_β	The Schrödinger representation attached to $\beta \in A$
σ_k	The divisor power sum $\sigma_k(n) = \sum_{d\mid n} d^k$, or more generally $\sigma_k(n,\chi) = \sum_{d\mid n} d^k \chi(n/d)$
T	The element $(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1)$ of $Mp_2(\mathbb{Z})$
τ	A complex variable $\tau = x + iy$ confined to the upper half-plane
u	The real part of z
v	The imaginary part of z, or a vector
x	The real part of τ
y	The imaginary part of τ
z	A complex variable $z = u + iv$
ζ	The Riemann zeta function $\zeta(s)$, or an element $\zeta = (\lambda,\mu,t)$ of the Heisenberg group, or $\zeta = e^{2\pi i z}$, depending on context
ζ_{Ig}	An Igusa (local) zeta function
$\square$	A rational square

Chapter 1

Introduction

This thesis contains some constructions of vector-valued modular forms with rational Fourier coefficients for the Weil representation attached to an even lattice, or equivalently an integral quadratic form Q. The Weil representation describes the behavior of a number of interesting functions under the modular group, including: theta functions of even lattices, half-integer weight modular forms satisfying the Kohnen plus-space condition, Dedekind's eta function and the generating series of partition numbers (as in section 3.2 of [19]), indefinite theta functions and mock modular forms in the sense of [75], and the input functions into the Borcherds and Kudla-Millson theta lifts (see e.g. [16]).

The application we will generally have in mind is the construction of Borcherds products. This began in [3] with the construction of meromorphic, orthogonal modular forms with Heegner divisors that have product expansions in which the exponents are themselves Fourier coefficients of modular forms of half-integral weight. A simple example is the weight zero meromorphic modular form of level one with a character,

$$j(\tau)^{1/3} = q^{-1/3}(1-q)^{-248}(1-q^2)^{26752}(1-q^3)^{-4096248} \times ...$$

where $j(\tau)$ is the j-invariant, in which the exponent of $(1-q^n)$ is the coefficient of q^{n^2} in the weight 1/2, level 4 modular form with a pole at ∞ below:

$$F(\tau) = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 - 4096248q^9 + O(q^{12}).$$

This construction was generalized and clarified in the subsequent paper [4]; from the point of view of [4], the "input functions" $F(\tau)$ should be thought of as modular forms for a Weil representation with singularities at cusps. In addition to direct applications to modular forms (the existence of modular forms with prescribed divisors can be used to determine generators and relations for algebras of modular forms, as in [26], [27], for example), Borcherds products have found important applications to the theory of Kac-Moody algebras through the work of Borcherds, and Gritsenko-Nikulin (for example, [34], [35]) and Scheithauer (for example, [55], [56]) and the moduli theory of K3 surfaces (for example, [36]) as well as other fields, and are therefore of wide interest.

The spaces of modular forms for Weil representations have good arithmetic properties that are not shared by general representations of $SL_2(\mathbb{Z})$. A fundamental result is that the Weil representation of an even lattice of level $N \in \mathbb{N}$ factors through a double-cover of $SL_2(\mathbb{Z}/N\mathbb{Z})$, so the theory of Hecke operators and newforms implies that in every weight there is a basis of modular forms for the Weil representation whose Fourier coefficients are algebraic integers. In fact, it was shown by McGraw [45] that one can always find a basis whose Fourier coefficients are *rational*, resolving a conjecture of Borcherds [5].

It turns out that one can construct such a rational basis of modular forms by setting $z = 0$ in Jacobi Eisenstein series of varying index. The Fourier coefficients of the Jacobi Eisenstein series can be calculated very quickly so the results here may be useful for computations involving Borcherds products. (Some examples in this direction are given in sections 3.6 and 5.9 and the appendices.)

The organization of this thesis is as follows:

Chapter 2 reviews some of the theory of even lattices, Weil representations, modular forms, Jacobi forms and Borcherds products. An important distinction is the focus on Jacobi forms of rational index; there are well-known examples of Jacobi forms of half-integer index (including the classical theta functions), but more general indices seem to have been mostly ignored in the literature.

Chapters 3 through 6 are essentially the papers [68],[69],[70],[72] with minor changes that improve continuity and that may add details to some of the proofs and examples. In chapter 3 we define Poincaré square series, which are vector-valued modular forms whose Petersson inner product with a cusp form g gives a special value of the symmetric square L-function attached to g. We prove that in weights $k \geq 5/2$, these modular forms have rational Fourier coefficients and contain all cusp forms within their span, and give a formula to compute them based on the Jacobi Eisenstein series. The resulting formula is a short sum over coefficients of

Eisenstein series and is therefore no harder to compute than the Eisenstein series itself. We describe how to compute the nontrivial part of these Eisenstein series (the local L-functions) using p-adic generating functions in the sense of [23] (in particular, we unravel the work of [23] to give an algorithm that is also valid at the prime $p = 2$); the results appear in sections 3.3 and 3.8 and are rather messy, but much faster than computing the local L-functions naively.

Chapter 4 describes the behavior of the formula of Bruinier and Kuss for the vector-valued Eisenstein series in weights $k \in \{1, 3/2, 2\}$. These are generally mock modular forms and we require the calculations of chapter 3 to determine their shadows. This motivates chapter 5, which works out the Fourier coefficients of Poincaré square series in weights $3/2$ and 2 via holomorphic projection. These are still rational, and in weight 2 they are still enough to produce all cusp forms; while this does not seem to be true in weight $3/2$.

Chapter 6 observes that many of the relations known to hold among Hurwitz class numbers are special cases of the main result of chapter 5 applied to certain two-dimensional lattices and the smallest possible index, and derives some relations that may be new.

Finally, chapter 7 gives analogous results in "antisymmetric" weights; that is, where the lattice signature (b^+, b^-) and weight k are related by $2k + b^+ - b^- \equiv 2\,(4)$ instead of $2k + b^+ - b^- \equiv 0\,(4)$. In this case, we can use the first development coefficients of Jacobi Eisenstein series to span all cusp forms, instead of the zero-values.

In the appendices we compute tables of paramodular and Hermitian modular forms that are Borcherds products, as applications of the algorithm of chapter 3.

Chapter 2

Background

2.1 Lattices

In this section we review some standard results on lattices and discriminant forms. The main reference for this material was [50].

Definition 1. Let V be a finite-dimensional (real) vector space with nondegenerate symmetric bilinear form $\langle -, - \rangle$.
(i) A **lattice** in V is a discrete subgroup $\Lambda \subseteq V$ such that $\mathrm{span}_{\mathbb{R}}(\Lambda) = V$.
(ii) Λ is **even** if $\langle v, v \rangle \in 2\mathbb{Z}$ for all $v \in \Lambda$.
(iii) Let Λ be an even lattice. The **dual lattice** is the subgroup

$$\Lambda' = \Big\{ w \in V : \ \langle v, w \rangle \in \mathbb{Z} \text{ for all } v \in \Lambda \Big\} \subseteq V.$$

We call Λ **unimodular** if $\Lambda' = \Lambda$.

Any lattice Λ is torsion-free (as a subgroup of a vector space). Since it is discrete, it must be finitely generated over $\mathbb{Z}$: if $e_1, ..., e_N \in \Lambda$ are any elements that span V then $\Lambda / \mathrm{span}_{\mathbb{Z}}(e_1, ..., e_N)$ is discrete and compact and therefore finite, so Λ is generated by $e_1, ..., e_N$ and that finite set. The structure theorem for finitely-generated abelian groups implies that $\Lambda \cong \mathbb{Z}^n$ is free of rank $n = \dim V$.

In this way one can always assume without loss of generality that $V = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$, with bilinear form given by some symmetric matrix $\mathbf{S}$ with integer entries and even diagonal:

$$\langle v, w \rangle = v^T \mathbf{S} w.$$

The dual lattice of Λ is then $\Lambda' = \mathbf{S}^{-1}\mathbb{Z}^n$.

It is sometimes more convenient to work with the associated quadratic form $Q(x) = \frac{1}{2}\langle x, x\rangle$. This determines $\langle -, -\rangle$ completely through the polarization identity:

$$\langle x, y\rangle = \frac{\langle x+y, x+y\rangle - \langle x, x\rangle - \langle y, y\rangle}{2} = Q(x+y) - Q(x) - Q(y), \quad x, y \in V.$$

Evenness of the lattice Λ is equivalent to $Q(x) \in \mathbb{Z}$ for all $x \in \Lambda$. In particular, the form

$$Q : \Lambda' \longrightarrow \mathbb{Q}$$

induces a well-defined quadratic form on the quotient group (which we also denote Q):

$$Q : \Lambda'/\Lambda \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad x \bmod \Lambda \mapsto Q(x) \bmod 1;$$

indeed, if $x, y \in \Lambda'$ are elements with $x - y \in \Lambda$ then polarization implies

$$Q(x) - Q(y) = Q(x-y) + \langle x-y, y\rangle \in \mathbb{Z} + \langle \Lambda, \Lambda'\rangle \subseteq \mathbb{Z}.$$

Therefore we attach to Λ a discriminant form:

Definition 2. A **discriminant form** (A, Q) is a finite abelian group A together with a nondegenerate quadratic form $Q : A \to \mathbb{Q}/\mathbb{Z}$, i.e. a function with the properties
(i) $Q(\lambda x) = \lambda^2 Q(x)$ for all $\lambda \in \mathbb{Z}$ and $x \in A$;
(ii) $\langle x, y\rangle = Q(x+y) - Q(x) - Q(y)$ is a nondegenerate bilinear form.

The setting of discriminant forms provides a convenient structure for our work on modular forms. In some sense it is no more general than the study of lattices:

Lemma 3. *(i) Let (A, Q) be a discriminant form. Then there is an even lattice $(\Lambda, \langle -, -\rangle)$ and an isomorphism $\phi : \Lambda'/\Lambda \to A$ of discriminant forms; i.e. an isomorphism of groups with*

$$Q(\phi(x + \Lambda)) = \frac{1}{2}\langle x, x\rangle + \mathbb{Z}$$

for all $x \in \Lambda'$.
(ii) Suppose Λ_1, Λ_2 are two even lattices that induce the same discriminant form (A, Q). Then there are even unimodular lattices U_1, U_2 such that $U_1 \oplus \Lambda_1 \cong U_2 \oplus \Lambda_2$.

Here $\oplus$ denotes the orthogonal direct sum of lattices: in particular, we add the quadratic forms i.e.

$$Q_{\Lambda_1 \oplus \Lambda_2} = Q_{\Lambda_1} + Q_{\Lambda_2}.$$

Proof. (i) See [65], theorem 6.
(ii) See [50], theorem 1.3.1. □

Any even unimodular lattice has signature (b^+, b^-) constrained by $b^+ - b^- \in 8\mathbb{Z}$. This means that every discriminant form (A, Q) has a well-defined **signature**

$$\mathrm{sig}(A) = \mathrm{sig}(A, Q) \in \mathbb{Z}/8\mathbb{Z},$$

defined by $\mathrm{sig}(A) = b^+ - b^-$ if Λ is any even lattice of signature (b^+, b^-) with discriminant form isomorphic to (A, Q). It is interesting to point out that this signature can be computed intrinsically in terms of the Gauss sum of (A, Q):

Proposition 4 (Milgram's formula)**.** *Let (A, Q) be a discriminant form; then*

$$\mathbf{e}\Big(\frac{1}{8}\mathrm{sig}(A)\Big) = \frac{1}{\sqrt{|A|}} \sum_{x \in A} \mathbf{e}\Big(Q(x)\Big).$$

Here $\mathbf{e}(t) = e^{2\pi i t}$ for $t \in \mathbb{C}/\mathbb{Z}$.

Proof. See [49], appendix 4. □

2.2 Schrödinger representations and the Weil representation

Let (A, Q) be a fixed discriminant form.

There is a natural 2-cocycle (the determinant form)

$$\omega\Big((x_1, y_1), (x_2, y_2)\Big) = x_1 y_2 - x_2 y_1$$

which determines a nontrivial cohomology class $[\omega] \in H^2(\mathbb{R}^2, \mathbb{R})$. (The cocycle condition

$$\omega(v_1 + v_2, v_3) + \omega(v_1, v_2) = \omega(v_1, v_2 + v_3) + \omega(v_2, v_3),\ v_1, v_2, v_3 \in \mathbb{R}^2$$

follows easily from multilinearity of the determinant.) The central extension of $\mathbb{R}^2$ associated to $[\omega]$ is called the **continuous Heisenberg group**: explicitly

$$\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2 \rtimes_\omega \mathbb{R},$$

which has underlying set $\mathbb{R}^2 \times \mathbb{R}$ and group operation

$$(v_1, t_1) \cdot (v_2, t_2) = (v_1 + v_2, t_1 + t_2 + \omega(v_1, v_2)).$$

We define the **integer Heisenberg group** $\mathcal{H}$ to be the subgroup of $\mathcal{H}_{\mathbb{R}}$ of tuples with integer entries.

There is an action of $\mathcal{H}_{\mathbb{R}}$ on $L^2(\mathbb{R})$ which is well-known from physics: the continuous Heisenberg group is related to the group generated by the exponentiated position and momentum operators and this action is their effect on wavefunctions. (Some details do not quite work here as the latter group is the extension by the cocycle $\frac{\omega}{2}$ rather than ω.) This is known classically as the **Schrödinger representation** of $\mathcal{H}_{\mathbb{R}}$. One would like to replace this by an action of the integer Heisenberg group on the group algebra $\mathbb{C}[A]$, i.e. the complex vector space generated by basis objects $\mathfrak{e}_\gamma$, $\gamma \in A$ together with the scalar product that makes $\mathfrak{e}_\gamma$, $\gamma \in A$ an orthonormal basis. (The group algebra is the analogue of $L^2(\mathbb{R})$ after interpreting elements $f = \sum_\gamma f(\gamma)\mathfrak{e}_\gamma$ as functions $f : A \to \mathbb{C}$.) The fact that one cannot divide by two in $\mathbb{Q}/\mathbb{Z}$ makes direct approaches somewhat complicated.

It seems necessary to use the fact that A is supplied with a quadratic form Q and not merely a bilinear form $\langle -, - \rangle$. For every $\beta \in A$ one can use $Q(\beta)$ to "divide by two" and define a variant σ_β of the Schrödinger representation. The family of representations $(\sigma_\beta)_{\beta \in A}$ can then be used to mimic the construction over $\mathbb{R}$ in an elementary way.

Proposition 5. *Let $\beta \in A$. Then*

$$\sigma_\beta : \mathcal{H} \longrightarrow \mathrm{GL}\, \mathbb{C}[A],$$

$$\sigma_\beta(\lambda, \mu, t)\mathfrak{e}_\gamma = \mathbf{e}\Big(\mu\langle \beta, \gamma \rangle + (t - \lambda\mu)Q(\beta)\Big)\mathfrak{e}_{\gamma - \lambda\beta}$$

is a unitary representation.

Proof. This is proved by direct computation: for any tuples $(\lambda_1, \mu_1, t_1), (\lambda_2, \mu_2, t_2) \in \mathcal{H}$ and any $\gamma \in A$,

$$\begin{aligned}
&\sigma_\beta(\lambda_1, \mu_1, t_1)\sigma_\beta(\lambda_2, \mu_2, t_2)\mathfrak{e}_\gamma \\
&= \mathbf{e}\Big(\mu_2\langle\beta,\gamma\rangle + (t_2 - \lambda_2\mu_2)Q(\beta)\Big)\sigma_\beta(\lambda_1, \mu_1, t_1)\mathfrak{e}_{\gamma-\lambda_1\beta} \\
&= \mathbf{e}\Big(\mu_2\langle\beta,\gamma\rangle + \mu_1\langle\beta,\gamma-\lambda_2\beta\rangle + (t_2 - \lambda_2\mu_2 + t_1 - \lambda_1\mu_1)Q(\beta)\Big)\mathfrak{e}_{\gamma-(\lambda_1+\lambda_2)\beta},
\end{aligned}$$

while

$$\begin{aligned}
&\sigma_\beta(\lambda_1+\lambda_2, \mu_1+\mu_2, t_1+t_2+\lambda_1\mu_2-\lambda_2\mu_1)\mathfrak{e}_\gamma \\
&= \mathbf{e}\Big((\mu_1+\mu_2)\langle\beta,\gamma\rangle + (t_1+t_2-\lambda_1\mu_1-\lambda_2\mu_2-2\lambda_2\mu_1)Q(\beta)\Big)\mathfrak{e}_{\gamma-(\lambda_1+\lambda_2)\beta},
\end{aligned}$$

which are equal. σ_β is unitary as one can check on the elements of the form $(\lambda, 0, 0)$, $(0, \mu, 0)$ and $(0, 0, t)$ which generate $\mathcal{H}$. $\square$

Lemma 6. *No nontrivial subgroups of* $\mathbb{C}[A]$ *are invariant under all Schrödinger representations* σ_β *simultaneously.*

The representations σ_β are by themselves not generally irreducible; for example, σ_0 is always trivial. This lemma is a reasonable substitute.

Proof. It will be helpful to interpret elements of $\mathbb{C}[A]$ as functions $A \to \mathbb{C}$; the function f corresponding to $\sum_\gamma f(\gamma)\mathfrak{e}_\gamma$. The scalar product is then the usual L^2-product.

Suppose $V \subseteq \mathbb{C}[A]$ is a nonzero subspace that is invariant under all σ_β, and suppose $g \in \mathbb{C}[A]$ is orthogonal to all of V. It follows that for any $\delta \in A$ and any $f \in V$,

$$\begin{aligned}
0 &= \sum_\beta \mathbf{e}\Big(-\langle\beta,\delta\rangle\Big)\langle\sigma_\beta(0,1,0)f, \gamma\rangle \\
&= \sum_{\beta,\gamma} \mathbf{e}\Big(\langle\beta,\gamma-\delta\rangle\Big)f(\gamma)g(\gamma) \\
&= f(\delta)g(\delta).
\end{aligned}$$

Applying this to $\sigma_\beta(1,0,0)f$ instead of f we obtain $f(\delta-\beta)g(\delta) = 0$ for all $\delta, \beta \in A$; since f can be taken nonzero, it follows that $g = 0$ and therefore $V = \mathbb{C}[A]$. $\square$

The classical Schrödinger representation can be used to construct the (projective) Weil representation of SL_2 on $L^2(\mathbb{R})$ as in [67]. We can use a similar argument to construct the (projective) Weil representation of $SL_2(\mathbb{Z})$ on $\mathbb{C}[A]$ out of the representations σ_β. (This is far easier than the situation of [67].) Recall that there is a natural action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ from the right (given by ignoring the final component and matrix multiplication):

$$(\lambda, \mu, t) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\lambda + c\mu, b\lambda + d\mu, t).$$

In particular, letting $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the familiar generators of $SL_2(\mathbb{Z})$ we find

$$(\lambda, \mu, t) \cdot S = (\mu, -\lambda, t), \;\; (\lambda, \mu, t) \cdot T = (\lambda, \lambda + \mu, t).$$

Lemma 7. *For any $M \in SL_2(\mathbb{Z})$, the twisted Schrödinger representations*

$$\sigma_\beta^M(\zeta) = \sigma_\beta(\zeta \cdot M), \;\; \zeta = (\lambda, \mu, t) \in \mathcal{H}$$

are simultaneously equivalent to σ_β; that is, there is an operator $\overline{\rho}(M) \in GL\,\mathbb{C}[A]$ such that

$$\overline{\rho}(M)^{-1}\sigma_\beta(\zeta)\overline{\rho}(M) = \sigma_\beta(\zeta \cdot M)$$

for all $\zeta \in \mathcal{H}$ and all $\beta \in A$.

Proof. This is a finite analogue of the Stone-von Neumann theorem. It is enough to prove this claim for the standard generators $M = S, T$.
(i) Let $M = T$. Then $\overline{\rho}(T)^{-1}\sigma_\beta(0,\mu,0)\overline{\rho}(T) = \sigma_\beta(0,\mu,0)$ is equivalent to T acting diagonally, i.e. $\overline{\rho}(T)\mathfrak{e}_\gamma = u(\gamma)\mathfrak{e}_\gamma$ for some $u(\gamma) \in \mathbb{C}^\times$. We need to check the claim under $\beta = (\lambda, 0, 0)$; i.e. we need to solve

$$u(\gamma - \lambda\beta)^{-1}u(\gamma) = \langle \sigma_\beta(\lambda, \lambda, 0)\mathfrak{e}_\gamma, \mathfrak{e}_{\gamma - \lambda\beta}\rangle = \mathbf{e}\Big(\lambda\langle \beta, \gamma\rangle - \lambda^2 Q(\beta)\Big).$$

We can write $\lambda\langle\beta,\gamma\rangle - \lambda^2 Q(\beta) = Q(\gamma) - Q(\gamma - \lambda\beta)$ and therefore take $\overline{\rho}(M)\mathfrak{e}_\gamma = \mathbf{e}\left(Q(\gamma)\right)\mathfrak{e}_\gamma$.

(ii) Replacing σ_β by σ_β^S has the effect of swapping the translation operators $\sigma_\beta(\lambda, 0, 0)$ with the modulation operators $\sigma_\beta(0, \mu, 0)$; it is well-known that the discrete Fourier transform has the same property so we are led to try

$$\overline{\rho}(S)\mathfrak{e}_\gamma = \frac{1}{\sqrt{|A|}}\sum_\delta \mathbf{e}\Big(-\langle\gamma,\delta\rangle\Big)\mathfrak{e}_\delta.$$

Indeed the convolution theorem implies $\overline{\rho}(S)^{-1}\sigma_\beta(\zeta)\overline{\rho}(S) = \sigma_\beta(\zeta \cdot S)$. $\square$

Schur's lemma implies that the equation

$$\phi^{-1}\sigma_\beta(\zeta)\phi = \sigma_\beta(\zeta) \text{ for all } \beta \in A,\ \zeta \in \mathcal{H}$$

forces $\phi \in GL\,\mathbb{C}[A]$ to be scalar: indeed if λ is an eigenvalue of ϕ then $\ker(\phi - \lambda\,\mathrm{id})$ is simultaneously invariant and nonzero so $\ker(\phi - \lambda\,\mathrm{id}) = \mathbb{C}[A]$ and $\phi = \lambda$. It follows from this that $\overline{\rho}$ determines a projective representation, i.e. a homomorphism

$$\overline{\rho} : SL_2(\mathbb{Z}) \longrightarrow PGL\,\mathbb{C}[A] = GL\,\mathbb{C}[A]/\mathbb{C}^\times, \quad M \mapsto \overline{\rho}(M) \bmod \mathbb{C}^\times.$$

The extent to which $\overline{\rho}$ fails to be a true representation can be measured on the generators S, T. Recall that $SL_2(\mathbb{Z})$ is presented by the relations $S^4 = \mathrm{id}$, $S^2 = (ST)^3$. Fourier inversion implies $\overline{\rho}(S)^2\mathfrak{e}_\gamma = \mathfrak{e}_{-\gamma}$. On the other hand

$$\begin{aligned}
&(\overline{\rho}(S)\overline{\rho}(T))^3\mathfrak{e}_\gamma \\
&= \frac{1}{\sqrt{|A|^3}} \sum_{\delta_1,\delta_2,\delta_3 \in A} \mathbf{e}\Big(Q(\gamma) - \langle \gamma, \delta_1\rangle + Q(\delta_1) - \langle \delta_1, \delta_2\rangle + Q(\delta_2) - \langle \delta_2, \delta_3\rangle\Big)\mathfrak{e}_{\delta_3} \\
&= \frac{1}{\sqrt{|A|^3}} \sum_{\delta_1,\delta_2,\delta_3} \mathbf{e}\Big(Q(\gamma - \delta_1 + \delta_2) - \langle \gamma + \delta_3, \delta_2\rangle\Big)\mathfrak{e}_{\delta_3} \\
&= \frac{1}{\sqrt{|A|}} \sum_{\delta_1} \mathbf{e}(Q(\delta_1)) \cdot \mathfrak{e}_{-\gamma} \\
&= \mathbf{e}\Big(\frac{1}{8}\mathrm{sig}(A)\Big)\mathfrak{e}_{-\gamma},
\end{aligned}$$

where in the last line we used Milgram's formula.

In particular, we can find representatives $\rho(S)$,$\rho(T)$ such that $\rho(S)^4 = \mathrm{id}$ and $\rho(S)^2 = (\rho(S)\rho(T))^3$ exactly when $\mathrm{sig}(A)$ is even, by choosing the different representatives $\rho(T) = \overline{\rho}(T)$ as above and

$$\rho(S) = \mathbf{e}\Big(-\frac{1}{8}\mathrm{sig}(A)\Big)\overline{\rho}(S);$$

but if $\mathrm{sig}(A)$ is odd then we can only solve this problem over the double cover $Mp_2(\mathbb{Z})$ presented by generators S, T with $S^2 = (ST)^3$, $S^8 = I$ instead. Traditionally $Mp_2(\mathbb{Z})$ is the **metaplectic group** of pairs (M, ϕ) with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and a holomorphic branch $\phi(\tau)$ of $\sqrt{c\tau + d}$ on the upper half-plane $\mathbb{H}$. This is summarized in the proposition below:

Proposition 8. *There is a unitary representation* $\rho : Mp_2(\mathbb{Z}) \to GL\,\mathbb{C}[A]$ *which is given on the standard generators*

$$S = \Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \Big), \;\; T = \Big(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \Big)$$

by

$$\rho(S)\mathfrak{e}_\gamma = \frac{1}{\sqrt{|A|}} \mathbf{e}\Big(-\frac{1}{8}\mathrm{sig}(A) \Big) \sum_{\delta \in A} \mathbf{e}\Big(-\langle \delta, \gamma \rangle \Big) \mathfrak{e}_\delta,$$

$$\rho(T)\mathfrak{e}_\gamma = \mathbf{e}\Big(Q(\gamma) \Big) \mathfrak{e}_\gamma.$$

If $\mathrm{sig}(A)$ *is even then* ρ *factors through* $SL_2(\mathbb{Z})$*. Moreover if* $\mathrm{sig}(A) \equiv 0$ *mod* 4 *then* ρ *factors through* $PSL_2(\mathbb{Z})$*.*

Proof. This follows from the previous considerations after multiplying by the necessary scalars. It is easy to see that $\rho(S)$ and $\rho(T)$ are unitary. $\square$

Remark 9. The construction via Schrödinger representations suggests the following computation. Suppose $N \in \mathbb{N}$ is the smallest integer such that $NQ(\gamma) \in \mathbb{Z}$ for all $\gamma \in A$ (i.e. the **level** of (A, Q)). By polarization it follows that $N\langle \gamma, \beta \rangle \in \mathbb{Z}$ for all $\gamma, \beta \in A$. Therefore, if $M \in SL_2(\mathbb{Z})$ comes from the **principal congruence subgroup of level** N, i.e. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, d \equiv 1$, $b, c \equiv 0 \bmod N$, then $\sigma_\beta(\zeta \cdot M) = \sigma_\beta(\zeta)$ for all $\beta \in A$ and $\zeta \in \mathcal{H}$. By our form of Schur's lemma, $\rho(M)$ is a scalar. In particular, since theta functions are modular forms for ρ (see the next section), we obtain a simple proof that the theta function of a positive-definite lattice of level N is a modular form of level N (and some multiplier system). Actually, ρ factors through a double-cover of $SL_2(\mathbb{Z}/N\mathbb{Z})$ but this argument does not seem to yield that as easily.

The Weil representations (both the classical form and the finite analogue described above) are of considerable interest and therefore closed formulas for $\rho(M)$ for arbitrary $M \in Mp_2(\mathbb{Z})$ have been given. Here the papers [56] (especially section 4) and [63] are worth mentioning. We follow Bruinier and Kuss [18] and rely instead on a formula of Shintani [60] which will be important in later chapters.

Proposition 10. *Let* $M = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\right) \in Mp_2(\mathbb{Z})$ *where the branch of the square root satisfies* $\mathrm{re}[c\tau + d] > 0$ *for* $\tau \in \mathbb{H}$. *Denote by* $\rho(M)_{\beta,\gamma}$ *the components*

$$\rho(M)_{\beta,\gamma} = \langle \rho(M)\mathfrak{e}_\gamma, \mathfrak{e}_\beta \rangle.$$

(i) If $c = 0$*, then*

$$\rho(M)_{\beta,\gamma} = \mathbf{e}\Big(\frac{1}{8}\mathrm{sig}(A)(\mathrm{sgn}(d) - 1) + abQ(\beta)\Big)\delta_{\beta,a\gamma}.$$

(ii) If $c \neq 0$*, then let* Λ *be an even lattice of signature* (b^+, b^-) *with discriminant form* (A, Q)*; then*

$$\rho(M)_{\beta,\gamma} = \frac{1}{|c|^{(b^+ + b^-)/2}\sqrt{|A|}}\mathbf{e}\Big(-\frac{\mathrm{sgn}(c)}{8}\mathrm{sig}(A)\Big) \times$$
$$\times \sum_{v \in \Lambda/c\Lambda} \mathbf{e}\Big(\frac{aQ(v+\beta) - \langle \gamma, v + \beta\rangle + dQ(\gamma)}{c}\Big).$$

Here, δ is the Kronecker delta: $\delta_{\beta,a\gamma} = 1$ if $\beta = a\gamma$ and 0 otherwise.

2.3 The dual Weil representation

Fix a discriminant form (A, Q).

At most points in this work we will be more interested in the dual ρ^* of the Weil representation of (A, Q), rather than ρ itself. As a unitary representation, ρ^* is obtained from ρ essentially by taking complex conjugates. However we want to mention here that there are also less obvious relations between ρ and ρ^*. To clarify notation we specify $\rho_Q = \rho$.

Remark 11. The dual Weil representation of (A, Q) is exactly the Weil representation of $(A, -Q)$.

Proof. It is easy to see that $-Q$ is a valid $\mathbb{Q}/\mathbb{Z}$-valued nondegenerate quadratic form on A just as Q is, of signature

$$\mathrm{sig}(A, -Q) = -\mathrm{sig}(A, Q) \bmod 8.$$

Taking conjugates in the formulas

$$\rho_Q(T)\mathfrak{e}_\gamma = \mathbf{e}\big(Q(\gamma)\big)\mathfrak{e}_\gamma,$$

$$\rho_Q(S)\mathfrak{e}_\gamma = \frac{1}{\sqrt{|A|}}\mathbf{e}\Big(-\frac{1}{8}\mathrm{sig}(A)\Big)\sum_{\delta\in A}\mathbf{e}\Big(-\langle\delta,\gamma\rangle\Big)\mathfrak{e}_\delta$$

shows that $\rho_Q^*(T) = \rho_{-Q}(T)$ and $\rho_Q^*(S) = \rho_{-Q}(S)$; since S and T are generators this implies that $\rho_Q^* = \rho_{-Q}$ everywhere. □

Remark 12. There is an involution $\sim$ of $Mp_2(\mathbb{Z})$ given by

$$\tilde{S} = S^{-1},\ \ \tilde{T} = T^{-1}.$$

This is well-defined because it respects the relations:

$$(\tilde{S}\tilde{T})^3 = (S^{-1}T^{-1})^3 = S^{-1}(ST)^{-3}S = S^{-1}S^{-2}S = S^{-2} = \tilde{S}^2$$

and

$$\tilde{S}^8 = (S^8)^{-1} = I.$$

One can work out that the action in general is $\widetilde{(M,\phi)} = (\tilde{M},\tilde{\phi})$, where

$$\tilde{M} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \text{ if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and where $\tilde{\phi}(\tau) = \overline{\phi(-\overline{\tau})}$. This intertwines ρ and ρ^* in the sense that

$$\rho(\tilde{M}) = \rho^*(M),$$

as one can check on the generators $M = S, T$ as usual.

2.4 Modular forms

Fix a discriminant form (A, Q).

Definition 13. Let $k \in \frac{1}{2}\mathbb{Z}$. A **modular form of weight** k for the (dual) Weil representation of (A, Q) is a holomorphic function $f : \mathbb{H} \to \mathbb{C}[A]$ with the following properties:
(i) f transforms under the action of $Mp_2(\mathbb{Z})$ by

$$f(M\cdot\tau) = (c\tau+d)^k\rho^*(M)f(\tau),\ \ M \in Mp_2(\mathbb{Z}).$$

(If k is half-integer then the branch of the square root is prescribed by M as an element of $Mp_2(\mathbb{Z})$.)
(ii) f is holomorphic in ∞; i.e. if we expand f as a Fourier series,

$$f(\tau) = \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} c(n,\gamma) q^n \mathfrak{e}_\gamma, \quad q = e^{2\pi i \tau},$$

then all coefficients $c(n,\gamma)$ for $n < 0$ are zero.

It is convenient to use Petersson's slash notation

$$f\Big|_{k,\rho^*} M = (c\tau + d)^{-k} \rho(M)^{-1} f(\tau), \quad M \in \tilde{\Gamma} = Mp_2(\mathbb{Z}),$$

where condition (i) can be abbreviated as $f|_{k,\rho^*} M = f$.

The existence of such a Fourier expansion (in particular the condition $n \in \mathbb{Z} - Q(\gamma)$) follows from $f(\tau+1) = \rho^*(T) f(\tau)$. We denote the vector space of modular forms by $M_k(\rho^*)$, and the subspace of **cusp forms** (which are modular forms for which $c(0,\gamma) = 0$ for all γ, since we are dealing with the full modular group) is denoted $S_k(\rho^*)$. Both spaces are finite-dimensional and for $k > 2$ their dimensions can be calculated using the Riemann-Roch formula. For several reasons we will only consider modular forms of weights k satisfying

$$2k + \mathrm{sig}(A) \equiv 0 \bmod 4$$

(e.g. in the case of classical scalar-valued modular forms, where Q is unimodular, we consider only $k \in 2\mathbb{Z}$); in this case, an explicit Riemann-Roch calculation that is well-adapted to computation was given by Bruinier in section 2 of [13]. (The formula below is slightly modified from [13] but essentially the same.)

Proposition 14. *Let $G(a,A)$ denote the Gauss sum*

$$G(a,A) = \sum_{\gamma \in A} \mathbf{e}\Big(aQ(\gamma)\Big), \quad a \in \mathbb{Z},$$

and denote by B the sawtooth function

$$B(x) = x - \frac{\lfloor x \rfloor - \lfloor -x \rfloor}{2};$$

i.e. $B(x) = x - 1/2$ *for* $0 < x < 1$*, extended to be* 1*-periodic with* $B(x) = 0$ *for* $x \in \mathbb{Z}$*. Additionally define*

$$B_1 = \sum_{\gamma \in A} B\left(Q(\gamma)\right), \quad B_2 = \sum_{\substack{\gamma \in A \\ 2\gamma = 0}} B\left(Q(\gamma)\right).$$

Let $d = \#(A/\pm I)$ *denote the number of pairs* $\pm\gamma$ *and let*

$$\alpha_4 = \#\{\gamma \in A : \; Q(\gamma) \in \mathbb{Z}\}/\pm I$$

denote the number of pairs $\pm\gamma$ *with* $Q(\gamma) \in \mathbb{Z}$*. Then*

$$\begin{aligned} \dim M_k(\rho^*) = & \frac{d(k-1)}{12} \\ & + \frac{1}{4\sqrt{|A|}} \mathbf{e}\Big(\frac{2k + \mathrm{sig}(A)}{8}\Big) \mathrm{re}[G(2,A)] \\ & - \frac{1}{3\sqrt{3|A|}} \mathrm{re}\Big[\mathbf{e}\Big(\frac{4k + 3\mathrm{sig}(A) - 10}{24}\Big)(G(1,A) + G(-3,A))\Big] \\ & + \frac{\alpha_4 - B_1 - B_2}{2}, \end{aligned}$$

and $\dim S_k(\rho^*) = \dim M_k(\rho^*) - \alpha_4$*.*

This formula tends to fail in weight $k \leq 2$, where it (like most formulas) instead produces the "Euler characteristic" $\dim M_k(\rho^*) - \dim S_{2-k}(\rho)$. Ehlen and Skoruppa [30] have described an algorithm that computes dimensions in weight $k = 2$ and $k = 3/2$ that in practice seems quite efficient, relying on the known structure for $M_0(\rho^*)$ (which consists of constant Weil invariants) and $M_{1/2}(\rho^*)$ (where the components are theta series and related oldforms by the Serre-Stark theorem [58], and which was computed more precisely in [61]).

The inner product structure is important:

Definition 15. The **Petersson scalar product** on $S_k(\rho^*)$ is

$$(f, g) = \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} \langle f(\tau), g(\tau) \rangle y^{k-2} \, \mathrm{d}x \, \mathrm{d}y, \quad \tau = x + iy.$$

(This is well-defined because $\langle f(\tau), g(\tau) \rangle y^{k-2} \mathrm{d}x\mathrm{d}y$ is invariant under $Mp_2(\mathbb{Z})$.) Recall that $\langle -, - \rangle$ is the scalar product on $\mathbb{C}[A]$.

This gives numerous ways to define cusp forms indirectly: there are natural functionals on cusp forms such as extraction of Fourier coefficients or evaluation at points on $\mathbb{H}$. In weight $k \geq 5/2$, the **Poincaré series**

$$\begin{aligned} P_{k,n,\gamma}(\tau) &= \sum_{M \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \Big(q^n \mathfrak{e}_\gamma \Big) \Big|_{k,\rho} M \\ &= \frac{1}{2} \sum_{c,d} (c\tau + d)^{-k} e^{2\pi i n \frac{a\tau + b}{c\tau + d}} \rho^*(M)^{-1} \mathfrak{e}_\gamma, \end{aligned}$$

where $\tilde{\Gamma}_\infty$ is the subgroup of $\tilde{\Gamma}$ generated by T and $Z = S^2 = (ST)^3$, and c, d run through all pairs of coprime integers, are up to scalar multiple the cusp forms that extract Fourier coefficients:

Proposition 16. *For any cusp form* $f(\tau) = \sum_{\gamma,n} c(n,\gamma) q^n \mathfrak{e}_\gamma \in S_k(\rho^*)$,

$$(f, P_{k,n,\gamma}) = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} c(n,\gamma).$$

It follows easily that the Poincaré series $P_{k,n,\gamma}$ span $S_k(\rho^*)$ as (n,γ) runs through all valid indices $n \in \mathbb{Z} + Q(\gamma)$, as any cusp form orthogonal to all of them must be identically zero.

Proof. This is an argument due to Rankin: an expression of the form $\int_{SL_2(\mathbb{Z})\backslash \mathbb{H}} \sum_{M \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}}$ can be formally replaced by the integral $\int_{\tilde{\Gamma}_\infty \backslash \mathbb{H}}$, which is an integral over the rectangle $-1/2 < x < 1/2$ and $0 < y < \infty$ and therefore may be easier to compute. The absolute convergence of all expressions involved can be reduced to the convergence of $\sum_{(m,n) \neq (0,0)} \frac{1}{|m\tau + n|^{5/2}}$, which itself follows from the integral criterion and $\int_{x^2+y^2 \geq 1} \frac{1}{(x^2+y^2)^{5/2}} \, \mathrm{d}x \, \mathrm{d}y = 2\pi \int_1^\infty r^{-3/2} \, \mathrm{d}r < \infty$. By Rankin's method we find

$$\begin{aligned} (f, P_{k,n,\gamma}) &= \int_{\tilde{\Gamma} \backslash \mathbb{H}} \sum_{M \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \Big\langle f \Big|_{k,\rho} M(\tau), q^n \mathfrak{e}_\gamma \Big|_{k,\rho} M(\tau) \Big\rangle y^{k-2} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{-1/2}^{1/2} \int_0^\infty \sum_{j \in \mathbb{Q}} \sum_{\beta \in A} \langle c(j,\beta) q^j \mathfrak{e}_\beta, q^n \mathfrak{e}_\gamma \rangle y^{k-2} \, \mathrm{d}y \, \mathrm{d}x \\ &= \sum_{j \in \mathbb{Z} + Q(\gamma)} c(j,\gamma) \int_{-1/2}^{1/2} e^{2\pi i (j-n) x} \, \mathrm{d}x \int_0^\infty e^{2\pi i (j+n) y} y^{k-2} \, \mathrm{d}y \\ &= c(n,\gamma) \int_0^\infty e^{-4\pi n y} y^{k-2} \, \mathrm{d}y \\ &= \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} c(n,\gamma). \end{aligned}$$

∎

The Poincaré series were studied in [12] where expressions for their Fourier coefficients are given, but these expressions are rather unwieldy. Everything simplifies considerably when $\beta \in A$ satisfies $Q(\beta) = 0$ and $n = 0$: we obtain the **Eisenstein series**

$$E_{k,\beta}(\tau) = \sum_{M \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \mathfrak{e}_\beta \Big|_{k,\rho} M$$

(where $\mathfrak{e}_\beta$ is interpreted as a constant function). For $\beta = 0$ the series $E_{k,0}$ has rational Fourier coefficients that were given explicitly in [18]. More generally, a recent preprint of Schwagenscheidt [57] shows that all $E_{k,\beta}$ have rational Fourier coefficients and gives a formula to compute them directly. The functional characterization of Poincaré series remains true in some sense: the integral $(f, E_{k,\beta})$ converges for every cusp form $f \in S_k(\rho^*)$ and has value zero. However we will never need the series $E_{k,\beta}$ for any $\beta \neq 0$ so we will not consider this further.

Another class of modular forms is worth mentioning:

Definition 17. Suppose (A, Q) is the discriminant form of a positive-definite lattice Λ. The **theta function** of Λ is

$$\vartheta_\Lambda(\tau) = \sum_{x \in \Lambda'} q^{Q(x)} \mathfrak{e}_{x+\Lambda}.$$

If Λ has dimension e then ϑ_Λ is a weight $e/2$ modular form for the Weil representation of (A, Q). (The transformation under T is clear, and the transformation under S follows from the Poisson summation formula.) Every discriminant form arises from a positive-definite lattice (and this can be proved algorithmically, essentially by repeatedly replacing negative-norm vectors by their complement in an E8-lattice; see algorithm 2.3 of [53]), so one can always construct modular forms in certain weights by theta functions. There are generalizations of ϑ_Λ to include homogeneous polynomials which are harmonic under Q; these generalizations will produce cusp forms, while ϑ_Λ as defined above always has a constant term. Also, theta functions can be used to construct modular forms in weights where Eisenstein and Poincaré series do not converge.

2.5 Jacobi forms

Jacobi forms are functions of two variables (τ, z), with $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, which generalize both modular forms and elliptic functions. Most of the basic theory is due to Eichler and Zagier [31]. Here the group of transformations is the **Jacobi group**, which is the semidirect product of $Mp_2(\mathbb{Z})$ by its right-action on the integer Heisenberg group:

$$\mathcal{J} = \mathcal{H} \rtimes Mp_2(\mathbb{Z}).$$

One can identify $\mathcal{J}$ as a parabolic subgroup of $Mp_4(\mathbb{Z})$ (the metaplectic cover of the symplectic group $Sp_4(\mathbb{Z})$) through the embedding

$$\Big(\lambda, \mu, t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) \mapsto \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & t \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

under which the branch $\phi(\tau)$ of $\sqrt{c\tau + d}$ is sent to

$$\tilde{\phi}\Big(\begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix}\Big) = \phi(\tau_1).$$

By restricting the action of $Sp_4(\mathbb{Z})$ on the Siegel upper half-space $\mathbb{H}_2$ we obtain the action of $\mathcal{J}$ on $\mathbb{H} \times \mathbb{C}$:

$$\Big(\lambda, \mu, t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) \cdot (\tau, z) = \Big(\frac{a\tau + b}{c\tau + d}, \frac{\lambda\tau + z + \mu}{c\tau + d}\Big).$$

Suppose $\rho : \mathcal{J} \to GL(V)$ is a representation of $\mathcal{J}$ whose kernel has finite index. A **Jacobi form** of weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \mathbb{Q}_{>0}$ for ρ is a holomorphic function $\Phi : \mathbb{H} \times \mathbb{C} \to V$ satisfying the following properties:

(i) For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$,

$$\Phi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k \mathbf{e}\Big(\frac{mcz^2}{c\tau + d}\Big)\rho(M)\Phi(\tau, z);$$

(ii) For any $\zeta = (\lambda, \mu, t) \in \mathcal{H}$,

$$\Phi(\tau, z + \lambda\tau + \mu) = \mathbf{e}\Big(- m\lambda^2\tau - 2m\lambda z - m(\lambda\mu + t)\Big)\rho(\zeta)\Phi(\tau, z);$$

(iii) If we write out the Fourier series of Φ as

$$\Phi(\tau, z) = \sum_{n,r} v(n,r) q^n \zeta^r, \quad q = e^{2\pi i \tau},\ \zeta = e^{2\pi i z}$$

with coefficient vectors $v(n,r) \in V$, then $v(n,r) = 0$ whenever $n < r^2/4m$.

The Jacobi forms we will consider transform under representations ρ^*_β which arise as semidirect products of the dual Schrödinger representations σ^*_β by the dual Weil representation ρ^*:

$$\rho^*_\beta : \mathcal{J} \longrightarrow GL\,\mathbb{C}[A], \quad \rho^*_\beta(\zeta, M) = \rho^*(M)\sigma^*_\beta(\zeta).$$

(These are well-defined because $\rho^*(M)^{-1}\sigma^*_\beta(\zeta)\rho^*(M) = \sigma^*_\beta(\zeta \cdot M)$.) Jacobi forms for ρ_β can be written out as

$$\Phi(\tau, z) = \sum_{\gamma \in A} \sum_{n,r \in \mathbb{Q}} c(n,r,\gamma) q^n \zeta^r \mathfrak{e}_\gamma,$$

with the transformation laws resulting in some important restrictions:

(i) The transformation under $\zeta = (0,0,1) \in \mathcal{H}$ gives

$$\Phi(\tau, z) = \mathbf{e}(-mt)\sigma^*_\beta(0,0,t)\Phi(\tau,z) = \mathbf{e}\Big(t(Q(\beta) - m)\Big)\Phi(\tau, z),$$

so there are no nonzero Jacobi forms unless $m \in \mathbb{Z} - Q(\beta)$.

(ii) The transformation under T implies that $c(n,r,\gamma) = 0$ unless $n \in \mathbb{Z} - Q(\gamma)$ (the same restriction as we have for modular forms).

(iii) The transformation under $\zeta = (0,1,0)$ gives

$$\begin{aligned}
\sum_{\gamma \in A} \sum_{n,r \in \mathbb{Q}} c(n,r,\gamma)\mathbf{e}(r) q^n \zeta^r \mathfrak{e}_\gamma &= \Phi(\tau, z+1) \\
&= \sigma^*_\beta(0,1,0)\Phi(\tau,z) \\
&= \sum_{\gamma \in A} \mathbf{e}\Big(-\langle \beta, \gamma \rangle\Big) \sum_{n,r} c(n,r,\gamma) q^n \zeta^r \mathfrak{e}_\gamma,
\end{aligned}$$

so $c(n,r,\gamma) = 0$ unless $r \in \mathbb{Z} - \langle \beta, \gamma \rangle$.

(iv) The transformation under $Z = S^2$ restricts the weight to $2k + \mathrm{sig}(A) \in 2\mathbb{Z}$. As before we will generally only consider weights that satisfy

$$2k + \mathrm{sig}(A) \in 4\mathbb{Z}$$

as the $\mathfrak{e}_0$-component of any Jacobi form will otherwise vanish identically. In this case the transformation under Z forces

$$c(n,r,\gamma) = c(n,-r,-\gamma) \text{ for all } n,r,\gamma.$$

(v) The transformation under $\zeta = (\lambda, 0, 0)$ implies

$$\begin{aligned}\sum_{n,r,\gamma} c(n,r,\gamma) q^{n+r\lambda}\zeta^r \mathfrak{e}_\gamma &= \Phi(\tau, z+\lambda\tau) \\ &= q^{-m\lambda^2}\zeta^{-2m\lambda}\sigma_\beta^*(\lambda,0,0)\Phi(\tau,z) \\ &= \sum_{n,r,\gamma} c(n,r,\gamma) q^{n-m\lambda^2}\zeta^{r-2m\lambda}\mathfrak{e}_{\gamma-\lambda\beta}\end{aligned}$$

and therefore

$$c(n,r,\gamma) = c(n+r\lambda+m\lambda^2, r+2m\lambda, \gamma+\lambda\beta) \text{ for all } \lambda \in \mathbb{Z}.$$

Note that in general the coefficient of $c(n,r,\gamma)$ does not only depend on $4mn - r^2$ (unless $\beta = 0$), unlike the scalar-valued result of Eichler and Zagier [31].

As before, it is convenient to use Petersson's slash notation: we write

$$\begin{aligned}&\Phi\Big|_{k,m,\rho_\beta}(\zeta, M)(\tau,z) \\ &= (c\tau+d)^{-k}\mathbf{e}\Big(m\lambda^2\tau + 2m\lambda z + m(\lambda\mu+t) - \frac{cm(z+\lambda\tau+\mu)^2}{c\tau+d}\Big)\times \\ &\qquad \times \rho_\beta^*(\zeta,M)^{-1}\Big[\Phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\Big)\Big],\end{aligned}$$

such that the transformation law of Jacobi forms can be summarized as

$$\Phi\Big|_{k,m,\rho_\beta^*}(\zeta, M) = \Phi$$

for all $(\zeta, M) \in \mathcal{J}$.

Our main use of Jacobi forms will be as a means of constructing modular forms: if $\Phi(\tau, z)$ is a Jacobi form of weight k and index m for ρ_β then its zero-value $\Phi(\tau, 0)$ is easily seen to be a modular form of the same weight k for the Weil representation.

2.6 Borcherds lifts

This section is meant to give an introduction to or review of the singular theta correspondence of [4] which produces orthogonal modular forms with known divisors and modular product expansions. At several points the references [4] and [12] are quite technical, so we will only state the general idea of the results and omit many details .

Suppose $\Lambda \subseteq V$ is an even lattice of signature (b^+, b^-). We define the **Grassmannian** of Λ to be the set $\mathrm{Gr}(\Lambda)$ of all positive-definite subspaces of V of maximal dimension b^+. $\mathrm{Gr}(\Lambda)$ receives the structure of a smooth manifold in the usual way: after identifying elements $W \in \mathrm{Gr}(\Lambda)$ with the orthogonal projection π_W to W, it is a differentiable submanifold of the vector space $\mathrm{End}(V)$. In particular, it makes sense to speak of real-analytic functions on $\mathrm{Gr}(\Lambda)$.

Let (A, Q) be the attached discriminant form to Λ. The **Siegel theta function** of Λ is the function

$$\Theta : \mathbb{H} \times \mathrm{Gr}(\Lambda) \longrightarrow \mathbb{C}[A],$$
$$\Theta(\tau, v) = \sum_{x \in \Lambda'} \mathbf{e}\Big[\tau Q(x_v) + \overline{\tau} Q(x_{v^\perp})\Big] \mathfrak{e}_{x+\Lambda},$$

where for a subspace $v \in \mathrm{Gr}(\Lambda)$ and element $x \in V$ we let x_v and $x_{v^\perp}$ denote the orthogonal projections of x onto v and $v^\perp$. This satisfies the usual theta transformation formula:

Proposition 18. *For any* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$,

$$\Theta(M \cdot \tau, v) = (c\tau + d)^{b^+/2} (c\overline{\tau} + d)^{b^-/2} \rho(M) \Theta(\tau, v).$$

Proof. This follows from the Poisson summation formula. See [4], theorem 4.1. □

Definition 19. Let $F(\tau)$ be a real-analytic function on $\mathbb{H}$ with the property that $q^N F(\tau)$ is bounded at ∞ for some $N \in \mathbb{N}$, and assume F transforms as a modular form of weight $(b^+ - b^-)/2$ for the Weil representation (not its dual!):

$$F(M \cdot \tau) = (c\tau + d)^{(b^+ - b^-)/2} \rho(M) F(\tau), \quad M \in Mp_2(\mathbb{Z}).$$

The **theta lift** of F is the regularized integral

$$\Phi_\Lambda(v; F) = \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}}^{reg} \langle F(\tau), \Theta(\tau, v) \rangle y^{b^+/2 - 2} \, \mathrm{d}x \, \mathrm{d}y.$$

The notation $\int^{reg}$ should be understood as follows. For a suitable function $f(\tau)$ that is invariant under $SL_2(\mathbb{Z})$, we set

$$\int^{reg} f(\tau)y^{-2}\,\mathrm{d}x\,\mathrm{d}y = \mathrm{CT}_{s=0}\Big[\lim_{h\to\infty}\int_{\mathcal{F}_h} f(\tau)y^{-s-2}\,\mathrm{d}x\,\mathrm{d}y\Big],$$

where

$$\mathcal{F}_h = \{\tau = x+iy \in \mathbb{H} :\ -1/2 \le x \le 1/2,\ y \le h,\ x^2+y^2 \ge 1\}$$

is the fundamental domain truncated above by $y = h$, and $\mathrm{CT}_{s=0}$ denotes the constant term in the Laurent series in the variable s centered at 0.

Suppose $\sigma \in O(\Lambda)$ is any lattice automorphism, i.e. $\sigma \in GL(V)$ preserves $\langle -,-\rangle$ and maps Λ into itself. For any $x \in \Lambda'$ and $y \in \Lambda$ we find

$$\langle \sigma(x), y\rangle = \langle x, \sigma^{-1}(y)\rangle \in \langle \Lambda', \Lambda\rangle \subseteq \mathbb{Z},$$

so σ preserves Λ'; in particular, it induces an automorphism of the discriminant form (A, Q) and therefore an action on $\mathbb{C}[A]$:

$$\sigma \cdot \mathfrak{e}_{x+\Lambda} = \mathfrak{e}_{\sigma x+\Lambda}, \quad x \in \Lambda'.$$

For any subspace $v \in \mathrm{Gr}(\Lambda)$ and $x \in \Lambda'$, we find $\sigma(x_v) = \sigma(x)_{\sigma(v)}$; therefore, σ acts on the Siegel theta function Θ by

$$\sigma^{-1}\Big[\Theta(\tau,\sigma(v))\Big] = \sum_{x\in\Lambda'} \mathbf{e}\Big[\tau Q(\sigma(x_v)) + \overline{\tau}Q(\sigma(x_{v^\perp}))\Big]\mathfrak{e}_{x+\Lambda} = \Theta(\tau, v),$$

i.e. $\Theta(\tau,\sigma(v)) = \sigma\Theta(\tau,v)$. Therefore,

$$\Phi_\Lambda(\sigma(v),\sigma F) = \int^{reg} \langle \sigma F(\tau), \sigma\Theta(\tau,v)\rangle y^{b^+/2-2}\,\mathrm{d}x\,\mathrm{d}y = \Phi_\Lambda(v,F),$$

as one can check this equation rigorously for large enough s and then apply unique analytic continuation.

In particular, $\Phi_\Lambda(v,F)$ is invariant under the subgroup of $O(M)$ that fixes F. The singularities of $\Phi_\Lambda(v,F)$ were worked out in section 6 of [4]: they are supported on rational quadratic divisors

$$\lambda^\perp = \{v \in \mathrm{Gr}(\Lambda) :\ v \perp \lambda\}$$

with vectors $\lambda \in \Lambda$ for which $Q(\lambda) < 0$.

Suppose Λ has signature $(2, b^-)$; for simplicity, we assume that the underlying vector space is $\mathbb{R}^{2+b^-}$ and that $\Lambda = \mathbb{Z}^{2+b^-}$ with a block Gram matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{S} & 0 \\ 1 & 0 & 0 \end{pmatrix}$ for some Lorentzian matrix $\mathbf{S}$. Then $\mathrm{Gr}(\Lambda)$ is a complex manifold (in fact, a Hermitian symmetric space of type IV in Cartan's classification): namely, after fixing a continuous orientation on $\mathrm{Gr}(\Lambda)$, associating an oriented orthonormal basis (x, y) of $v \in \mathrm{Gr}(\Lambda)$ to the span of $x + iy$ identifies $\mathrm{Gr}(\Lambda)$ with the set of norm-zero lines in $\mathbb{CP}^{2+b^-}$. The choice of orientation is equivalent to a choice of **positive cone** C of $\mathbf{S}$, i.e. a component of those vectors $y \in \mathbb{R}^{b^-}$ with $y^T \mathbf{S} y > 0$. We define the **orthogonal upper half-space**

$$\mathbb{H}_{\mathbf{S}} = \{z = x + iy \in \mathbb{C}^{b^-} \, : \, y \in C\}.$$

This embeds into $\mathrm{Gr}(\Lambda)$ via

$$\mathbb{H}_{\mathbf{S}} \longrightarrow \mathrm{Gr}(\Lambda), \quad z \mapsto \mathrm{span}(-\frac{1}{2} z^T \mathbf{S} z, z, 1),$$

and so we get an action of $O^+(\Lambda)$ on $\mathbb{H}_{\mathbf{S}}$ as follows: $M \cdot z = w$ if and only if

$$M \begin{pmatrix} -\frac{1}{2} z^T \mathbf{S} z \\ z \\ 1 \end{pmatrix} = j(M; z) \begin{pmatrix} -\frac{1}{2} w^T \mathbf{S} w \\ w \\ 1 \end{pmatrix}$$

for some $j(M; z) \in \mathbb{C}^\times$. It is easy to see that $j(M; z)$ defines a cocycle; we choose it to be the **factor of automorphy** and we define orthogonal modular forms as holomorphic functions satisfying the usual transformations:

$$\Psi(M \cdot z) = \chi(M) j(M; z)^k \Psi(z), \quad M \in O^+(\Lambda)$$

for some character χ and some $k \in \mathbb{Z}$, together with a growth condition (which is redundant for $b^- \geq 3$ by Koecher's principle). Note that the full group $O(\Lambda)$ will generally not preserve the positive cone; the subgroup $O^+(\Lambda)$ consists exactly of those transformations that do preserve it. The main theorem is then:

Proposition 20. *Suppose*

$$F(\tau) = \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} + Q(\gamma)} c(n, \gamma) q^n \mathfrak{e}_\gamma$$

is a nearly-holomorphic modular form (i.e. holomorphic on $\mathbb{H}$ and finite order in ∞) of weight $\frac{2-b^-}{2}$ for the Weil representation of Λ. Then there is a meromorphic orthogonal modular form Ψ_F of weight $\frac{c(0,0)}{2}$ and some unitary character for the subgroup $O^+(\Lambda;F)$ of transformations that preserve F. Its divisor is supported on rational quadratic divisors $\lambda^\perp$ (with $\lambda \in \Lambda$, $Q(\lambda) < 0$) with

$$\mathrm{ord}(\Psi_F; \lambda^\perp) = \sum_{\substack{x \in \mathbb{R}_{>0} \\ x\lambda \in \Lambda'}} c(Q(x\lambda), x\lambda).$$

*The divisors $\lambda^\perp$ divide $\mathbb{H}_{\mathbf{S}}$ into **Weyl chambers**; and on each Weyl chamber whose closure contains the norm-zero vector $(1,0,0) \in \Lambda$, Ψ_F is given by the product*

$$\Psi_F(z) = \mathbf{e}(\rho(W)^T \mathbf{S} z) \prod_{\substack{\lambda \in \mathbf{S}^{-1}\mathbb{Z}^{b^-} \\ \lambda^T \mathbf{S} W > 0}} \Big(1 - \mathbf{e}(\lambda^T \mathbf{S} z)\Big)^{c(Q(\lambda),\lambda)},$$

*where $\rho(W)$ is the **Weyl vector** of W (see [4], 10.4).*

Proof sketch. Let W be a Weyl chamber. Interpret $z \in W$ as $v \in Gr(\Lambda)$ and define $\Psi_F(z)$ by the product above. Then the regularized theta integral $\Phi_\Lambda(v;F)$ as calculated in sections 6,7,9,10,13 of [4] is

$$-\frac{1}{4}\Phi_\Lambda(v;F) = \log|\Psi_F(z)| + \frac{c(0,0)}{2}\Big(\frac{\Gamma'(1) + \log(2\pi y^T \mathbf{S} y)}{2}\Big).$$

This is invariant under $O^+(\Lambda;F)$, so it remains invariant after exponentiating:

$$\exp\Big(-\frac{1}{4}\Phi_\Lambda(v,F)\Big) = \mathrm{const} \times |\Psi_F(z)|(y^T \mathbf{S} y)^{c(0,0)/4}.$$

Since $f(z) = y^T \mathbf{S} y$ transforms under $O^+(\Lambda)$ by

$$f(M \cdot z) = |j(M;z)|^{-2} f(z),$$

it follows that

$$|\Psi_F(M \cdot z)| = |j(M;z)|^{c(0,0)/2} |\Psi_F(z)|, \quad M \in O^+(\Lambda;F).$$

Therefore $\Psi_F(M \cdot z) = j(M;z)^{c(0,0)/2} \chi(M) \Psi_F(z)$ for some unitary character χ. □

2.7 The obstruction principle

To summarize the previous section, the input objects into Borcherds lifts are **nearly-holomorphic modular forms** of negative (or at least small) weight, which are functions that are holomorphic on $\mathbb{H}$ and transform like modular forms but are allowed to have a pole of finite order in ∞ (i.e. their Fourier series may have principal parts). In order to understand the possible divisors and weights of automorphic products we need to understand which principal parts

$$\sum_{\gamma \in A} \sum_{\substack{n \in \mathbb{Z}+Q(\gamma) \\ n<0}} c(n,\gamma) q^n \mathfrak{e}_\gamma + c(0,0)\mathfrak{e}_0$$

extend to nearly-holomorphic modular forms. Borcherds gave an answer to this in [5]:

Proposition 21. *A principal part*

$$\sum_{\gamma \in A} \sum_{\substack{n \in \mathbb{Z}+Q(\gamma) \\ n<0}} c(n,\gamma) q^n \mathfrak{e}_\gamma + c(0,0)\mathfrak{e}_0$$

with $c(n,\gamma) = c(n,-\gamma)$ *extends to a nearly-holomorphic modular form of weight* $2-k$ *for the Weil representation if and only if*

$$c(0,0)a(0,0) + \sum_{\gamma \in A} \sum_{\substack{n \in \mathbb{Z}+Q(\gamma) \\ n<0}} c(n,\gamma) a(-n,\gamma) = 0$$

for all holomorphic *modular forms of weight* k *for the dual of the Weil representation that have the form*

$$a(0,0)\mathfrak{e}_0 + \sum_{\gamma \in A} \sum_{\substack{n \in \mathbb{Z}-Q(\gamma) \\ n>0}} a(n,\gamma) q^n \mathfrak{e}_\gamma.$$

Proof. In [5] this is proved as an application of Serre duality. Also, Bruinier [12] has given a different argument that constructs the nearly-holomorphic form explicitly from the shadows of harmonic Maass-Poincaré series. □

Example 22. Suppose we are in the scalar-valued case (i.e. we consider a unimodular quadratic form) and the weight is $k = 6$. The only holomorphic modular form is

$$E_6(\tau) = 1 - 504q - ...$$

so by the obstruction principle we can find a nearly-holomorphic modular form of weight -4 that begins

$$F(\tau) = q^{-1} + 504 + ...$$

Of course F must be

$$\frac{E_8}{\Delta} = q^{-1} + 504 + 73764q + 2695040q^2 + ...$$

Remark 23. The obstruction principle also holds in the "reverse" direction: nearly-holomorphic modular forms act as obstructions to the existence of modular forms of the dual weight, i.e. they determine all linear coefficient relations that modular forms are forced to satisfy. In other words, they determine all linear relations among Poincaré series. This interpretation of the obstruction principle was made by Rhoades in [54].

2.8 Harmonic Maass forms

Harmonic Maass forms are a generalization of modular forms where the condition of holomorphy is weakened to annihilation under the weight-k hyperbolic Laplace operator

$$\Delta_k f(\tau) = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f(\tau) - 2iky\frac{\partial}{\partial \overline{\tau}} f(\tau)$$

which is invariant under the weight k action of $SL_2(\mathbb{R})$. Harmonic weak Maass forms are similarly a generalization of nearly-holomorphic modular forms. There has been a lot of interest in harmonic weak Maass forms since the work of Zwegers [75] which shows that Ramanujan's mock theta functions arise as holomorphic parts of such forms in the following sense.

Any harmonic weak Maass form is given by a Fourier series

$$f(\tau) = \sum_{n=-\infty}^{\infty} c(n,y)q^n, \quad c(n,y) = \int_{\mathbb{R}/\mathbb{Z}+iy} f(\tau)e^{-2\pi i n\tau}\,\mathrm{d}\tau,$$

where $c(n, y)$ are constrained by

$$y^2\Big[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big]\left(c(n,y)e^{2\pi i n(x+iy)}\right) = iky\Big[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big]\left(c(n,y)e^{2\pi i n(x+iy)}\right).$$

This has linearly independent solutions $c(n,y) = \text{const}$ and $c(n,y) = \int_1^y t^{-k}e^{4\pi nt}\,\mathrm{d}t$ for $y \in (0,\infty)$, and using the weak growth condition one can show that f has the form

$$f(\tau) = \sum_{n=-N}^{\infty} a(n)q^n + \sum_{n=-N}^{\infty} b(n)\phi_n(y)\overline{q}^n,\ q = e^{2\pi i\tau},\ \overline{q} = e^{-2\pi i\overline{\tau}},\ y = \mathrm{Im}(\tau),$$

where $\phi_n(y)$ is a special function that solves $\phi_n'(y) - 4\pi n\phi_n(y) = y^{-k}$ (and when k is a negative integer, it is the polynomial $(4\pi n)^{-1}y^{-k}\sum_{j=0}^{\infty}(-k)_j(-4\pi ny)^{-j}$). This is often expressed in terms of Whittaker functions or incomplete Gamma functions ([10], section 6.3). We will generally work in weight 3/2 and express all Fourier series in terms of the special function $\beta(x) = \frac{1}{16\pi}\int_1^\infty u^{-3/2}e^{-xu}\,\mathrm{d}u$ of [38].

The holomorphic part is then $\sum_{n=-N}^{\infty} a(n)q^n$. It transforms under $SL_2(\mathbb{Z})$ with a complicated cocycle. On the other hand, the coefficients $b(n)$ are easier to understand, as one can see using the **Bruinier-Funke operator** [16]:

$$\xi f(\tau) = 2iy^k\overline{\frac{\partial}{\partial\overline{\tau}}f(\tau)}.$$

Differentiating the weight k transformation law of f shows that

$$\xi f\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{2-k}\xi f;$$

moreover, applying ξ termwise to the Fourier series of f shows that

$$\xi f(\tau) = \sum_{n=-N}^{\infty}\overline{b(n)}q^n,$$

so $b(n)$ are coefficients of a weight $2-k$ nearly-holomorphic modular form, called the **shadow** of $\sum_{n=-N}^{\infty} a(n)q^n$.

There is an obvious generalization of harmonic weak Maass forms to vector-valued forms ([12], [16]). In this case, if f is a Maass form for a representation ρ then its shadow ξf is a modular form for the dual ρ^*.

2.9 L-functions

At several points we will need to consider the L-function

$$L(s,\chi_D) = \sum_{n=1}^{\infty}\chi_D(n)n^{-s}$$

attached to the Dirichlet character mod $|D|$,

$$\chi_D(n) = \left(\frac{D}{n}\right),$$

where D is a discriminant (i.e. $D \equiv 0, 1 \bmod 4$). It will be useful to recall the following properties of Dirichlet L-functions.

(i) Let χ be a Dirichlet character. Then $L(s, \chi)$ converges absolutely in some half-plane $\mathrm{Re}[s] > s_0$ and is given by an Euler product

$$L(s, \chi) = \prod_{p \,\mathrm{prime}} (1 - \chi(p)p^{-s})^{-1}$$

there.

(ii) $L(s, \chi)$ has a meromorphic extension to all $\mathbb{C}$ and satisfies the functional equation

$$\Gamma(s) \cos\left(\frac{\pi(s - \delta)}{2}\right) L(s, \chi) = \frac{\tau(\chi)}{2i^\delta}(2\pi/f)^s L(1 - s, \overline{\chi}),$$

where f is the conductor of χ, $\tau(\chi) = \sum_{a=1}^{f} \chi(a)e^{2\pi i a/f}$ is the Gauss sum of χ, and

$$\delta = \begin{cases} 1 : & \chi(-1) = -1; \\ 0 : & \chi(-1) = 1. \end{cases}$$

(iii) $L(s, \chi)$ is never zero at $s = 1$, and is holomorphic there unless χ is a trivial character, in which case it has a simple pole.

(iv) The values $L(1 - n, \chi)$, $n \in \mathbb{N}$ are rational numbers, given by

$$L(1 - n, \chi) = -\frac{B_{n,\chi}}{n},$$

where $B_{n,\chi} \in \mathbb{Q}$ is a generalized Bernoulli number.

We refer to section 4 of [66] for these and other results on Dirichlet L-functions.

Chapter 3

Poincaré square series

This chapter is taken from the paper [68].

3.1 Poincaré square series

Fix a discriminant form (A, Q) and let ρ^* denote the dual Weil representation.

Definition 24. Let $\beta \in A$ and $m \in \mathbb{Z} - Q(\beta)$. The **Poincaré square series** $Q_{k,m,\beta}$ is the series

$$Q_{k,m,\beta} = \sum_{\lambda \in \mathbb{Z}} P_{k,\lambda^2 m,\lambda\beta}.$$

Here, we set $P_{k.0,0}$ to be the Eisenstein series $E_k = E_{k,0}$. In other words, $Q_{k,m,\beta}$ is the unique modular form such that $Q_{k,m,\beta} - E_k$ is a cusp form and

$$(f, Q_{k,m,\beta}) = \frac{2 \cdot \Gamma(k-1)}{(4m\pi)^{k-1}} \sum_{\lambda=1}^{\infty} \frac{c(\lambda^2 m, \lambda\beta)}{\lambda^{2k-2}}$$

for all cusp forms $f(\tau) = \sum_{\gamma,n} c(n,\gamma) q^n \mathfrak{e}_\gamma$.

The name "Poincaré square series" appears to be due to Ziegler in [74], where he refers to a scalar-valued Siegel modular form with an analogous definition by that name.

Remark 25. The components of any cusp form $f = \sum_{n,\gamma} c(n,\gamma)\mathfrak{e}_\gamma$ can be considered as scalar-valued modular forms of higher level. Although the Ramanujan-Petersson conjecture is still open in half-integer weight, nontrivial bounds on the growth of $c(n,\gamma)$ are known. For example, Bykovskii [21] gives the bound $c(n,\gamma) = O(n^{k/2-5/16+\varepsilon})$ for all n and any $\varepsilon > 0$. This implies that the series

$$\sum_{\lambda \neq 0} (f, P_{k,\lambda^2 m,\lambda\beta}) = \sum_{\lambda \neq 0} \frac{\Gamma(k-1)}{(4\pi\lambda^2 m)^{k-1}} c(\lambda^2 m, \lambda\beta)$$

converges for $k \geq 5/2$. Since $S_k(\rho^*)$ is finite-dimensional, the weak convergence of $\sum_{\lambda \neq 0} P_{k,\lambda^2 m,\lambda\beta}$ actually implies its uniform convergence on compact subsets of $\mathbb{H}$. On the other hand, the estimate

$$\begin{aligned}\sum_{\lambda \in \mathbb{Z}} \Big| \mathbf{e}\Big(m\lambda^2 \frac{a\tau + b}{c\tau + d} \Big) \Big| &= \sum_{\lambda \in \mathbb{Z}} e^{-2\pi m \lambda^2 \frac{y}{|c\tau+d|^2}} \\ &\approx \int_{-\infty}^{\infty} e^{-2\pi m t^2 \frac{y}{|c\tau+d|^2}} \, \mathrm{d}t \\ &= \frac{|c\tau + d|}{\sqrt{2my}}, \quad y = \mathrm{Im}(\tau)\end{aligned}$$

implies that as a triple series,

$$Q_{k,m,\beta}(\tau) = \frac{1}{2} \sum_{\lambda \in \mathbb{Z}} \sum_{\gcd(c,d)=1} (c\tau + d)^{-k} \mathbf{e}\Big(m\lambda^2 \frac{a\tau + b}{c\tau + d} \Big) \, \rho^*(M)^{-1} \mathfrak{e}_{\lambda\beta}$$

converges absolutely only when $k > 3$.

Proposition 26. *The span of all Poincaré square series $Q_{k,m,\beta}$, $m \in \mathbb{N}$, $\beta \in \Lambda'/\Lambda$ contains all of $S_k(\rho^*)$.*

Proof. Since $\mathrm{Span}(Q_{k,m,\beta})$ is finite-dimensional, it is enough to find all Poincaré series as weakly convergent infinite linear combinations of $Q_{k,m,\beta}$. Möbius inversion implies the formal identity

$$P_{k,m,\beta} = \frac{1}{2}\Big(P_{k,m,\beta} + P_{k,m,-\beta} \Big) = \frac{1}{2} \sum_{d \in \mathbb{N}} \mu(d) \Big[Q_{k,d^2 m, d\beta} - E_k \Big].$$

The series on the right converges (weakly) in $S_k(\rho^*)$ because we can bound

$$\Big| (f, Q_{k,d^2 m,d\beta}) \Big| \leq \sum_{\lambda \in \mathbb{Z}} \frac{\Gamma(k-1)}{(4\pi\lambda^2 d^2 m)^{k-1}} \Big| c(\lambda^2 d^2 m, \lambda d\beta) \Big| \leq C \cdot d^{-9/8+\varepsilon}$$

for an appropriate constant C and all cusp forms $f(\tau) = \sum_\gamma \sum_n c(n,\gamma) q^n \mathfrak{e}_\gamma$, where we again use the bound $c(n,\gamma) = O(n^{k/2-5/16+\varepsilon})$. ∎

3.2 The Jacobi Eisenstein series

Fix a lattice Λ. Let $\mathcal{J}_\infty$ denote the subgroup of $\mathcal{J}$ that fixes the constant function $\mathfrak{e}_0$ under the action $|_{k,m,\rho^*_\beta}$. This is independent of β and it is the group generated by $T, Z \in \tilde{\Gamma}$ and the elements of the form $(0, \mu, t) \in \mathcal{H}$ in the Heisenberg group.

Definition 27. The **Jacobi Eisenstein series** twisted at $\beta \in \Lambda'$ of weight k and index $m \in \mathbb{Z} - Q(\beta)$ is

$$E_{k,m,\beta}(\tau, z) = \sum_{(M,\zeta) \in \mathcal{J}_\infty \backslash \mathcal{J}} \mathfrak{e}_0 \Big|_{k,m,\rho^*_\beta} (M, \zeta)(\tau, z).$$

It is clear that this is a Jacobi form of weight k and index m for the representation ρ^*_β. More explicitly, we can write it in the form

$$\begin{aligned} &E_{k,m,\beta}(\tau, z) \\ =\ &\frac{1}{2} \sum_{c,d} (c\tau + d)^{-k} \sum_{\lambda \in \mathbb{Z}} \mathbf{e}\Big(m\lambda^2 (M \cdot \tau) + \frac{2m\lambda z}{c\tau + d} - \frac{cmz^2}{c\tau + d} \Big) \rho^*(M)^{-1} \sigma^*_\beta(\lambda, 0, 0)^{-1} \mathfrak{e}_0. \end{aligned}$$

Remark 28. This series converges absolutely when $k > 3$. In that case the zero-value $E_{k,m,\beta}(\tau, 0)$ is the Poincaré square series $Q_{k,m,\beta}(\tau)$, as one can see by swapping the order of the sum over (c, d) and the sum over λ.

$E_{k,m,\beta}$ has a Fourier expansion of the form

$$E_{k,m,\beta}(\tau, z) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - Q(\gamma)} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} c(n, r, \gamma) q^n \zeta^r \mathfrak{e}_\gamma.$$

We will calculate its coefficients. The contribution from $c = 0$ and $d = \pm 1$ is

$$\sum_{\lambda \in \mathbb{Z}} \mathbf{e}\Big(m\lambda^2 \tau + 2m\lambda z \Big) \mathfrak{e}_{\lambda\beta}.$$

We denote the contribution from all other terms by $c'(n, r, \gamma)$; so

$$E_{k,m,\beta}(\tau, z) = \sum_{\lambda \in \mathbb{Z}} \mathbf{e}\Big(m\lambda^2 \tau + 2m\lambda z \Big) \mathfrak{e}_{\lambda\beta} + \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - Q(\gamma)} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} c'(n, r, \gamma) q^n \zeta^r \mathfrak{e}_\gamma.$$

Write $\tau = x + iy$ and $z = u + iv$. Then $c'(n, r, \gamma)$ is given by the integral

$$\begin{aligned}
&c'(n,r,\gamma)\\
&= \frac{1}{2}\int_0^1\int_0^1 \sum_{\substack{c\neq 0\\ \gcd(c,d)=1}} \sum_{\lambda} \Big[(c\tau+d)^{-k}\mathbf{e}\Big(m\lambda^2(M\cdot\tau) + \frac{2m\lambda z}{c\tau+d} - \frac{cmz^2}{c\tau+d}\Big)\Big]\times\\
&\qquad \times \mathbf{e}(-n\tau - rz)\langle \rho^*(M)^{-1}\sigma_\beta^*(\lambda,0,0)^{-1}\mathfrak{e}_0, \mathfrak{e}_\gamma\rangle \,\mathrm{d}x\,\mathrm{d}u\\
&= \frac{1}{2}\sum_{c\neq 0}\sum_{d\,(c)^*}\sum_{\lambda} \rho(M)_{\lambda\beta,\gamma}\int_{-\infty}^{\infty}\int_0^1 \Big[(c\tau+d)^{-k}\times\\
&\qquad \times \mathbf{e}\Big(-n\tau - rz + m\lambda^2(M\cdot\tau) + \frac{2m\lambda z}{c\tau+d} - \frac{cmz^2}{c\tau+d}\Big)\Big]\,\mathrm{d}u\,\mathrm{d}x.
\end{aligned}$$

Here, the notation $\sum_{d\,(c)^*}$ implies that the sum is taken over representatives of $(\mathbb{Z}/c\mathbb{Z})^\times$. The double integral simplifies to

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^1 (c\tau+d)^{-k}\mathbf{e}\Big(-n\tau - rz + m\lambda^2(M\cdot\tau) + \frac{2m\lambda z}{c\tau+d} - \frac{cmz^2}{c\tau+d}\Big)\,\mathrm{d}u\,\mathrm{d}x\\
&= c^{-k}\mathbf{e}\Big(\frac{am\lambda^2+nd}{c}\Big)\int_{-\infty}^{\infty}\tau^{-k}\int_0^1 \mathbf{e}\Big(-n\tau - rz - m(cz-\lambda)^2/(c^2\tau)\Big)\,\mathrm{d}u\,\mathrm{d}x
\end{aligned}$$

by substituting $\tau - d/c$ into τ.

The inner integral over u is easiest to evaluate within the sum over λ. Namely,

$$\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}} \rho(M)_{\lambda\beta,\gamma}\mathbf{e}\Big(\frac{am\lambda^2}{c}\Big)\int_0^1 \mathbf{e}\Big(-rz - m\frac{(cz-\lambda)^2}{c^2\tau}\Big)\,\mathrm{d}u\\
&= \sum_{\lambda\in\mathbb{Z}} \rho(M)_{\lambda\beta,\gamma}\mathbf{e}\Big(\frac{am\lambda^2 - r\lambda}{c}\Big)\int_{-\lambda/c}^{1-\lambda/c}\mathbf{e}\Big(-rz - mz^2/\tau\Big)\,\mathrm{d}u
\end{aligned}$$

after substituting $z + \lambda/c$ into z. Note that

$$\begin{aligned}
&\rho(M)_{\lambda\beta,\gamma}\mathbf{e}\Big(\frac{am\lambda^2 - r\lambda}{c}\Big)\\
&= \frac{\sqrt{i}^{(b^- - b^+)\mathrm{sgn}(c)}}{|c|^{(b^-+b^+)/2}\sqrt{|\Lambda'/\Lambda|}}\sum_{v\in\Lambda/c\Lambda}\mathbf{e}\Big(\frac{aQ(v+\lambda\beta) - \langle v+\lambda\beta,\gamma\rangle + dQ(\gamma) + am\lambda^2 - r\lambda}{c}\Big)\\
&= \frac{\sqrt{i}^{(b^- - b^+)\mathrm{sgn}(c)}}{|c|^{(b^-+b^+)/2}\sqrt{|\Lambda'/\Lambda|}}\times\\
&\quad\times\sum_{v\in\Lambda/c\Lambda}\mathbf{e}\Big(\frac{a\lambda^2[m+Q(\beta)] + \lambda[a\langle v,\beta\rangle - \langle\beta,\gamma\rangle - r] + aQ(v) - \langle v,\gamma\rangle + dQ(\gamma)}{c}\Big)
\end{aligned}$$

depends only on the remainder of λ mod c, because $m+Q(\beta)$ and $r+\langle\beta,\gamma\rangle$ are integers. Continuing, we see that

$$\begin{aligned}&\sum_{\lambda\in\mathbb{Z}}\rho(M)_{\lambda\beta,\gamma}\mathbf{e}\Big(\frac{am\lambda^2-r\lambda}{c}\Big)\int_{-\lambda/c}^{1-\lambda/c}\mathbf{e}\Big(-rz-mz^2/\tau\Big)\,\mathrm{d}u\\ =\;&\frac{\sqrt{i}^{(b^- - b^+)\mathrm{sgn}(c)}}{|c|^{(b^-+b^+)/2}\sqrt{|\Lambda'/\Lambda|}}\times\\ &\times\sum_{\substack{v\in\Lambda/c\Lambda\\ \lambda\in\mathbb{Z}/c\mathbb{Z}}}\mathbf{e}\Big(\frac{a\lambda^2[m+Q(\beta)]+\lambda[a\langle v,\beta\rangle-\langle\beta,\gamma\rangle-r]+aQ(v)-\langle v,\gamma\rangle+dQ(\gamma)}{c}\Big)\times\\ &\times\int_{-\infty}^{\infty}\mathbf{e}\Big(-rz-mz^2/\tau\Big)\,\mathrm{d}u.\end{aligned}$$

The Gaussian integral is well-known:

$$\int_{-\infty}^{\infty}\mathbf{e}\Big(-rz-mz^2/\tau\Big)\,\mathrm{d}u=\mathbf{e}\Big(r^2\tau/4m\Big)\sqrt{\tau/2im}.$$

We are left with

$$\begin{aligned}c'(n,r,\gamma)=\frac{1}{2\sqrt{2im}}\sum_{c\neq 0}&\frac{\sqrt{i}^{(b^- - b^+)\mathrm{sgn}(c)}}{|c|^{(b^-+b^+)/2}\sqrt{|\Lambda'/\Lambda|}}c^{-k}K_c(\beta,m,\gamma,n,r)\times\\ &\times\int_{-\infty}^{\infty}\tau^{1/2-k}\mathbf{e}\Big(\tau(r^2/4m-n)\Big)\,\mathrm{d}x,\end{aligned}$$

where $K_c(\beta,m,\gamma,n,r)$ is a Kloosterman sum:

$$\begin{aligned}&K_c(\beta,m,\gamma,n,r)\\ =&\sum_{d\,(c)^*}\sum_{\substack{v\in\Lambda/c\Lambda\\ \lambda\in\mathbb{Z}/c\mathbb{Z}}}\mathbf{e}\Big(\frac{a\lambda^2[m+Q(\beta)]+\lambda[a\langle v,\beta\rangle-\langle\beta,\gamma\rangle-r]+aQ(v)-\langle v,\gamma\rangle+dQ(\gamma)+dn}{c}\Big)\\ =&\sum_{\substack{v\in\Lambda/c\Lambda\\ \lambda\in\mathbb{Z}/c\mathbb{Z}}}\sum_{d\,(c)^*}\mathbf{e}\Big(\frac{d}{c}\Big[\lambda^2(m+Q(\beta))+\lambda(\langle v,\beta\rangle-\langle\gamma,\beta\rangle-r)+Q(v)-\langle v,\gamma\rangle+Q(\gamma)+n\Big]\Big)\\ =&\sum_{\substack{v\in\Lambda/c\Lambda\\ \lambda\in\mathbb{Z}/c\mathbb{Z}}}\sum_{d\,(c)^*}\mathbf{e}\Big(\frac{d}{c}\Big[Q(v+\lambda\beta-\gamma)+m\lambda^2-r\lambda+n\Big]\Big).\end{aligned}$$

(In the second equality we have replaced v and λ by $d\cdot v$ and $d\cdot\lambda$.)

The integral $\int_{-\infty}^{\infty} \tau^{1/2-k}\mathbf{e}\Big(\tau(r^2/4m-n)\Big)\,\mathrm{d}x$ is 0 when $r^2/4m-n\ge 0$, since the integral is independent of $y=\mathrm{Im}(\tau)$ and tends to 0 as $y\to\infty$. When $r^2/4m-n<0$, we deform the contour to a keyhole and use Hankel's integral

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\oint_\gamma e^\tau \tau^{-s}\,\mathrm{d}\tau$$

to conclude that

$$\int_{-\infty}^{\infty} \tau^{1/2-k}\mathbf{e}\Big(\tau(r^2/4m-n)\Big)\,\mathrm{d}x = \frac{2\pi i\cdot(2\pi i(r^2/4m-n))^{k-3/2}}{\Gamma(k-1/2)}$$

and therefore

$$\begin{aligned} c'(n,r,\gamma) &= \frac{(2\pi i)^{k-1/2}(r^2/4m-n)^{k-3/2}}{2\cdot\Gamma(k-1/2)\sqrt{2im|\Lambda'/\Lambda|}}\sum_{c\ne 0}\frac{\sqrt{i}^{(b^--b^+)\mathrm{sgn}(c)}}{|c|^{(b^-+b^+)/2}}c^{-k}K_c(\beta,m,\gamma,n,r) \\ &= \frac{(-i)^k\pi^{k-1/2}(4mn-r^2)^{k-3/2}}{2^{k-3}m^{k-1}\Gamma(k-1/2)\sqrt{|\Lambda'/\Lambda|}}\sum_{c\ne 0}\frac{\sqrt{i}^{(b^--b^+)\mathrm{sgn}(c)}}{|c|^{(b^-+b^+)/2}}c^{-k}K_c(\beta,m,\gamma,n,r). \end{aligned}$$

We can use

$$\sqrt{i}^{(b^--b^+)\mathrm{sgn}(c)}\mathrm{sgn}(c)^k(-i)^k = (-1)^{(2k-b^-+b^+)/4}$$

and the fact that $K_c(\beta,m,\gamma,n,r)=K_{-c}(\beta,m,\gamma,n,r)$ to write this as

$$c'(n,r,\gamma) = \frac{(-1)^{(2k-b^-+b^+)/4}\pi^{k-1/2}(4mn-r^2)^{k-3/2}}{2^{k-2}m^{k-1}\Gamma(k-1/2)\sqrt{|\Lambda'/\Lambda|}}\sum_{c=1}^{\infty}c^{-k-e/2}K_c(\beta,m,\gamma,n,r).$$

Remark 29. Using the evaluation of the Ramanujan sum,

$$\sum_{d\,(c)^*}\mathbf{e}\Big(\frac{d}{c}N\Big) = \sum_{a|(c,N)}\mu(c/a)a,$$

where μ is the Möbius function, it follows that

$$\begin{aligned} &K_c(\beta,m,\gamma,n,r) \\ &= \sum_{a|c}\mu(c/a)a\cdot\#\Big\{(v,\lambda)\in(\Lambda\oplus\mathbb{Z})/(c):\ Q(v+\lambda\beta-\gamma)+m\lambda^2-r\lambda+n=0\,(c)\Big\} \\ &= \sum_{a|c}\mu(c/a)a(c/a)^{e+1}\times \\ &\qquad\times\#\Big\{(v,\lambda)\in(\Lambda\oplus\mathbb{Z})/(a):\ Q(v+\lambda\beta-\gamma)+m\lambda^2-r\lambda+n=0\,(c)\Big\} \\ &= c^{e+1}\sum_{a|c}\mu(c/a)a^{-e}\mathbf{N}(a), \end{aligned}$$

where we define

$$\mathbf{N}(a) = \#\Big\{(v,\lambda) \in (\Lambda \oplus \mathbb{Z})/a(\Lambda \oplus \mathbb{Z}) : \; Q(v + \lambda\beta - \gamma) + m\lambda^2 - r\lambda + n \equiv 0 \, (a)\Big\}$$

and we use the fact that this congruence depends only on the remainder of v and λ mod a (rather than c).

Remark 30. If we identify $\Lambda = \mathbb{Z}^n$ and write $Q(v) = \frac{1}{2}v^T S v$ with a symmetric integer matrix S with even diagonal (its Gram matrix), then we can rewrite

$$\begin{aligned} &\lambda^2 m + Q(v + \lambda\beta - \gamma) - r\lambda + n \\ &= \frac{1}{2}(\tilde{v} - \tilde{\gamma})^T \begin{pmatrix} S & S\beta \\ (S\beta)^T & 2(m + Q(\beta)) \end{pmatrix} (\tilde{v} - \tilde{\gamma}) + \tilde{n} \end{aligned}$$

with $\tilde{v} = (v, \lambda)$ and $\tilde{\gamma} = (\gamma, -\frac{r}{2(m+Q(\beta))})$ and $\tilde{n} = n + \frac{r}{2(m+Q(\beta))}\langle \gamma, \beta \rangle - \frac{r^2}{4(m+Q(\beta))}$. Therefore, $\mathbf{N}(a)$ equals the representation number $N_{\tilde{\gamma},\tilde{n}}(a)$ in the notation of [18]. The analysis there does not seem to apply to this situation because $\tilde{\gamma}$ has no reason to be in the dual lattice of this larger quadratic form, and because $\tilde{n}$ can be negative or even zero.

In the particular case $\beta = 0$, the coefficient $c(n,r,\gamma)$ does in fact occur as the coefficient of

$$(\tilde{n}, \tilde{\gamma}) = (n - r^2/4m, (\gamma, r/2m))$$

in the Eisenstein series $E_{k-1/2,0}$ attached to the lattice with Gram matrix $\begin{pmatrix} S & 0 \\ 0 & 2m \end{pmatrix}$. This can be seen as a case of the theta decomposition, which gives more generally an isomorphism between Jacobi forms for a trivial action of the Heisenberg group and vector-valued modular forms, and identifies Jacobi Eisenstein series with vector-valued Eisenstein series. Actually, a form of theta decomposition appears to hold for arbitrary β; I hope to clarify this in a future note.

Remark 31. We consider the Dirichlet series

$$\tilde{L}(n,r,\gamma,s) = \sum_{c=1}^{\infty} c^{-s} K_c(\beta, m, \gamma, n, r).$$

Since K_c is c^{e+1} times the convolution of $\mu(a)$ and $a^{-e}\mathbf{N}(a)$, it follows formally that

$$\tilde{L}(n,r,\gamma,s+e+1) = \zeta(s)^{-1} L(n,r,\gamma,s+e),$$

where we define

$$L(n, r, \gamma, s) = \sum_{c=1}^{\infty} c^{-s}\mathbf{N}(c).$$

($\mathbf{N}(c)$, depending on n, r, γ, denotes the representation count modulo c considered earlier.) Since $\mathbf{N}(a)$ is multiplicative (for coprime a_1, a_2, a pair (v, λ) solves the congruence modulo a_1a_2 if and only if it does so modulo both a_1 and a_2), $L(n, r, \gamma, s)$ can be written as an Euler product

$$L(n, r, \gamma, s) = \prod_{p \, \mathrm{prime}} L_p(n, r, \gamma, s) \;\text{ with }\; L_p(n, r, \gamma, s) = \sum_{\nu=0}^{\infty} \mathbf{N}(p^\nu)p^{-\nu s}.$$

The functions L_p are always rational functions in p^{-s} and in particular they have a meromorphic extension to $\mathbb{C}$; and it follows that $c'(n, r, \gamma)$ is the value of the analytic continuation of

$$\frac{(-1)^{(2k-b^-+b^+)/4}\pi^{k-1/2}(4mn - r^2)^{k-3/2}}{2^{k-2}m^{k-1}\Gamma(k-1/2)\zeta(s-e)\sqrt{|\Lambda'/\Lambda|}} \prod_{p \, \mathrm{prime}} L_p(n, r, \gamma, s)$$

at $s = k + e/2 - 1$.

3.3 Evaluation of Euler factors

In this section we review the calculation of Igusa zeta functions of quadratic polynomials due to Cowan, Katz and White in [23] and apply it to calculate the Euler factors $L_p(n, r, \gamma, k + e/2 - 1)$.

Definition 32. Let $f \in \mathbb{Z}_p[X_1, ..., X_e]$ be a polynomial of e variables. The **Igusa zeta function** of f at a prime p is the p-adic integral

$$\zeta_{Ig}(f; p; s) = \int_{\mathbb{Z}_p^e} |f(x)|^s \, \mathrm{d}x, \quad s \in \mathbb{C}.$$

In other words,

$$\zeta_{Ig}(f; p; s) = \sum_{\nu=0}^{\infty} \mathrm{Vol}\Big(\{x \in \mathbb{Z}_p^e : \; |f(x)|_p = p^{-\nu}\}\Big)p^{-\nu s},$$

where Vol denotes the Haar measure on $\mathbb{Z}_p^e$ normalized such that $\mathrm{Vol}(\mathbb{Z}_p^e) = 1$.

Igusa proved [41] that $\zeta_{Ig}(f;p;s)$, which is a priori only a formal power series in p^{-s}, is in fact a rational function of p^{-s}. In particular, it has a meromorphic continuation to all of $\mathbb{C}$.

Our interest in the Igusa zeta function is due to the identity of generating functions

$$\frac{1-p^{-s}\zeta_{Ig}(f;p;s)}{1-p^{-s}} = \sum_{\nu=0}^{\infty} N_f(p^\nu) p^{-\nu(s+e)},$$

where $N_f(p^\nu)$ denotes the number of solutions

$$N_f(p^\nu) = \#\Big\{ x \in \mathbb{Z}^e/p^\nu \mathbb{Z}^e : \ f(x) \equiv 0 \bmod p^\nu \Big\}.$$

In particular,

$$L_p(n,r,\gamma,s) = \frac{1-p^{-s+e+1}\zeta_{Ig}(f;p;s-e-1)}{1-p^{-s+e+1}}$$

for the polynomial of $(e+1)$ variables

$$f(v,\lambda) = \lambda^2 m + Q(v+\lambda\beta-\gamma) - r\lambda + n.$$

The calculation of $\zeta_{Ig}(f;p;s)$ will be stated for quadratic polynomials in the form

$$f = \bigoplus_{i\in\mathbb{N}_0} p^i Q_i \oplus L + c,$$

where Q_i are unimodular quadratic forms, L is a linear form involving at most one variable, and $c \in \mathbb{Z}_p$. The notation $\bigoplus$ implies that no two terms in this sum contain any variables in common. To any quadratic polynomial g, there exists a polynomial f as above that is "isospectral" to g at p, in the sense that $N_f(p^\nu) = N_g(p^\nu)$ for all $\nu \in \mathbb{N}_0$. Consult section 4.9 of [23] for an algorithm to compute f. We will say that polynomials f as above are in **normal form**.

Proposition 33. *Let p be an odd prime. Let $f(X) = \bigoplus_{i\in\mathbb{N}_0} p^i Q_i(X) \oplus L(X) + c$ be a $\mathbb{Z}_p$-integral quadratic polynomial in normal form, and fix $\omega \in \mathbb{N}_0$ such that $Q_i = 0$ for $i > \omega$. Define*

$$r_i = \mathrm{rank}(Q_i) \ \text{ and } \ d_i = \mathrm{disc}(Q_i), \ i \in \mathbb{N}_0$$

and

$$\mathbf{r}_{(j)} = \sum_{\substack{0\le i\le j \\ i\equiv j\ (2)}} r_i \ \text{ and } \ \mathbf{d}_{(j)} = \prod_{\substack{0\le i\le j \\ i\equiv j\ (2)}} d_i, \ j \in \mathbb{N}_0,$$

and also define

$$\mathbf{p}_{(j)} = p^{\sum_{0\le i<j}\mathbf{r}_{(i)}},\ j\in\mathbb{N}_0.$$

Define the auxiliary functions $I_a(r,d)(s)$ *by*

$$I_a(r,d)(s) = \begin{cases} (1-p^{-s-r})\frac{p-1}{p-p^{-s}} : & r\,\text{odd},\ p|a; \\ \Big[1+p^{-s-(r+1)/2}\Big(\dfrac{ad(-1)^{(r+1)/2}}{p}\Big)\Big]\dfrac{p-1}{p-p^{-s}} - p^{-r} - \\ \qquad -p^{-(r+1)/2}\Big(\dfrac{ad(-1)^{(r+1)/2}}{p}\Big) \quad : & r\,\text{odd},\ p\nmid a; \\ \Big[1-p^{-r/2}\Big(\frac{(-1)^{r/2}d}{p}\Big)\Big]\cdot\Big[1+p^{-s-r/2}\Big(\frac{(-1)^{r/2}d}{p}\Big)\Big]\frac{p-1}{p-p^{-s}} : & r\,\text{even},\ p|a; \\ \Big[1-p^{-r/2}\Big(\frac{(-1)^{r/2}d}{p}\Big)\Big]\cdot\Big[\frac{p-1}{p-p^{-s}}+p^{-r/2}\Big(\frac{(-1)^{r/2}d}{p}\Big)\Big] : & r\,\text{even},\ p\nmid a, \end{cases}$$

where $\left(\frac{a}{p}\right)$ *is the quadratic reciprocity symbol on* $\mathbb{Z}_p$*. Then:*
(i) If $L=0$ *and* $c=0$*, let* $r=\sum_{i\in\mathbb{N}_0} r_i$*; then*

$$\zeta_{Ig}(f;p;s) = \sum_{0\le\nu<\omega-1}\frac{I_0(\mathbf{r}_{(\nu)},\mathbf{d}_{(\nu)})}{\mathbf{p}_{(\nu)}}p^{-\nu s} + \\ + \Big[\frac{I_0(\mathbf{r}_{(\omega-1)},\mathbf{d}_{(\omega-1)})}{\mathbf{p}_{(\omega-1)}}p^{-(\omega-1)s} + \frac{I_0(\mathbf{r}_{(\omega)},\mathbf{d}_{(\omega)})}{\mathbf{p}_{(\omega)}}p^{-\omega s}\Big]\cdot(1-p^{-2s-r})^{-1}.$$

(ii) If $L(x)=bx$ *with* $b\neq 0$ *and* $v_p(c)\ge v_p(b)$*, let* $\lambda=v_p(b)$*; then*

$$\zeta_{Ig}(f;p;s) = \sum_{0\le\nu<\lambda}\frac{I_0(\mathbf{r}_{(\nu)},\mathbf{d}_{(\nu)})}{\mathbf{p}_{(\nu)}}p^{-\nu s} + \frac{p^{-\lambda s}}{\mathbf{p}_{(\lambda)}}\cdot\frac{p-1}{p-p^{-s}}.$$

(iii) If $L=0$ *and* $c\neq 0$*, or if* $L(x)=bx$ *with* $v_p(b)>v_p(c)$*, let* $\kappa=v_p(c)$*; then*

$$\zeta_{Ig}(f;p;s) = \sum_{0\le\nu\le\kappa}\frac{I_{c/p^\nu}(\mathbf{r}_{(\nu)},\mathbf{d}_{(\nu)})}{\mathbf{p}_{(\nu)}}p^{-\nu s} + \frac{1}{\mathbf{p}_{(\kappa+1)}}p^{-\kappa s}.$$

Proof. This is theorem 2.1 of [23]. We have replaced the variable t there by p^{-s}. □

Remark 34. Since the constant term here is never 0, we are always in either case (ii) or case (iii). It follows that the only possible pole of $\zeta_{Ig}(f;p;s)$ is at $s=-1$, and therefore the only possible poles of $L_p(s)$ are at e or $e+1$. Therefore, the value $k+e/2-1$ is not a pole of L_p, with the weights $k=e/2+1$ or $k=e/2+2$ as the only possible exceptions. In fact, $k=e/2+1$ can occur as a pole but this is ultimately canceled out by the corresponding Euler factor of $\zeta(k-e/2-1)$ in the denominator of $c'(n,r,\gamma)$, and $k=e/2+2$ never occurs as a pole (as one can show by bounding $\mathbf{N}$). Case (i) will turn out to be useful to compute the Poincaré square series in weights $3/2, 2, 5/2$.

An easy, if unsatisfying, proof that $e/2+2$ could not occur as a pole is that the problem can be avoided entirely by appending hyperbolic planes (or other unimodular lattices) to Λ, which does not change the discriminant group and therefore does not change the coefficients of $E_{k,m,\beta}$, but makes e arbitrarily large.

Remark 35. Identify $\Lambda=\mathbb{Z}^n$ and $Q(v)=\frac{1}{2}v^TSv$ where S is the Gram matrix. We will use proposition 33 to calculate

$$L_p(n,r,\gamma,k+e/2-1)=\frac{1-p^{-k+e/2+2}\zeta_{Ig}(f;p;k-e/2-2)}{1-p^{-k+e/2+2}}$$

for "generic" primes p - these are primes $p\neq 2$ at which

$$\det(S),\quad d_\beta^2 m,\quad \text{or}\quad \tilde{n}:=d_\beta^2 d_\gamma^2(n-r^2/4m)$$

have valuation 0. Here, d_β and d_γ denote the denominators of β and γ, respectively. Since $p\nmid\det(S)$, it follows that d_β and d_γ are invertible mod p; so we can multiply the congruence

$$\lambda^2 m+Q(v+\lambda\beta-\gamma)-r\lambda+n\equiv 0\ (p^\nu)$$

by $d_\beta^2 d_\gamma^2$ and replace $d_\beta d_\gamma v+\lambda d_\beta d_\gamma\beta-d_\beta d_\gamma\gamma$ by v to obtain

$$\mathbf{N}(p^\nu)=\#\Big\{(v,\lambda):\ d_\beta^2 d_\gamma^2 m\lambda^2+Q(v)-d_\beta^2 d_\gamma^2 r\lambda+d_\beta^2 d_\gamma^2 n\equiv 0\,(p^\nu)\Big\}.$$

Here, $d_\beta^2 m, d_\gamma^2 n, d_\beta d_\gamma r\in\mathbb{Z}$. By completing the square and replacing $\lambda-d_\beta\frac{d_\gamma d_\beta r}{2d_\beta^2 m}$ by λ, we see that

$$\begin{aligned}\mathbf{N}(p^\nu)&=\#\Big\{(v,\lambda)\in(\mathbb{Z}/p^\nu\mathbb{Z})^{e+1}:\ Q(v)+d_\beta^2 m\lambda^2+d_\beta^2 d_\gamma^2(n-r^2/4m)\equiv 0\,(p^\nu)\Big\}\\&=\#\Big\{(v,\lambda)\in(\mathbb{Z}/p^\nu\mathbb{Z})^{e+1}:\ v^TSv+2d_\beta^2 m\lambda^2+2\tilde{n}\equiv 0\,(p^\nu)\Big\}.\end{aligned}$$

The polynomial $f(v,\lambda) = v^T Sv + 2d_\beta^2 m\lambda^2 + 2\tilde{n}$ is p-integral and in isospectral normal form so proposition 33 (specifically, case (iii)) applies. The Igusa zeta function is

$$\zeta_{Ig}(f;p;s) = \frac{1}{p^{e+1}} + I_{2\tilde{n}}(e+1, |\det(S)|)(s).$$

For even e, this is

$$\zeta_{Ig}(f;p;s) = \Big[1 + p^{-e/2-1-s}\Big(\frac{\mathcal{D}'}{p}\Big)\Big] \cdot \frac{p-1}{p-p^{-s}} - p^{-e/2-1}\Big(\frac{\mathcal{D}'}{p}\Big),$$

where $\mathcal{D}' = md_\beta^2(-1)^{e/2+1}\tilde{n}\det(S)$, and with some algebraic manipulation we find

$$\begin{aligned}
&\frac{1 - p^{-s}\zeta_{Ig}(f;p;s)}{1-p^{-s}} \\
=\ &\frac{p - p^{-s} - p^{-s}(p-1)\Big(1 + p^{-e/2-1-s}\left(\frac{\mathcal{D}'}{p}\right)\Big) - p^{-e/2-1-s}(p-p^{-s})\left(\frac{\mathcal{D}'}{p}\right)}{(1-p^{-s})(p-p^{-s})} \\
=\ &\frac{p - p^{1-s} + p^{-s-e/2-1}(p-p^{1-s})\left(\frac{\mathcal{D}'}{p}\right)}{(1-p^{-s})(p-p^{-s})} \\
=\ &\frac{1}{1-p^{-s-1}}\Big[1 + \Big(\frac{\mathcal{D}'}{p}\Big)p^{-s-e/2-1}\Big],
\end{aligned}$$

and therefore

$$L_p(n,r,\gamma,k+e/2-1) = \frac{1}{1-p^{-k+e/2+1}}\Big[1 + \Big(\frac{\mathcal{D}'}{p}\Big)p^{1-k}\Big)\Big].$$

For odd e, it is

$$\zeta_{Ig}(f;p;s) = \frac{p-1}{p-p^{-s}} + p^{-(e+1)/2}\Big(\frac{D'}{p}\Big)\Big[1 - \frac{p-1}{p-p^{-s}}\Big],$$

where $D' = 2md_\beta^2(-1)^{(e+1)/2}\det(S)$, and a similar calculation shows that

$$\frac{1-p^{-s}\zeta_{Ig}(f;p;s)}{1-p^{-s}} = \frac{1}{1-p^{-s-1}}\Big[1 - \Big(\frac{D'}{p}\Big)p^{-s-(e+1)/2-1}\Big]$$

and therefore

$$L_p(n,r,\gamma,k+e/2-1) = \frac{1}{1-p^{-k+e/2+1}}\Big[1 - \Big(\frac{D'}{p}\Big)p^{1/2-k}\Big].$$

Proposition 36. *Define the constant*

$$\alpha_{k,m}(n,r) = \frac{(-1)^{(2k+b^+-b^-)/4}\pi^{k-1/2}(4mn-r^2)^{k-3/2}}{2^{k-2}m^{k-1}\Gamma(k-1/2)\sqrt{|\det(S)|}}.$$

Define the set of "bad primes" to be

$$\{2\} \cup \Big\{p\,\mathrm{prime}:\ p|\det(S) \text{ or } p|d_\beta^2 m \text{ or } v_p(\tilde{n}) \neq 0\Big\}.$$

(i) If e is even, then define

$$\mathcal{D} = \mathcal{D}' \cdot \prod_{\mathrm{bad}\,p} p^2 = md_\beta^2(-1)^{e/2+1}\tilde{n}\det(S)\prod_{\mathrm{bad}\,p} p^2.$$

For $4mn - r^2 > 0$,

$$c(n,r,\gamma) = \frac{\alpha_{k,m}(n,r)L(k-1,\chi_{\mathcal{D}})}{\zeta(2k-2)}\prod_{\mathrm{bad}\,p}\Big[\frac{1-p^{-k+e/2+1}}{1-p^{2-2k}}L_p(n,r,\gamma,k+e/2-1)\Big].$$

(ii) If e is odd, then define

$$D = D' \cdot \prod_{\mathrm{bad}\,p} p^2 = 2md_\beta^2(-1)^{(e+1)/2}\det(S)\prod_{\mathrm{bad}\,p} p^2.$$

For $4mn - r^2 > 0$,

$$c(n,r,\gamma) = \frac{\alpha_{k,m}(n,r)}{L(k-1/2,\chi_D)}\prod_{\mathrm{bad}\,p}\Big[(1-p^{-k+e/2+1})L_p(n,r,\gamma,k+e/2-1)\Big].$$

Here, $L(s,\chi_{\mathcal{D}})$ and $L(s,\chi_D)$ denote the L-series

$$L(s,\chi_{\mathcal{D}}) = \sum_{c=1}^\infty c^{-s}\Big(\frac{\mathcal{D}}{c}\Big),\quad L(s,\chi_D) = \sum_{c=1}^\infty c^{-s}\Big(\frac{D}{c}\Big),$$

where $\left(\frac{D}{c}\right)$ and $\left(\frac{\mathcal{D}}{c}\right)$ are the Kronecker symbols.

Proof. This follows immediately from the Euler products

$$L(s,\chi_{\mathcal{D}}) = \prod_p \Big(1 - \Big(\frac{\mathcal{D}}{p}\Big)p^{-s}\Big)^{-1},\quad L(s,\chi_D) = \prod_p \Big(1 - \Big(\frac{D}{p}\Big)p^{-s}\Big)^{-1},$$

which are valid because $\mathcal{D}$ and D are discriminants (congruent to 0 or 1 mod 4) and therefore $\left(\frac{\mathcal{D}}{a}\right)$ and $\left(\frac{D}{a}\right)$ define Dirichlet characters of a modulo $|\mathcal{D}|$ resp. $|D|$. □

In particular, $c(n,r,\gamma)$ is always rational.

The factors $L_p(n,r,\gamma,k+e/2-1)$ are easy to evaluate for bad primes $p \neq 2$ using proposition 33. To calculate the factor at $p=2$, we need a longer formula. This is postponed to section 3.8.

3.4 Poincaré square series of weight $5/2$

An application of the Hecke trick shows that the Poincaré square series of weight 3 is still the zero-value of the Jacobi Eisenstein series of weight 3. This result is not surprising and the derivation is essentially the same as the weight $5/2$ case below, so we omit the details. However, the result in the case $k = 5/2$ is somewhat more complicated. I realized later that this would be easier to derive using holomorphic projection as in chapter 5 later on but have still included the original proof from [68] here.

Definition 37. For $k = 5/2$, we define the nonholomorphic Jacobi Eisenstein series of weight $5/2$, twisted at $\beta \in \Lambda'/\Lambda$, of index $m \in \mathbb{Z} - Q(\beta)$, by

$$E^*_{5/2,m,\beta}(\tau, z, s) = \frac{1}{2} \sum_{c,d} (c\tau + d)^{-5/2} |c\tau + d|^{-2s} \times \\ \times \sum_{\lambda \in \mathbb{Z}} \mathbf{e}\Big(m\lambda^2 (M \cdot \tau) + \frac{2m\lambda z}{c\tau + d} - \frac{cmz^2}{c\tau + d} \Big) \rho^*(M)^{-1} \sigma^*_\beta(\lambda, 0, 0)^{-1} \mathfrak{e}_0.$$

This defines a holomorphic function of s in the half-plane $\mathrm{Re}[s] > 0$.

We write the Fourier series of $E^*_{5/2,m,\beta}$ in the form

$$E^*_{5/2,m,\beta}(\tau, z, s) = \sum_{n,r,\gamma} c(n, r, \gamma, s, y) q^n \zeta^r \mathfrak{e}_\gamma.$$

(Here, the coefficients depend on y, since $E^*_{5/2,m,\beta}$ is not holomorphic in τ.) As before, the contribution from $c = 0$ and $d = \pm 1$ is

$$\sum_{\lambda \in \mathbb{Z}} \mathbf{e}\Big(m\lambda^2 \tau + 2m\lambda z \Big) \mathfrak{e}_{\lambda\beta}.$$

We denote the contribution from all other terms by $c'(n, r, \gamma, s, y)$, so

$$E^*_{5/2,m,\beta}(\tau, z, s) = \sum_{\lambda \in \mathbb{Z}} \mathbf{e}\Big(m\lambda^2 \tau + 2m\lambda z \Big) \mathfrak{e}_{\lambda\beta} + \sum_{n,r,\gamma} c'(n, r, \gamma, s, y) q^n \zeta^r \mathfrak{e}_\gamma.$$

A derivation similar to section 3.2 gives

$$c'(n, r, \gamma, s, y) = \frac{1}{2\sqrt{2im}} \sum_{c \neq 0} \frac{\sqrt{i}^{(b^- - b^+)\mathrm{sgn}(c)}}{|c|^{e/2} \sqrt{|\Lambda'/\Lambda|}} c^{-5/2} |c|^{-2s} K_c(\beta, m, \gamma, n, r) \times \\ \times \int_{-\infty + iy}^{\infty + iy} \tau^{-2} |\tau|^{-2s} \mathbf{e}\Big(\tau(r^2/4m - n) \Big) \, \mathrm{d}x.$$

Substituting $\tau = y(t+i)$ in the integral yields

$$\begin{aligned}&\int_{-\infty+iy}^{\infty+iy} \tau^{-2}|\tau|^{-2s}\mathbf{e}\Big(\tau(r^2/4m-n)\Big)\,\mathrm{d}x \\ &= y^{-1-2s}\mathbf{e}\Big(iy(r^2/4m-n)\Big)\int_{-\infty}^{\infty}(t+i)^{-2}(t^2+1)^{-s}\mathbf{e}\Big(yt(r^2/4m-n)\Big)\,\mathrm{d}t.\end{aligned}$$

We use

$$\sqrt{i}^{(b^- - b^+)\mathrm{sgn}(c)}\mathrm{sgn}(c)^{-5/2} = (-1)^{(5-b^-+b^+)/4} i^{5/2} = (-1)^{(1-b^-+b^+)/4}\sqrt{i}$$

and conclude that

$$\begin{aligned}c'(n,r,\gamma,s,y) &= \frac{(-1)^{(1+b^+-b^-)/4}}{\sqrt{2m|\Lambda'/\Lambda|}} I(y, r^2/4m-n, s)\sum_{c=1}^{\infty} c^{-5/2-2s-e/2}K_c(\beta,m,\gamma,n,r) \\ &= \frac{(-1)^{(1+b^+-b^-)/4}}{\sqrt{2m|\Lambda'/\Lambda|}} I(y, r^2/4m-n, s)\tilde{L}(n,r,\gamma,5/2+e/2+2s),\end{aligned}$$

where $I(y,N,s)$ denotes the integral

$$I(y,N,s) = I(2,y,N,s) = y^{-1-2s}e^{-2\pi Ny}\int_{-\infty}^{\infty}(t+i)^{-2}(t^2+1)^{-s}\mathbf{e}(Nyt)\,\mathrm{d}t,$$

and

$$\tilde{L}(n,r,\gamma,s) = \sum_{c=1}^{\infty} c^{-s}K_c(\beta,m,\gamma,n,r)$$

as before.

Remark 38. When $r^2 \neq 4mn$, we were able to express $\tilde{L}(s)$ up to finitely many holomorphic factors as $\frac{1}{L(s-e/2-1/2,\chi_D)}$, and it follows that $\tilde{L}(s)$ is holomorphic in $5/2+e/2$. In particular, if $r^2 \neq 4mn$, then the coefficient $c'(n,r,\gamma,0,y)$ is independent of y and given by

$$c'(n,r,\gamma,0,y) = \frac{\alpha_{k,m}(n,r)}{L(2,\chi_D)}\prod_{\mathrm{bad}\,p}\Big[\frac{1-p^{-3/2+e/2}}{1-\Big(\frac{D}{p}\Big)p^{-2}}L_p(n,r,\gamma,3/2+e/2)\Big] \text{ if } 4mn-r^2>0,$$

and $c'(n,r,\gamma,0,y)=0$ if $4mn-r^2<0$, just as for $k\geq 3$. This analysis does not apply when $r^2=4mn$ and indeed $\tilde{L}$ may have a (simple) pole in $s=5/2+e/2$ in that case.

We will study the coefficients $c'(n,r,\gamma,0,y)$ when $4mn=r^2$. The integral $I(y,0,s)$ is zero at $s=0$, and its derivative there is

$$\frac{\partial}{\partial s}\Big|_{s=0} I(y,0,s) = -y^{-1}\int_{-\infty}^{\infty}(t+i)^{-2}\log(t^2+1)\,\mathrm{d}t = -\frac{\pi}{y}.$$

This cancels the possible pole of $\tilde{L}(5/2+e/2+2s)$ at 0, and therefore we need to know the residue of $\tilde{L}(5/2+e/2+2s)$ there. As before, $\tilde{L}$ factors as

$$\tilde{L}(n,r,\gamma,5/2+e/2+2s) = \zeta(2s+3/2-e/2)^{-1}L(n,r,\gamma,3/2+e/2+2s)$$

where $L(s)$ has an Euler product

$$L(n,r,\gamma,s) = \prod_{p\,\mathrm{prime}} L_p(n,r,\gamma,s), \text{ with } L_p(n,r,\gamma,s) = \sum_{\nu=0}^{\infty} \mathbf{N}(p^\nu)p^{-\nu s},$$

and $\mathbf{N}(p^\nu)$ is the number of zeros of the polynomial $f(v,\lambda) = Q(v+\lambda\beta-\gamma)+m\lambda^2-r\lambda+n$ mod p^ν.

Remark 39. Identify $\Lambda = \mathbb{Z}^n$ and $Q(v) = \frac{1}{2}v^TSv$ where S is the Gram matrix. We will calculate L_p when $4mn-r^2=0$ for primes p dividing neither $\det(S)$ nor $d_\beta^2 m$. In this case, it follows that

$$\mathbf{N}(p^\nu) = \#\Big\{(v,\lambda)\in(\mathbb{Z}/p^\nu\mathbb{Z})^{e+1} :\; v^TSv+2d_\beta^2 m\lambda^2 \equiv 0\,(p^\nu)\Big\}.$$

We are in case (i) of proposition 33 and it follows that

$$\zeta_{Ig}(f;p;s) = \Big[1-p^{-(e+1)/2}\Big(\frac{D'}{p}\Big)\Big]\cdot\Big[1+p^{-s-(e+1)/2}\Big(\frac{D'}{p}\Big)\Big]\cdot\frac{p-1}{(p-p^{-s})(1-p^{-2s-e-1})}$$

with $D' = 2md_\beta^2(-1)^{(e+1)/2}\det(S)$. After some algebraic manipulation, we find that

$$\frac{1-p^{-s}\zeta_{Ig}(f;p;s)}{1-p^{-s}} = \frac{1-\Big(\frac{D'}{p}\Big)p^{-s-1-(e+1)/2}}{(1-p^{-s-1})(1-\Big(\frac{D'}{p}\Big)p^{-s-(e+1)/2})},$$

so

$$L_p(n,r,\gamma,3/2+e/2+2s) = \frac{1-\Big(\frac{D'}{p}\Big)p^{-2-2s}}{(1-p^{e/2-3/2-2s})(1-\Big(\frac{D'}{p}\Big)p^{-1-2s})}.$$

This immediately implies the following lemma:

Lemma 40. *In the situation treated in this section, define $D = D'\cdot\prod_{\mathrm{bad}\,p}p^2$; then*

$$\tilde{L}(n,r,\gamma,5/2+e/2+2s) = \frac{L(2s+1,\chi_D)}{L(2s+2,\chi_D)}\prod_{\mathrm{bad}\,p}\Big[(1-p^{e/2-3/2-2s})L_p(n,r,\gamma,3/2+e/2+2s)\Big].$$

Notice that $L(2s+1,\chi_D)$ is holomorphic in $s=0$ unless D is a square, in which case it is the Riemann zeta function with finitely many Euler factors missing.

Proposition 41. *If $4mn-r^2=0$, then $c'(n,r,\gamma,0,y)=0$ unless D is a square, in which case*

$$c'(n,r,\gamma,0,y)=\frac{(-1)^{(1+b^+-b^-)/4}}{\sqrt{2m|\Lambda'/\Lambda|}}\cdot\frac{3}{\pi y}\prod_{p|D}\Big[(1-p^{(e-3)/2})L_p(n,r,\gamma,(e+3)/2)\Big].$$

Proof. Assume that D is a square. As s approaches zero,

$$\begin{aligned}&\lim_{s\to 0}c'(n,r,\gamma,s,y)\\&=\frac{(-1)^{(1+b^+-b^-)/4}}{\sqrt{2m|\Lambda'/\Lambda|}}\cdot\frac{\partial}{\partial s}\Big|_{s=0}I(y,0,s)\cdot\mathrm{Res}\Big(\tilde{L}(n,r,\gamma,5/2+e/2+2s);s=0\Big).\end{aligned}$$

We calculated

$$\frac{\partial}{\partial s}\Big|_{s=0}I(y,0,s)=-\frac{\pi}{y}$$

earlier. The residue of $\tilde{L}(5/2+e/2+2s)$ at 0 is

$$\frac{1}{L(2,\chi_D)}\prod_{\mathrm{bad}\,p}\Big[(1-p^{e/2-3/2})L_p(n,r,\gamma,3/2+e/2)\Big]\cdot\mathrm{Res}(L(2s+1,\chi_D);s=0),$$

and using

$$L(2s+1,\chi_D)=\zeta(2s+1)\prod_{p|D}(1-p^{-2s-1})$$

and the fact that $\zeta(s)$ has residue 1 at $s=1$, it follows that

$$\mathrm{Res}(L(2s+1,\chi_D);s=0)=\frac{1}{2}\prod_{p|D}(1-p^{-1}).$$

We write

$$L(2,\chi_D)=\zeta(2)\prod_{p|D}(1-p^{-2})=\frac{\pi^2}{6}\prod_{p|D}(1-p^{-2}).$$

Since the "bad primes" are exactly the primes dividing D (by construction of D), we find

$$\begin{aligned}&\mathrm{Res}\Big(\tilde{L}(n,r,\gamma,5/2+e/2+2s);s=0\Big)\\&=\frac{3}{\pi^2}\prod_{p|D}\Big[\frac{(1-p^{e/2-3/2})(1-p^{-1})}{1-p^{-2}}L_p(n,r,\gamma,3/2+e/2)\Big],\end{aligned}$$

which gives the formula. ∎

Let A_n denote the constant

$$A_n = \frac{(-1)^{(5+b^+-b^-)/4}}{\sqrt{2m|\Lambda'/\Lambda|}} \cdot \frac{3}{\pi} \prod_{p|D} \Big[\frac{(1-p^{(e-3)/2})(1-p^{-1})}{1-p^{-2}} L_p(n,r,\gamma,(e+3)/2) \Big],$$

such that $E^*_{5/2,m,\beta}(\tau,z) + \frac{1}{y}\vartheta$ is holomorphic, where ϑ is the theta function

$$\vartheta(\tau,z) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{\substack{4mn-r^2=0 \\ n \in \mathbb{Z}-Q(\gamma) \\ r \in \mathbb{Z}-\langle \gamma,\beta \rangle}} A_n q^n \zeta^r \mathfrak{e}_\gamma.$$

Even when D is not square, this becomes true after defining $A_n = 0$ for all n.

Lemma 42.

$$\vartheta(\tau,z) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{\substack{4mn-r^2=0 \\ n \in \mathbb{Z}-Q(\gamma) \\ r \in \mathbb{Z}-\langle \gamma,\beta \rangle}} A_n q^n \zeta^r \mathfrak{e}_\gamma$$

is a Jacobi form of weight $1/2$ *and index* m *for the representation* ρ^*_β.

Proof. We give a proof relying on the transformation law of $E^*_{5/2,m,\beta}$. Denote by

$$E_{5/2,m,\beta}(\tau,z) = E^*_{5/2,m,\beta}(\tau,z,0) + \frac{1}{y}\vartheta(\tau,z)$$

the holomorphic part of $E^*_{5/2,m,\beta}$. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}$,

$$\begin{aligned}
&E_{5/2,m,\beta}\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) - \frac{|c\tau+d|^2}{y}\vartheta\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) \\
&= E^*_{5/2,m,\beta}\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, 0\Big) \\
&= (c\tau+d)^{5/2}\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\rho^*(M)E^*_{5/2,m,\beta}(\tau,z,0) \\
&= (c\tau+d)^{5/2}\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\rho^*(M)E_{5/2,m,\beta}(\tau,z) - \frac{(c\tau+d)^{5/2}}{y}\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\rho^*(M)\vartheta(\tau,z).
\end{aligned}$$

In particular,

$$\begin{aligned}
&\frac{1}{y}|c\tau+d|^2\vartheta\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) - (c\tau+d)^{5/2}\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\rho^*(M)\vartheta(\tau,z) \\
&= E_{5/2,m,\beta}\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) - (c\tau+d)^{5/2}\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\rho^*(M)E_{5/2,m,\beta}(\tau,z).
\end{aligned}$$

Using the identity $\frac{|c\tau+d|^2}{y} = \frac{(c\tau+d)^2}{y} - 2ic(c\tau+d)$ and differentiating both sides of this equation with respect to $\overline{\tau}$ leads to

$$(c\tau+d)^2\vartheta\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) - (c\tau+d)^{5/2}\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\rho^*(M)\vartheta(\tau,z) = 0,$$

which implies the modularity of ϑ under M. One can verify the transformation law under the Heisenberg group by a similar argument: for any $\lambda, \mu \in \mathbb{Z}$ we find

$$\begin{aligned} &E_{5/2,m,\beta}\Big(\tau, z+\lambda\tau+\mu\Big) - \frac{1}{y}\vartheta(\tau, z+\lambda\tau+\mu) \\ &= E^*_{5/2,m,\beta}(\tau, z+\lambda\tau+\mu, 0) \\ &= \mathbf{e}\Big(-m(\lambda^2\tau + 2\lambda z + \lambda\mu)\Big)\sigma^*_\beta(\lambda,\mu,0)\Big[E_{5/2,m,\beta}(\tau,z) + \frac{1}{y}\vartheta(\tau,z)\Big], \end{aligned}$$

so differentiating with respect to $\overline{\tau}$ and multiplying both sides of this by $2iy^2$ gives

$$\vartheta(\tau, z+\lambda\tau+\mu) = \mathbf{e}\Big(-m(\lambda^2\tau+2\lambda z+\lambda\mu)\Big)\sigma^*_\beta(\lambda,\mu,0)\vartheta(\tau,z). \qquad \square$$

We can now compute $Q_{5/2,m,\beta}$. Let $\vartheta(\tau)$ denote the zero-value $\vartheta(\tau,0)$.

Proposition 43. *The Poincaré square series of weight* $5/2$ *is*

$$Q_{5/2,m,\beta}(\tau) = E_{5/2,m,\beta}(\tau,0) + 4i\vartheta'(\tau).$$

Proof. Using the modularity of $E^*_{5/2,m,\beta}$ and ϑ, we find that $E_{5/2,m,\beta}(\tau,0)$ transforms under $\tilde{\Gamma}$ by

$$E_{5/2,m,\beta}\Big(\frac{a\tau+b}{c\tau+d}, 0\Big) = \rho^*(M)\Big[(c\tau+d)^{5/2}E_{5/2,m,\beta}(\tau,0) - 2ic(c\tau+d)^{3/2}\vartheta(\tau)\Big].$$

Differentiating the equation $\vartheta(M\cdot\tau) = (c\tau+d)^{1/2}\rho^*(M)\vartheta(\tau)$ gives the similar equation

$$\vartheta'(M\cdot\tau) = \rho^*(M)\Big[(c\tau+d)^{5/2}\vartheta'(\tau) + \frac{1}{2}c(c\tau+d)^{3/2}\vartheta(\tau)\Big].$$

This implies that $E_{5/2,m,\beta}(\tau,0) + 4i\vartheta'(\tau)$ is a modular form of weight $5/2$.

Now we prove that it equals $Q_{5/2,m,\beta}$ by showing that it satisfies the characterization of $Q_{5/2,m,\beta}$ with respect to the Petersson scalar product. First, we remark that $E^*_{5/2,m,\beta}(\tau,0,0)$, although not holomorphic, satisfies that characterization: for any cusp form $f(\tau) = \sum_\gamma \sum_n c(n,\gamma) q^n$, and any $\mathrm{Re}[s] > 0$,

$$\Big\langle f(\tau), E^*_{5/2,m,\beta}(\tau,0,s)\Big\rangle y^{1/2+2s}\,\mathrm{d}x\,\mathrm{d}y$$

is invariant under $\tilde{\Gamma}$, and we integrate:

$$\begin{aligned}
&\int_{\tilde{\Gamma}\backslash\mathbb{H}} \langle f(\tau), E^*_{5/2,m,\beta}(\tau,0,s)\rangle y^{1/2+2s}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{\gamma\in\Lambda'/\Lambda}\sum_{\lambda\in\mathbb{Z}}\sum_n \int_{-1/2}^{1/2}\int_0^\infty \langle c(n,\gamma)\mathfrak{e}_\gamma, \mathfrak{e}_{\lambda\beta}\rangle \mathbf{e}\Big(n(x+iy) - m\lambda^2(x-iy)\Big) y^{1/2+2s}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{\lambda\neq 0} c(\lambda^2 m, \lambda\beta)\int_0^\infty e^{-4\pi m\lambda^2 y} y^{1/2+2s}\,\mathrm{d}y \\
&= \sum_{\lambda\neq 0} c(\lambda^2 m, \lambda\beta)\frac{\Gamma(3/2+2s)}{(4\pi m\lambda^2)^{3/2+2s}}.
\end{aligned}$$

Taking the limit as $s \to 0$, we get

$$\lim_{s\to 0}\int_{\tilde{\Gamma}\backslash\mathbb{H}} \langle f(\tau), E^*_{5/2,m,\beta}(\tau,0,s)\rangle y^{1/2+2s}\,\mathrm{d}x\,\mathrm{d}y = \sum_{\lambda\neq 0} c(\lambda^2 m,\lambda\beta)\frac{\Gamma(3/2)}{(4\pi m\lambda^2)^{3/2}}.$$

The difference

$$\Big(E_{5/2,m,\beta}(\tau,0) + 4i\vartheta'(\tau)\Big) - E^*_{5/2,m,\beta}(\tau,0,0) = 4i\vartheta'(\tau) + \frac{1}{y}\vartheta(\tau)$$

is orthogonal to all cusp forms, because: when we integrate against a Poincaré series

$$P_{5/2,n,\gamma}(\tau) = \frac{1}{2}\sum_{c,d}(c\tau+d)^{-k}\mathbf{e}\Big(n(M\cdot\tau)\Big)\rho^*(M)^{-1}(\mathfrak{e}_\gamma),$$

we find that

$$\begin{aligned}
&\Big(4i\vartheta' + \frac{1}{y}\vartheta, P_{5/2,n,\gamma}\Big) \\
&= 4i\int_{-1/2}^{1/2}\int_0^\infty \langle \vartheta'(\tau), \mathbf{e}(n\tau)\mathfrak{e}_\gamma\rangle y^{1/2}\,\mathrm{d}y\,\mathrm{d}x + \int_{-1/2}^{1/2}\int_0^\infty \langle\vartheta(\tau), \mathbf{e}(n\tau)\mathfrak{e}_\gamma\rangle y^{-1/2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \sum_{r\in\mathbb{Z}-\langle\beta,\gamma\rangle} \delta_{4mn-r^2} A_n\Big(4i\cdot(2\pi i n)\frac{\Gamma(3/2)}{(4\pi n)^{3/2}} + \frac{\Gamma(1/2)}{(4\pi n)^{1/2}}\Big) \\
&= 0,
\end{aligned}$$

since $4i \cdot (2\pi i n)\frac{\Gamma(3/2)}{(4\pi n)^{3/2}} + \frac{\Gamma(1/2)}{(4\pi n)^{1/2}} = 0$ for all n. Here, δ_N denotes the delta function $\delta_N = \begin{cases} 1: & N = 0; \\ 0: & N \neq 0. \end{cases}$

Finally, the fact that $E_{5/2,m,\beta}(\tau, 0) + 4i\vartheta'(\tau)$ and $Q_{5/2,m,\beta}$ both have constant term $1 \cdot \mathfrak{e}_0$ implies that their difference is a cusp form that is orthogonal to all Poincaré series and therefore zero. □

Example 44. Consider the quadratic form with Gram matrix $S = \Big(-2\Big)$. The space of weight 5/2 modular forms is 1-dimensional, spanned by the Eisenstein series

$$E_{5/2}(\tau) = \Big(1 - 70q - 120q^2 - ...\Big)\mathfrak{e}_0 + \Big(-10q^{1/4} - 48q^{5/4} - 250q^{9/4} - ...\Big)\mathfrak{e}_{1/2}.$$

The nonmodular Jacobi Eisenstein series of index 1 and weight 5/2 is

$$\begin{aligned} E_{5/2,1,0}(\tau, z) = &\Big(1 + q(\zeta^{-2} - 16\zeta^{-1} - 16 - 16\zeta + \zeta^2) + \\ &+ q^2(\zeta^{-2} - 32\zeta^{-1} - 24 - 32\zeta + \zeta^2) + ...\Big)\mathfrak{e}_0 \\ &+ \Big(-4q^{1/4} + q^{5/4}(-4\zeta^{-2} - 8\zeta^{-1} - 24 - 8\zeta - 4\zeta^2) + ...\Big)\mathfrak{e}_{1/2}, \end{aligned}$$

and setting $z = 0$, we find

$$\begin{aligned} E_{5/2,1,0}(\tau, 0) = &\Big(1 - 46q - 120q^2 - 240q^3 - 454q^4 - ...\Big)\mathfrak{e}_0 + \\ &+ \Big(-4q^{1/4} - 48q^{5/4} - 196q^{9/4} - 240q^{13/4} - ...\Big)\mathfrak{e}_{1/2}. \end{aligned}$$

This differs from $E_{5/2}$ by the theta derivative

$$\Big(-24q - 96q^4 - ...\Big)\mathfrak{e}_0 + \Big(-6q^{1/4} - 54q^{9/4} - ...\Big)\mathfrak{e}_{1/2} = 4i\vartheta'(\tau).$$

For comparison, the Jacobi Eisenstein series of index 2 (which is a true Jacobi form) is

$$\begin{aligned} &E_{5/2,2,0}(\tau, z) \\ &= \Big(1 + q(-10\zeta^{-2} - 16\zeta^{-1} - 18 - 16\zeta - 10\zeta^2) + \\ &\quad + q^2(\zeta^{-4} - 16\zeta^{-3} - 12\zeta^{-2} - 16\zeta^{-1} - 34 - 16\zeta - 12\zeta^2 - 16\zeta^3 + \zeta^4) + \cdots\Big)\mathfrak{e}_0 \\ &+ \Big(q^{1/4}(-2\zeta^{-1} - 6 - 2\zeta) + \\ &\quad + q^{5/4}(-2\zeta^{-3} - 4\zeta^{-2} - 14\zeta^{-1} - 8 - 14\zeta - 4\zeta^2 - 2\zeta^3) + \cdots\Big)\mathfrak{e}_{1/2}, \end{aligned}$$

and we see that $E_{5/2,2}(\tau, 0) = Q_{5/2,2}(\tau) = E_{5/2}(\tau)$ as predicted.

3.5 Coefficient formula for $Q_{k,m,\beta}$

For convenience, the results of the previous sections are summarized here.

Proposition 45. *Let $k \ge 5/2$. The coefficients $c(n,\gamma)$ of the Poincaré square series $Q_{k,m,\beta}$,*

$$Q_{k,m,\beta}(\tau) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z}-Q(\gamma)} c(n,\gamma) q^n \mathfrak{e}_\gamma,$$

are given as follows:
(i) If $n<0$, then $c(n,\gamma)=0$.
(ii) If $n=0$, then $c(n,\gamma)=1$ if $\gamma=0$ and $c(n,\gamma)=0$ otherwise.
(iii) If $n>0$, then

$$c(n,\gamma) = \delta + \frac{(-1)^{(2k-b^-+b^+)/4}\pi^{k-1/2}}{2^{k-2}m^{k-1}\Gamma(k-1/2)\zeta(2k-2)\sqrt{|\det(S)|}} \times$$
$$\times \sum_{|r|<\sqrt{4mn}} \Big(L(k-1,\chi_{\mathcal{D}}) \prod_{\mathrm{bad}\,p} \Big[\frac{1-p^{-k+e/2+1}}{1-p^{2-2k}} L_p(n,r,\gamma,k+e/2-1) \Big] \Big)$$

if e is even, and

$$c(n,\gamma) = \varepsilon_{5/2} + \delta + \frac{(-1)^{(2k-b^-+b^+)/4}\pi^{k-1/2}}{2^{k-2}m^{k-1}\Gamma(k-1/2)\sqrt{|\det(S)|}} \times$$
$$\times \sum_{|r|<\sqrt{4mn}} \Big(\frac{1}{L(k-1/2,\chi_D)} \prod_{\mathrm{bad}\,p} \Big[(1-p^{-k+e/2+1}) L_p(n,r,\gamma,k+e/2-1) \Big] \Big)$$

if e is odd. Here, for each r, we define the set of "bad primes" to be

$$\{\mathrm{bad\,primes}\} = \{2\} \cup \Big\{ p\,\mathrm{prime}:\ p | d_\beta^2 m \det(S) \text{ or } v_p(d_\beta^2 d_\gamma^2 (n - r^2/4m)) \neq 0 \Big\},$$

and we define

$$\mathcal{D} = m d_\beta^4 d_\gamma^2 (-1)^{e/2+1}(n-r^2/4m)\det(S) \prod_{\mathrm{bad}\,p} p^2$$

if e is even and

$$D = 2m d_\beta^2 (-1)^{(e+1)/2} \det(S) \prod_{\mathrm{bad}\,p} p^2$$

if e is odd; $L(s,\chi_{\mathcal{D}})$ and $L(s,\chi_D)$ denote the L-series

$$L(s,\chi_{\mathcal{D}}) = \sum_{a=1}^\infty \Big(\frac{\mathcal{D}}{a}\Big) a^{-s}, \quad L(s,\chi_D) = \sum_{a=1}^\infty \Big(\frac{D}{a}\Big) a^{-s};$$

and L_p is the L-series

$$L_p(n,\gamma,r,s) = \sum_{\nu=0}^{\infty} \mathbf{N}(p^\nu)p^{-\nu s},$$

where

$$\mathbf{N}(p^\nu) = \#\Big\{(v,\lambda) \in \mathbb{Z}^{n+1}/p^\nu\mathbb{Z}^{n+1} :\ Q(v+\lambda\beta-\gamma) + m\lambda^2 - r\lambda + n = 0 \in \mathbb{Z}/p^\nu\mathbb{Z}\Big\}.$$

Finally,

$$\delta = \begin{cases} 2: & n = m\lambda^2 \text{ for some } \lambda \in \mathbb{Z}, \text{ and } \gamma = \lambda\beta; \\ 0: & \text{otherwise;} \end{cases}$$

and $\varepsilon_{5/2} = 0$ unless $k = 5/2$ and D is a rational square, in which case

$$\varepsilon_{5/2} = \sum_{\substack{r \in \mathbb{Z}-\langle\gamma,\beta\rangle \\ r^2 = 4mn}} \frac{24n\cdot(-1)^{(5+b^+-b^-)/4}}{\sqrt{2m\cdot\det(S)}} \prod_{\text{bad } p} \Big[\frac{(1-p^{(e-3)/2})(1-p^{-1})}{1-p^{-2}} L_p(n,r,\gamma,(e+3)/2)\Big].$$

Proof. For $k > 5/2$, since $Q_{k,m,\beta}(\tau) = E_{k,m,\beta}(\tau,0)$, we get the coefficients of $Q_{k,m,\beta}$ by summing the coefficients of $E_{k,m,\beta}$ over r. δ accounts for the contribution from the term

$$\sum_{\lambda\in\mathbb{Z}} \mathbf{e}\Big(m\lambda^2\tau + 2m\lambda z\Big)\mathfrak{e}_{\lambda\beta}.$$

When $k = 5/2$, $\varepsilon_{5/2}$ accounts for $4i$ times the derivative of the theta series

$$\vartheta(\tau) = \sum_{\gamma\in\Lambda'/\Lambda} \sum_{\substack{4mn-r^2=0 \\ n\in\mathbb{Z}-Q(\gamma) \\ r\in\mathbb{Z}-\langle\gamma,\beta\rangle}} A_n q^n \mathfrak{e}_\gamma.$$

□

3.6 Example: calculating an automorphic product

The notation in this section is taken from [4].

Since $Q_{k,m,\beta}$ can be calculated efficiently, we can automate the process of searching for automorphic products. This method can handle arbitrary even lattices (with no restrictions on the level or the dimension of the cusp space $S_k(\rho^*)$).

Let Λ be an even lattice of signature $(2,n)$. Recall that Borcherds' singular theta correspondence [4] sends a nearly-holomorphic modular form with integer coefficients

$$f(\tau) = \sum_\gamma \sum_n c(n,\gamma)\mathfrak{e}_\gamma$$

of weight $k = 1 - n/2$ for the Weil representation to a meromorphic automorphic form Ψ on the Grassmannian of Λ. The weight of Ψ is $\frac{c(0,0)}{2}$, and Ψ is holomorphic when $c(n,\gamma)$ is nonnegative for all γ and $n < 0$.

Automorphic products Ψ of **singular weight** $n/2 - 1$ are particularly interesting, since in this case most of the Fourier coefficients of Ψ must vanish: the nonzero Fourier coefficients correspond to vectors of norm zero.

Taking the scalar product on $\mathbb{C}[A]$ of nearly-holomorphic modular forms of weight k for ρ and weight $2-k$ for ρ^* gives a scalar-valued (nearly-holomorphic) modular form of weight 2, or equivalently an invariant differential form on $\mathbb{H}$, whose residue in ∞ must be 0. This implies that the constant term in the Fourier expansion must be zero. Also, the coefficients $c(n,\gamma)$ of a nearly-holomorphic modular form must satisfy $c(n,\gamma) = c(n,-\gamma)$ for all n and γ, due to the transformation law under Z. As shown in [5] and [12], this is the only obstruction for a sum $\sum_{n<0}\sum_\gamma c(n,\gamma)\mathfrak{e}_\gamma + c(0,0)\mathfrak{e}_0$ to occur as the principal part of a nearly-holomorphic modular form.

The lattice $A_1(-2) + A_1(-2) + II_{1,1} + II_{1,1}$ produces an automorphic product of singular weight. This product also arises through an Atkin-Lehner involution from an automorphic product attached to the lattice $A_1 \oplus A_1 \oplus II_{1,1} \oplus II_{1,1}(8)$, found by Scheithauer in [56].

Using the dimension formula (proposition 14) for the lattice $\Lambda = \mathbb{Z}^2$ with Gram matrix $\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$, we find

$$\dim M_3(\rho^*) = 4, \quad \dim S_3(\rho^*) = 2.$$

The Eisenstein series of weight 3 is

$$\begin{aligned} &E_{3,(0,0)}(\tau) \\ =& \Big(1 - 24q - 164q^2 - 192q^3 - ...\Big)\mathfrak{e}_{(0,0)} \\ +& \Big(-1/2q^{1/8} - 73/2q^{9/8} - 145q^{17/8} - ...\Big)(\mathfrak{e}_{(1/4,0)} + \mathfrak{e}_{(3/4,0)} + \mathfrak{e}_{(0,1/4)} + \mathfrak{e}_{(0,3/4)}) \\ +& \Big(-10q^{1/2} - 48q^{3/2} - 260q^{5/2} - ...\Big)(\mathfrak{e}_{(1/2,0)} + \mathfrak{e}_{(0,1/2)}) \\ +& \Big(-2q^{1/4} - 52q^{5/4} - 146q^{9/4} - ...\Big)(\mathfrak{e}_{(1/4,3/4)} + \mathfrak{e}_{(3/4,1/4)} + \mathfrak{e}_{(1/4,1/4)} + \mathfrak{e}_{(3/4,3/4)}) \\ +& \Big(-13q^{5/8} - 85q^{13/8} - 192q^{21/8} - ...\Big)(\mathfrak{e}_{(1/2,1/4)} + \mathfrak{e}_{(1/2,3/4)} + \mathfrak{e}_{(1/4,1/2)} + \mathfrak{e}_{(3/4,1/2)}) \\ +& \Big(-44q - 96q^2 - 288q^3 - ...\Big)\mathfrak{e}_{(1/2,1/2)}. \end{aligned}$$

We find two linearly independent cusp forms as differences between E_3 and particular Poincaré square series: for example,

$$\begin{aligned}
&\frac{2}{3}\Big(Q_{3,1/8,(1/4,0)} - E_3\Big)\\
&= \Big(q^{1/8} + 9q^{9/8} - 30q^{17/8} + ...\Big)(\mathfrak{e}_{(1/4,0)} + \mathfrak{e}_{(3/4,0)} - \mathfrak{e}_{(0,1/4)} - \mathfrak{e}_{(0,3/4)})\\
&+ \Big(6q^{5/8} - 10q^{13/8} - 42q^{29/8} - ...\Big)(\mathfrak{e}_{(1/2,1/4)} + \mathfrak{e}_{(1/2,3/4)} - \mathfrak{e}_{(1/4,1/2)} - \mathfrak{e}_{(3/4,1/2)})\\
&+ \Big(8q^{1/2} - 48q^{5/2} + 72q^{9/2} + ...\Big)(\mathfrak{e}_{(1/2,0)} - \mathfrak{e}_{(0,1/2)}),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{3}\Big(Q_{3,1/4,(1/4,1/4)} - E_3\Big)\\
&= \Big(q^{1/4} - 6q^{5/4} + 9q^{9/4} + 10q^{13/4} + ...\Big)(\mathfrak{e}_{(1/4,1/4)} + \mathfrak{e}_{(3/4,3/4)} - \mathfrak{e}_{(1/4,3/4)} - \mathfrak{e}_{(3/4,1/4)}).
\end{aligned}$$

The other Eisenstein series $E_{3,(1/2,1/2)}$ can be easily computed by averaging $E_{3,(0,0)}$ over the Schrödinger representation (as in the appendix), but Eisenstein series other than $E_{k,0}$ never represent new obstructions so we do not need them.

We see that the sum

$$q^{-1/8}(\mathfrak{e}_{(1/4,0)} + \mathfrak{e}_{(3/4,0)} + \mathfrak{e}_{(0,1/4)} + \mathfrak{e}_{(0,3/4)}) + 2\mathfrak{e}_{(0,0)}$$

occurs as the principal part of a nearly-holomorphic modular form, and the corresponding automorphic product has weight 1 (which is the singular weight for the lattice $\Lambda \oplus II_{1,1} \oplus II_{1,1}$ of signature $(2,4)$).

A brute-force way to calculate the nearly-holomorphic modular form F is to search for $\Delta \cdot F$ among cusp forms of weight 11 for ρ. Since ρ is also the dual Weil representation ρ^* of the lattice with Gram matrix $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, we can use the same formulas for Poincaré square series. This is somewhat messier since the cusp space is now 8-dimensional. Using the coefficients

$$\alpha_0 = \frac{12221466065269207652111168}{665492278281307137675},$$
$$\alpha_1 = -\frac{8147005528164244342336}{1996476834843921413025},$$
$$\alpha_2 = -\frac{5383641094234426568192}{133098455656261427535},$$
$$\alpha_3 = \frac{7719027691905873961829 2}{665492278281307137675},$$
$$\alpha_4 = -\frac{38164413333716056915312 64}{1996476834843921413025},$$

a calculation shows that

$$\begin{aligned} F &= \frac{\alpha_0 E_{11,0} + \alpha_1 Q_{11,1,0} + \alpha_2 Q_{11,2,0} + \alpha_3 Q_{11,3,0} + \alpha_4 Q_{11,4,0}}{\Delta} \\ &= \Big(2 + 8q + 24q^2 + 64q^3 + 152q^4 + ... \Big) (\mathfrak{e}_{(0,0)} - \mathfrak{e}_{(1/2,1/2)}) \\ &+ \Big(q^{-1/8} + 3q^{7/8} + 11q^{15/8} + 28q^{23/8} + ... \Big) (\mathfrak{e}_{(1/4,0)} + \mathfrak{e}_{(3/4,0)} + \mathfrak{e}_{(0,1/4)} + \mathfrak{e}_{(0,3/4)}) \\ &+ \Big(-2q^{3/8} - 6q^{11/8} - 18q^{19/8} - ... \Big) (\mathfrak{e}_{(1/4,1/2)} + \mathfrak{e}_{(3/4,1/2)} + \mathfrak{e}_{(1/2,1/4)} + \mathfrak{e}_{(1/2,3/4)}). \end{aligned}$$

Once enough coefficients have been calculated, it is not hard to identify these components: the coefficients come from the weight -1 eta products

$$\frac{2\eta(2\tau)^2}{\eta(\tau)^4} = 2 + 8q + 24q^2 + 64q^3 + 152q^4 + ...$$

and

$$\frac{\eta(\tau/2)^2}{\eta(\tau)^4} = q^{-1/8} - 2q^{3/8} + 3q^{7/8} - 6q^{11/8} + 11q^{15/8} - 18q^{19/8} + ...$$

We will calculate the automorphic product using theorem 13.3 of [4], following the pattern of the examples of [29]. Fix the primitive isotropic vector $z = (1,0,0,0,0,0)$ and $z' = (0,0,0,0,0,1)$ and the lattice $K = \Lambda \oplus II_{1,1}$. We fix as positive cone the component of positive-norm vectors containing those of the form $(+,*,*,+)$. This is split into Weyl chambers by the hyperplanes $\alpha^\perp$ with $\alpha \in \{\pm(0,1/4,0,0), \pm(0,0,1/4,0)\}$. These are all essentially the same so we will fix the Weyl chamber

$$W = \{(x_1,x_2,x_3,x_4) : \; x_1,\, x_2,\, x_3,\, x_4,\, x_1x_4 - 2x_2^2 - 2x_3^2 > 0\} \subseteq K \otimes \mathbb{R}.$$

The Weyl vector attached to F and W is the isotropic vector

$$\rho = \rho(K, W, F_K) = (1/4, 1/8, 1/8, 1/4),$$

which can be calculated with theorem 10.4 of [4].

The product

$$\Psi_z(Z) = \mathbf{e}\Big(\langle \rho, Z\rangle\Big) \prod_{\substack{\lambda \in K' \\ \langle \lambda, W\rangle > 0}} \Big(1 - \mathbf{e}((\lambda, Z))\Big)^{c(Q(\lambda),\lambda)}$$

has singular weight, and therefore its Fourier expansion has the form

$$\Psi_z(Z) = \sum_{\substack{\lambda \in K' \\ \langle \lambda, W\rangle > 0}} a(\lambda)\mathbf{e}\Big(\langle \lambda + \rho, Z\rangle\Big)$$

where $a(\lambda) = 0$ unless $\lambda + \rho$ has norm 0. Since $\Psi_z(w(Z)) = \det(w)\Psi_z(w)$ for all elements of the Weyl group $w \in G$, we can write this as

$$\Psi_z(w(Z)) = \sum_{w \in G} \det(w) \sum_{\substack{\lambda \in K' \\ \lambda + \rho \in \overline{W} \\ \langle \lambda, W\rangle > 0}} a(\lambda)\mathbf{e}\Big(\langle w(\lambda + \rho), Z\rangle\Big).$$

As in [29], any such λ must be a positive integer multiple of ρ; and in fact to be in K' it must be a multiple of 4ρ. Also, the only terms in the product that contribute to $a(\lambda)$ come from other positive multiples of 4ρ; i.e.

$$\mathbf{e}\Big(\langle \rho, Z\rangle\Big) \prod_{m>0} \Big[1 - \mathbf{e}\Big(\langle 4m\rho, Z\rangle\Big)\Big]^{c(0,4m\rho)} = \sum_{\substack{\lambda \in K' \\ \langle \lambda, W\rangle > 0 \\ \lambda + \rho \in \overline{W}}} a(\lambda)\mathbf{e}\Big(\langle \lambda + \rho, Z\rangle\Big).$$

Here, $c(0, 4m\rho) = 2 \cdot (-1)^m$, so

$$\sum_{\lambda} a(\lambda)\mathbf{e}\Big(\langle \lambda + \rho, Z\rangle\Big) = \mathbf{e}\Big(\langle \rho, Z\rangle\Big) \prod_{m>0} \Big[1 - \mathbf{e}\Big(\langle 4m\rho, Z\rangle\Big)\Big]^{2(-1)^m},$$

so we get the identity

$$\begin{aligned}\Psi_z(Z) &= \mathbf{e}\Big(\langle \rho, Z\rangle\Big) \prod_{\substack{\lambda \in K' \\ \langle \lambda, W\rangle > 0}} \Big(1 - \mathbf{e}(\langle \lambda, Z\rangle)\Big)^{c(Q(\lambda),\lambda)} \\ &= \sum_{w \in G} \det(w)\mathbf{e}\Big(\langle w(\rho), Z\rangle\Big) \prod_{m=1}^{\infty} \Big[1 - \mathbf{e}\Big(4m\langle w(\rho), Z\rangle\Big)\Big]^{2(-1)^m}.\end{aligned}$$

Note that the product on the right is an eta product

$$q\prod_{m=1}^{\infty}\Big[1-q^{4m}\Big]^{2(-1)^m}=\frac{\eta(8\tau)^4}{\eta(4\tau)^2},$$

so we can write this in the more indicative form

$$\mathbf{e}\Big(\langle\rho,Z\rangle\Big)\prod_{\substack{\lambda\in K'\\ \langle\lambda,W\rangle>0}}\Big(1-\mathbf{e}(\langle\lambda,Z\rangle)\Big)^{c(Q(\lambda),\lambda)}=\sum_{w\in G}\det(w)\frac{\eta(8\langle w(\rho),Z\rangle)^4}{\eta(4\langle w(\rho),Z\rangle)^2}.$$

3.7 Example: computing Petersson scalar products

One side effect of the computation of Poincaré square series is another way to compute the Petersson scalar product of (vector-valued) cusp forms numerically. This is rather easy so we will only give an example, rather than state a general theorem. Consider the weight 3 cusp form

$$\begin{aligned}\Theta(\tau)&=\sum_{n,\gamma}c(n,\gamma)q^n\mathfrak{e}_\gamma\\&=\Big(q^{1/6}+2q^{7/6}-22q^{13/6}+26q^{19/6}+...\Big)\times\\&\qquad\qquad\times(\mathfrak{e}_{(1/6,2/3)}+\mathfrak{e}_{(1/3,5/6)}+\mathfrak{e}_{(2/3,1/6)}+\mathfrak{e}_{(5/6,1/3)}-2\mathfrak{e}_{(1/6,1/6)}-2\mathfrak{e}_{(5/6,5/6)})\\&\quad+\Big(-6q^{1/2}+18q^{3/2}+0q^{5/2}-12q^{7/2}-...\Big)(\mathfrak{e}_{(1/2,0)}+\mathfrak{e}_{(0,1/2)}-2\mathfrak{e}_{(1/2,1/2)}),\end{aligned}$$

which is the theta series with respect to a harmonic polynomial for the lattice with Gram matrix $\begin{pmatrix}-4&-2\\-2&-4\end{pmatrix}$. The component functions are

$$q^{1/6}+2q^{7/6}-22q^{13/6}+26q^{19/6}+...=\eta(\tau/3)^3\eta(\tau)^3+3\eta(\tau)^3\eta(3\tau)^3$$

and

$$-6q^{1/2}+18q^{3/2}+0q^{5/2}-12q^{7/2}+...=-6\eta(\tau)^3\eta(3\tau)^3.$$

To compute the Petersson scalar product (Θ,Θ), we write Θ as a linear combination of Eisenstein series and Poincaré square series; for example,

$$\Theta=E_{3,0}-Q_{3,1/6,(1/6,1/6)}.$$

It follows that

$$\begin{aligned}(\Theta,\Theta) &= -(\Theta, Q_{3,1/6,(1/6,1/6)})\\ &= -\frac{9}{2\pi^2}\sum_{\lambda=1}^{\infty}\frac{c(\lambda^2/6,(\lambda/6,\lambda/6))}{\lambda^4}\\ &= \frac{9}{\pi^2}\Big[\sum_{\lambda\equiv 1,5\,(6)}\frac{a(\lambda^2/2)}{\lambda^4} - 6\sum_{\lambda\equiv 3\,(6)}\frac{a(\lambda^2/2)}{\lambda^4}\Big],\end{aligned}$$

where $a(n)$ is the coefficient of n in $\eta(\tau)^3\eta(3\tau)^3$. This series converges rather slowly but summing the first 150 terms seems to give the value $(\Theta,\Theta)\approx 0.24$. We get far better convergence for larger weights.

For scalar-valued forms (i.e. when the lattice Λ is unimodular), applying this method to Hecke eigenforms gives the same result as a well-known method involving the symmetric square L-function. For example, the discriminant

$$\Delta = q - 24q^2 + ... = \sum_{n=1}^{\infty} c(n)q^n \in S_{12}$$

can be written as

$$\Delta = \frac{53678953}{304819200}(Q_{12,1,0} - E_{12})$$

which gives the identity

$$(\Delta,\Delta) = \frac{131\cdot 593\cdot 691}{2^{23}\cdot 3\cdot 7\cdot \pi^{11}}\sum_{n=1}^{\infty}\frac{c(n^2)}{n^{22}}.$$

This identity is equivalent to the case $s = 22$ of equation (29) of [73]:

$$\sum_{n=1}^{\infty}\frac{c(n)^2}{n^{22}} = \frac{7\cdot 11\cdot 4^{22}\cdot \pi^{33}\cdot \zeta(11)}{2\cdot 23\cdot 691\cdot 22!\cdot \zeta(22)}(\Delta,\Delta),$$

since

$$\sum_{n=1}^{\infty}\frac{c(n)^2}{n^{22}} = \zeta(11)\sum_{n=1}^{\infty}\frac{c(n^2)}{n^{22}},$$

which can be proved directly using the fact that Δ is a Hecke eigenform.

3.8 Calculating the Euler factors at $p = 2$

We will summarize the calculations of Appendix B in [23] as they apply to our situation.

Proposition 46. *Let $f(X) = \bigoplus_{i\in\mathbb{N}_0} 2^i Q_i(X) \oplus L + c$ be a $\mathbb{Z}_2$-integral quadratic polynomial in normal form, and assume that all Q_i are given by $Q_i(v) = v^T S_i v$ for a symmetric (not necessarily even) $\mathbb{Z}_2$-integral matrix S_i. For any $j \in \mathbb{N}_0$, define*

$$\mathbf{Q}_{(j)} := \bigoplus_{\substack{0\le i\le j \\ i\equiv j\,(2)}} Q_i, \quad \mathbf{r}_{(j)} = \operatorname{rank}(\mathbf{Q}_{(j)}), \quad \mathbf{p}_{(j)} = 2^{\sum_{0\le i<j}\mathbf{r}_{(i)}}.$$

Let $\omega \in \mathbb{N}_0$ be such that $Q_i = 0$ for all $i > \omega$. Then:

(i) If $L = 0$ and $c = 0$, let $r = \sum_i \operatorname{rank}(Q_i)$; then the Igusa zeta function for f at 2 is

$$\begin{aligned}
&\zeta_{Ig}(f;2;s) \\
&= \sum_{0\le\nu<\omega-1} \frac{2^{-\nu s}}{\mathbf{p}_{(\nu)}} I_0(\mathbf{Q}_{(\nu)}, \mathbf{Q}_{(\nu+1)}, Q_{\nu+2}) + \\
&\quad + \Big[\frac{2^{-s(\omega-1)}}{\mathbf{p}_{(\omega-1)}} I_0(\mathbf{Q}_{(\omega-1)}, \mathbf{Q}_{(\omega)}, 0) + \frac{2^{-\omega s}}{\mathbf{p}_{(\omega)}} I_0(\mathbf{Q}_{(\omega)}, \mathbf{Q}_{(\omega-1)}, 0)\Big] \cdot (1 - 2^{-2s-r})^{-1}.
\end{aligned}$$

(ii) If $L(x) = bx$ for some $b \neq 0$ with $v_2(b) = \lambda$ and if $v_2(b) \le v_2(c)$, then

$$\begin{aligned}
\zeta_{Ig}(f;2;s) = &\sum_{0\le\nu<\lambda-2} \frac{2^{-\nu s}}{\mathbf{p}_{(\nu)}} I_0(\mathbf{Q}_{(\nu)}, \mathbf{Q}_{(\nu+1)}, Q_{\nu+2}) + \\
&+ \sum_{\max\{0,\lambda-2\}\le\nu<\lambda} \frac{2^{-\nu s}}{\mathbf{p}_{(\nu)}} I_0^{\lambda-\nu}(\mathbf{Q}_{(\nu)}, \mathbf{Q}_{(\nu+1)}, Q_{i+2}) + \frac{2^{-\lambda s}}{\mathbf{p}_{(\lambda)}} \cdot \frac{1}{2 - 2^{-s}}.
\end{aligned}$$

(iii) If $L(x) = bx$ with $b \neq 0$ and $v_2(c) < v_2(b) \le v_2(c) + 2$, let $\kappa = v_2(c)$; then

$$\begin{aligned}
\zeta_{Ig}(f;2;s) = &\sum_{0\le\nu<\lambda-2} \frac{2^{-\nu s}}{\mathbf{p}_{(\nu)}} I_{c/2^\nu}(\mathbf{Q}_{(\nu)}, \mathbf{Q}_{(\nu+1)}, Q_{\nu+2}) \\
&+ \sum_{\max\{0,\lambda-2\}\le\nu\le\kappa} \frac{2^{-\nu s}}{\mathbf{p}_{(\nu)}} I_{c/2^\nu}^{\lambda-\nu}(\mathbf{Q}_{(\nu)}, \mathbf{Q}_{(\nu+1)}, Q_{\nu+2}) + \frac{1}{\mathbf{p}_{(\kappa+1)}} 2^{-\kappa s}.
\end{aligned}$$

(iv) If $L = 0$ or $L(x) = bx$ with $v_2(b) > v_2(c) + 2$, let $\kappa = v_2(c)$; then

$$\zeta_{Ig}(f;2;s) = \sum_{0\le\nu\le\kappa} \frac{2^{-\nu s}}{\mathbf{p}_{(\nu)}} I_{c/2^\nu}(\mathbf{Q}_{(\nu)}, \mathbf{Q}_{(\nu+1)}, Q_{\nu+2}) + \frac{1}{\mathbf{p}_{(\kappa+1)}} 2^{-\kappa s}.$$

Here, $I_a^b(Q_0, Q_1, Q_2)(s)$ are helper functions that we describe below, and we set $I_a(Q_0, Q_1, Q_2) = I_a^\infty(Q_0, Q_1, Q_2)$. Note that not every unimodular quadratic form Q_i over $\mathbb{Z}_2$ can be written in the form $Q_i(v) = v^T S_i v$; but $2 \cdot Q_i$ can always be written in this form, and replacing f by $2 \cdot f$ only multiplies $\zeta_{Ig}(f; 2; s)$ by 2^{-s}, so this does not lose generality.

Every unimodular quadratic form over $\mathbb{Z}_2$ that has the form $Q_i(v) = v^T S_i v$ is equivalent to a direct sum of at most two one-dimensional forms $a \cdot \mathrm{Sq}(x) = ax^2$; at most one elliptic plane $\mathrm{Ell}(x, y) = 2x^2+2xy+2y^2$; and any number of hyperbolic planes $\mathrm{Hyp}(x, y) = 2xy$. This decomposition is not necessarily unique. It will be enough to fix one such decomposition.

The following proposition explains how to compute $I_a^b(Q_0, Q_1, Q_2)(s)$.

Proposition 47. *Define the function*

$$\mathrm{Ig}(a, b, \nu) = \begin{cases} \frac{2^{-\nu s}}{2-2^{-s}} : & v_2(a) \geq \min(b, \nu); \\ 2^{-v_2(a)s} : & v_2(a) < \min(b, \nu). \end{cases}$$

(Here, $v_2(0) = \infty$.) For a unimodular quadratic form Q of rank r, fix a decomposition into hyperbolic planes, at most one elliptic plane and at most two square forms as above. Let $\varepsilon = 1$ if Q contains no elliptic plane and $\varepsilon = -1$ otherwise. Define functions $H_1(a, b, Q)$, $H_2(a, b, Q)$ and $H_3(a, b, Q)$ as follows:
(i) If Q contains no square forms, then

$$\begin{aligned} H_1(a, b, Q) &= (1 - 2^{-r})\mathrm{Ig}(a, b, 1); \\ H_2(a, b, Q) &= \Big(1 - 2^{-r/2}\varepsilon\Big) \cdot \Big(\mathrm{Ig}(a, b, 1) + 2^{-r/2}\varepsilon\mathrm{Ig}(a, b, 2)\Big); \\ H_3(a, b, Q) &= 0. \end{aligned}$$

(ii) If Q contains one square form cx^2, then

$$\begin{aligned} H_1(a, b, Q) &= \mathrm{Ig}(a, b, 0) - 2^{-r}\mathrm{Ig}(a, b, 1); \\ H_2(a, b, Q) &= (1 - 2^{-(r-1)/2}\varepsilon)\mathrm{Ig}(a, b, 0) - 2^{-r}\mathrm{Ig}(a, b, 2)+ \\ &\quad + 2^{-(r+1)/2}\varepsilon(\mathrm{Ig}(a, b, 2) + \mathrm{Ig}(a + c, b, 2)); \\ H_3(a, b, Q) &= 2^{-r}(\mathrm{Ig}(a + c, b, 3) - \mathrm{Ig}(a + c, b, 2)). \end{aligned}$$

(iii) If Q contains two square forms cx^2, dx^2 and $c + d \equiv 0\,(4)$, then

$$\begin{aligned} H_1(a, b, Q) &= \mathrm{Ig}(a, b, 0) - 2^{-r}\mathrm{Ig}(a, b, 1); \\ H_2(a, b, Q) &= \mathrm{Ig}(a, b, 0) - 2^{-r/2}\varepsilon\mathrm{Ig}(a, b, 1) + (2^{-r/2}\varepsilon - 2^{-r})\mathrm{Ig}(a, b, 2); \\ H_3(a, b, Q) &= (-1)^{(c+d)/4}2^{-r}(\mathrm{Ig}(a, b, 3) - \mathrm{Ig}(a, b, 2)). \end{aligned}$$

(iv) If Q contains two square forms cx^2, dx^2 and $c+d \not\equiv 0\,(4)$, then

$$\begin{aligned} H_1(a,b,Q) &= \mathrm{Ig}(a,b,0) - 2^{-r}\mathrm{Ig}(a,b,1); \\ H_2(a,b,Q) &= (1-2^{-(r-2)/2}\varepsilon)\mathrm{Ig}(a,b,0) + 2^{-r/2}\varepsilon(\mathrm{Ig}(a,b,1) + \mathrm{Ig}(a+c,b,2)) - \\ &\quad - 2^{-r}\mathrm{Ig}(a,b,2); \\ H_3(a,b,Q) &= -2^{1-r}\mathrm{Ig}(a,b,1) + 2^{-r}(\mathrm{Ig}(a,b,2) + \mathrm{Ig}(a+c+d,b,3)). \end{aligned}$$

Let $\varepsilon_1 = 1$ if Q_1 contains no elliptic plane and $\varepsilon_1 = -1$ otherwise, and let r_1 denote the rank of Q_1. Then $I_a^b(Q_0,Q_1,Q_2)$ is given as follows:
(1) If both Q_1 and Q_2 contain at least one square form, then

$$I_a^b(Q_0,Q_1,Q_2) = H_1(a,b,Q_0).$$

(2) If Q_1 contains no square forms but Q_2 contains at least one square form, then

$$I_a^b(Q_0,Q_1,Q_2) = H_2(a,b,Q_0).$$

(3) If both Q_1 and Q_2 contain no square forms, then

$$I_a^b(Q_0,Q_1,Q_2) = H_2(a,b,Q_0) + 2^{-r_1/2}\varepsilon_1 H_3(a,b,Q_0).$$

(4) If Q_1 contains one square form cx^2, and Q_2 contains no square forms, then

$$I_a^b(Q_0,Q_1,Q_2) = H_1(a,b,Q_0) + 2^{-(r_1+1)/2}\varepsilon_1(H_3(a,b,Q_0) + H_3(a+2c,b,Q_0)).$$

(5) If Q_1 contains two square forms cx^2 and dx^2 such that $c+d \equiv 0\,(4)$, and Q_2 contains no square forms, then

$$I_a^b(Q_0,Q_1,Q_2) = H_1(a,b,Q_0) + 2^{-r_1/2}\varepsilon_1 H_3(a,b,Q_0).$$

(6) If Q_1 contains two square forms cx^2 and dx^2 such that $c+d \not\equiv 0\,(4)$, and Q_2 contains no square forms, then

$$I_a^b(Q_0,Q_1,Q_2) = H_1(a,b,Q_0) + 2^{-r_1/2}\varepsilon_1 H_3(a+c,b,Q_0).$$

Proof. In the notation of [23],

$$\mathrm{Ig}(a,b,\nu) = \mathrm{Ig}(z^{a+2^b\mathbb{Z}_2+2^\nu\mathbb{Z}_2})$$

and

$$H_1(a, b, Q) = \mathrm{Ig}\Big(z^{a+2^b\mathbb{Z}_2}\hat{H}_Q(z)\Big)$$

and

$$H_2(a, b, Q) = \mathrm{Ig}\Big(z^{a+2^b\mathbb{Z}_2}\tilde{H}_Q(z)\Big)$$

and

$$H_3(a, b, Q) = \mathrm{Ig}\Big(z^{a+2^b\mathbb{Z}_2}(H_Q(z) - \tilde{H}_Q(z))\Big).$$

This calculation of $I_a^b(Q_0, Q_1, Q_2)$ is available in Appendix B of [23]. Finally, the calculation of $\zeta_{Ig}(f; 2; s)$ is given in theorem 4.5 loc. cit. $\square$

Chapter 4

Vector-valued Eisenstein series of small weight

This chapter is taken from the paper [70].

4.1 Introduction

In [18], Bruinier and Kuss give an expression for the Fourier coefficients of the Eisenstein series E_k of weight $k \geq 5/2$ for the Weil representation attached to a discriminant form. These coefficients involve special values of L-functions and zero counts of polynomials modulo prime powers, and they also make sense for $k \in \{1, 3/2, 2\}$. Unfortunately, the q-series E_k obtained in this way often fail to be modular forms. In particular, in weight $k = 3/2$ and $k = 2$, the Eisenstein series may be a mock modular form that requires a real-analytic correction in order to transform as a modular form. Many examples of this phenomenon of the Eisenstein series are well-known (although perhaps less familiar in a vector-valued setting). We will list a few examples of this:

Example 48. The Eisenstein series of weight 2 for a unimodular lattice Λ is the quasimodular form

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - ...$$

where $\sigma_1(n) = \sum_{d|n} d$, which transforms under the modular group by

$$E_2\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2 E_2(\tau) + \frac{6}{\pi i}c(c\tau+d).$$

Example 49. The Eisenstein series of weight 3/2 for the quadratic form $Q_2(x) = x^2$ is essentially Zagier's mock Eisenstein series:

$$E_{3/2}(\tau) = \Big(1 - 6q - 12q^2 - 16q^3 - ...\Big)\mathfrak{e}_0 + \Big(-4q^{3/4} - 12q^{7/4} - 12q^{11/4} - ...\Big)\mathfrak{e}_{1/2},$$

in which the coefficient of $q^{n/4}\mathfrak{e}_{n/2}$ is -12 times the Hurwitz class number $H(n)$. It transforms under the modular group by

$$E_{3/2}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{3/2}\rho^*\Big(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\Big)\Big[E_{3/2}(\tau) - \frac{3}{\pi}\sqrt{\frac{i}{2}}\int_{d/c}^{i\infty}(\tau+t)^{-3/2}\vartheta(t)\,\mathrm{d}t\Big],$$

where ϑ is the theta series

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/4}\mathfrak{e}_{n/2}.$$

Example 50. In the Eisenstein series of weight 3/2 for the quadratic form $Q_3(x) = 6x^2$, the components of $\mathfrak{e}_{1/12}$, $\mathfrak{e}_{5/12}$, $\mathfrak{e}_{7/12}$ and $\mathfrak{e}_{11/12}$ are

$$\Big(-3q^{23/24} - 5q^{47/24} - 7q^{71/24} - 8q^{95/24} - 10q^{119/24} - 10q^{143/24} - ...\Big)\mathfrak{e}_\gamma$$

for $\gamma \in \{1/12, 5/12, 7/12, 11/12\}$. We verified by computer that the coefficient of $q^{n-1/24}$ above is (-1) times the degree of the n-th partition class polynomial considered by Bruinier and Ono [19] for $1 \leq n \leq 750$, which is not surprising in view of example 2 since this degree also counts equivalence classes of certain binary quadratic forms. This Eisenstein series is not a modular form.

Example 51. The Eisenstein series of weight 3/2 for the quadratic form $Q_4(x,y,z) = x^2+y^2-z^2$ is a mock modular form that is related to the functions considered by Bringmann and Lovejoy [8] in their work on overpartitions. More specifically, the component of $\mathfrak{e}_{(0,0,0)}$ in $E_{3/2}$ is

$$1 - 2q - 4q^2 - 8q^3 - 10q^4 - ... = 1 - \sum_{n=1}^\infty |\overline{\alpha}(n)|q^n,$$

where $\overline{\alpha}(n)$ is the difference between the number of even-rank and odd-rank overpartitions of n. Similarly, the $M2$-rank differences considered in [8] occur in the Eisenstein series of weight 3/2 for the quadratic form $Q_5(x,y,z) = 2x^2 + 2y^2 - z^2$, whose $\mathfrak{e}_{(0,0,0)}$-component is

$$1 - 2q - 4q^2 - 2q^4 - 8q^5 - 8q^6 - 8q^7 - ...$$

as pointed out in [71].

Example 52. Unlike the previous examples, the Eisenstein series of weight 3/2 for the quadratic form $Q_6(x,y,z) = -x^2 - y^2 - z^2$ is a true modular form; in fact, it is the theta series for the cubic lattice and the Fourier coefficients of its $\mathfrak{e}_{(0,0,0)}$-component count the representations of integers as sums of three squares. From our point of view the difference between this and the previous examples is because Q_6 has a relatively small number of isotropic vectors modulo large powers of 2 (such that certain local L-functions will be holomorphic).

Among negative-definite lattices of small dimension there are lots of examples where the Eisenstein series equals the theta series. (Note that we find theta series for negative-definite lattices instead of positive-definite because we consider the dual Weil representation ρ^*.) When the lattice is even-dimensional this immediately leads to formulas for representation numbers in terms of twisted divisor sums. These formulas are of course well-known but the vector-valued derivations of these formulas seem more natural than the usual derivation as identities among scalar-valued forms of higher level. We give several examples of this throughout this chapter.

In the last section we make some remarks about the case $k = 1/2$, where the formula of [18] no longer makes sense and so the methods here break down.

4.2 The real-analytic Eisenstein series

Fix an even lattice Λ and let ρ^* be the dual Weil representation on $\mathbb{C}[\Lambda'/\Lambda]$.

Definition 53. The **real-analytic Eisenstein series** of weight k is

$$E_k^*(\tau, s) = \sum_{M \in \tilde{\Gamma}_\infty \backslash \Gamma} (y^s \mathfrak{e}_0)|_k M = \frac{y^s}{2} \sum_{c,d} (c\tau + d)^{-k} |c\tau + d|^{-2s} \rho^*(M)^{-1} \mathfrak{e}_0.$$

Here, (c,d) runs through all pairs of coprime integers and M is any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$ with bottom row (c,d); and the branch of $(c\tau+d)^{-k}$ is determined by M as an element of $Mp_2(\mathbb{Z})$ as usual.

This series converges absolutely and locally uniformly in the half-plane $\mathrm{Re}[s] > 1-k/2$ and defines a holomorphic function in s. For fixed s, it transforms under the metaplectic group by

$$E_k^*\Big(M\cdot\tau, s\Big) = (c\tau+d)^k \rho^*(M) E_k^*(\tau,s)$$

for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$. These series were considered by Bruinier and Kühn [17] in weight $k \geq 2$ who also give expressions for their Fourier expansions. (More generally they consider the series obtained after replacing $\mathfrak{e}_0$ with $\mathfrak{e}_\beta$ for an element $\beta \in \Lambda'/\Lambda$ with $Q(\beta) \in \mathbb{Z}$. We do not do this because it seems to make the formulas below considerably more complicated, and because for many discriminant forms Λ'/Λ one can obtain the real-analytic Eisenstein series associated to any β from the $E_k^*(\tau,s)$ above by a simple "averaging" argument as in section 3.8.)

The series $E_k^*(\tau,s)$ can be analytically extended beyond the half-plane $\mathrm{Re}[s] > 1-k/2$. We will focus here on weights $k \in \{1, 3/2, 2\}$, in which the Fourier series is enough to give an explicit analytic continuation to $s=0$. First we work out an expression for the Fourier series (in particular, our result below differs in appearance from [17] because we use a different computation of the Euler factors). Writing

$$E_k^*(\tau,s) = \mathfrak{e}_0 + \sum_{\gamma\in\Lambda'/\Lambda}\sum_{n\in\mathbb{Z}-Q(\gamma)} c(n,\gamma,s,y)q^n\mathfrak{e}_\gamma,$$

a computation analogous to section 1.2.3 of [12] using the exact formula for the coefficients

$\rho(M)_{0,\gamma}$ of the Weil representation cited there shows that

$$\begin{aligned} c(n,\gamma,s,y) &= \frac{y^s}{2}\sum_{c\neq 0}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\rho(M)_{0,\gamma}\int_{-\infty+iy}^{\infty+iy}(c\tau+d)^{-k}|c\tau+d|^{-2s}\mathbf{e}(-n\tau)\,\mathrm{d}x \\ &= y^s\sum_{c=1}^{\infty}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\rho(M)_{0,\gamma}c^{-k-2s}\mathbf{e}\Big(\frac{nd}{c}\Big)\int_{-\infty+iy}^{\infty+iy}\tau^{-k}|\tau|^{-2s}\mathbf{e}(-n\tau)\,\mathrm{d}x \\ &= \frac{\sqrt{i}^{b^--b^+}}{\sqrt{|\Lambda'/\Lambda|}}y^s\tilde{L}(n,\gamma,k+e/2+2s)I(k,y,n,s), \end{aligned} \tag{4.1}$$

where M is any element of $Mp_2(\mathbb{Z})$ whose bottom row is (c,d). Here, $\tilde{L}(n,\gamma,s)$ is the L-series

$$\begin{aligned} \tilde{L}(n,\gamma,s) &= \sum_{c=1}^{\infty}c^{-s+e/2}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\rho(M)_{0,\gamma}\mathbf{e}\Big(\frac{nd}{c}\Big) \\ &= \sum_{c=1}^{\infty}c^{-s}\sum_{\substack{v\in\Lambda/c\Lambda \\ d\in(\mathbb{Z}/c\mathbb{Z})^\times}}\mathbf{e}\Big(\frac{aQ(v)-\langle\gamma,v\rangle+dQ(\gamma)-nd}{c}\Big) \\ &= \sum_{c=1}^{\infty}c^{-s}\sum_{a|c}\Big[\mu(c/a)a(c/a)^e\cdot\#\Big\{v\in\Lambda/a\Lambda:\ Q(v-\gamma)+n\equiv 0\,(\mathrm{mod}\ a)\Big\}\Big] \\ &= \zeta(s-e)^{-1}L(n,\gamma,s-1), \end{aligned}$$

where $L(n,\gamma,s)$ is

$$L(n,\gamma,s)=\sum_{a=1}^{\infty}a^{-s}\mathbf{N}(a)=\prod_{p\,\mathrm{prime}}\Big(\sum_{\nu=0}^{\infty}p^{-\nu s}\mathbf{N}(p^\nu)\Big)=\prod_{p\,\mathrm{prime}}L_p(n,\gamma,s),$$

and $\mathbf{N}(p^\nu)$ is the number of zeros $v\in\Lambda/p^\nu\Lambda$ of the quadratic polynomial $Q(v-\gamma)+n$; and $I(k,y,n,s)$ is the integral

$$\begin{aligned} I(k,y,n,s) &= \int_{-\infty+iy}^{\infty+iy}\tau^{-k}|\tau|^{-2s}\mathbf{e}(-n\tau)\,\mathrm{d}x \\ &= y^{1-k-2s}e^{2\pi ny}\int_{-\infty}^{\infty}(t+i)^{-k}(t^2+1)^{-s}\mathbf{e}(-nyt)\,\mathrm{d}t,\quad \tau=y(t+i). \end{aligned}$$

Remark 54. Both the L-series term $\tilde{L}(n,\gamma,s)$ and the integral term $I(k,y,n,s)$ of (1) have meromorphic continuations to all $s\in\mathbb{C}$. First we remark that the integral $I(k,y,n,s)$ was considered by Gross and Zagier [37], section IV.3., where it was shown that for $n \neq 0$, $I(k, y, n, s)$ is a finite linear combination of K-Bessel functions (we will not need the exact expression) and its value at $s = 0$ is given by

$$I(k, y, n, 0) = \begin{cases} 0 : & n < 0; \\ (-2\pi i)^k n^{k-1} \frac{1}{\Gamma(k)} : & n > 0; \end{cases} \tag{4.2}$$

if $n \neq 0$; and when $n = 0$,

$$I(k, y, 0, s) = \pi(-i)^k 2^{2-k-2s} y^{1-k-2s} \frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)}. \tag{4.3}$$

In particular, the zero value of the latter expression is

$$I(k, y, 0, 0) = \begin{cases} 0 : & k \neq 1; \\ -i\pi : & k = 1. \end{cases}$$

The Euler factors $L_p(n, \gamma, s) = \sum_{\nu=0}^{\infty} p^{-\nu s} \mathbf{N}(p^\nu)$ are known to be rational functions in p^{-s} that can be calculated using the methods of [23] (as in sections 3.3 and 3.9). For generic primes (primes $p \neq 2$ that do not divide $|\Lambda'/\Lambda|$, or the numerator or denominator of n if $n \neq 0$) the result is that

$$L_p(n, \gamma, s) = \begin{cases} \frac{1}{1-p^{e-1-s}} \Big[1 - \Big(\frac{D}{p}\Big) p^{e/2-s} \Big] : & n \neq 0; \\ \\ \frac{1 - \Big(\frac{D'}{p}\Big) p^{e/2-1-s}}{(1-p^{e-1-s}) \Big[1 - \Big(\frac{D'}{p}\Big) p^{e/2-s} \Big]} : & n = 0; \end{cases}$$

if e is even and

$$L_p(n, \gamma, s) = \begin{cases} \frac{1}{1-p^{e-1-s}} \Big[1 + \Big(\frac{\mathcal{D}'}{p}\Big) p^{(e-1)/2-s} \Big] : & n \neq 0; \\ \\ \frac{1-p^{e-1-2s}}{(1-p^{e-1-s})(1-p^{e-2s})} : & n = 0; \end{cases}$$

if e is odd. Here, D' and $\mathcal{D}'$ are defined by

$$D' = (-1)^k |\Lambda'/\Lambda| \text{ and } \mathcal{D}' = 2nd_\gamma^2 (-1)^{k-1/2} |\Lambda'/\Lambda|.$$

In particular, if we define $D = D' \cdot \prod_{\mathrm{bad}\, p} p^2$ and $\mathcal{D} = \mathcal{D}' \cdot \prod_{\mathrm{bad}\, p} p^2$, where the bad primes are 2 and any prime dividing $|\Lambda'/\Lambda|$ or n, then we get the meromorphic continuations

$$\tilde{L}(n,\gamma,s) = \begin{cases} \frac{1}{L(s-e/2,\chi_D)} \prod_{\mathrm{bad}\, p}(1-p^{e-s})L_p(n,\gamma,s-1): & n \neq 0; \\ \\ \frac{L(s-1-e/2,\chi_D)}{L(s-e/2,\chi_D)} \prod_{\mathrm{bad}\, p}(1-p^{e-s})L_p(s-1): & n = 0; \end{cases}$$

if e is even and

$$\tilde{L}(n,\gamma,s) = \begin{cases} \frac{L(s-(e+1)/2,\chi_{\mathcal{D}})}{\zeta(2s-1-e)} \prod_{\mathrm{bad}\, p} \frac{1-p^{e-s}}{1-p^{e+1-2s}} L_p(n,\gamma,s-1): & n \neq 0; \\ \\ \frac{\zeta(2s-2-e)}{\zeta(2s-1-e)} \prod_{\mathrm{bad}\, p} \frac{(1-p^{e-s})(1-p^{e+2-2s})}{1-p^{e+1-2s}} L_p(s-1): & n = 0; \end{cases}$$

if e is odd.

Remark 55. We denote by E_k the series

$$E_k(\tau) = \mathfrak{e}_0 + \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n>0} c(n,\gamma,0,y) q^n \mathfrak{e}_\gamma.$$

The formula (2) gives $I(k,y,n,0) = (-2\pi i)^k n^{k-1} \frac{1}{\Gamma(k)}$ independently of y, and so $E_k(\tau)$ is holomorphic. When $k > 2$, this is just the zero-value $E_k(\tau) = E_k^*(\tau,0)$ and therefore E_k is a modular form. In small weights this tends to fail because the terms

$$\lim_{s \to 0} \tilde{L}(n,\gamma,k+e/2+2s) I(k,y,n,s)$$

may have a pole of $\tilde{L}$ canceling the zero of I for $n \leq 0$, resulting in nonzero (and often nonholomorphic) contributions to $E_k^*(\tau,0)$.

Remark 56. Suppose the dimension e is even; then we can apply theorem 4.8 of [18] to get a simpler coefficient formula. (The condition $k = e/2$ there is only necessary for their computation of local L-factors, which we do not use.) It follows that the coefficient $c(n,0)$ of $q^n \mathfrak{e}_0$ in E_k is

$$c(n,0) = \frac{(2\pi)^k (-1)^{(2k+b^+-b^-)/4}}{L(k,\chi_D)\sqrt{|\Lambda'/\Lambda|}\Gamma(k)} \cdot \sigma_{k-1}(n,\chi_D) \cdot \prod_{p | D'} \Big[(1-p^{e/2-k}) L_p(n,0,k+e/2-1)\Big],$$

where $\sigma_{k-1}(n, \chi_D)$ is the twisted divisor sum

$$\sigma_{k-1}(n, \chi_D) = \sum_{d|n} \chi_D(n/d) d^{k-1}$$

and $D' = 4|\Lambda'/\Lambda|$. For a fixed lattice Λ, the expression

$$\prod_{p|D'} \Big[(1 - p^{e/2-k}) L_p(n, 0, k + e/2 - 1) \Big]$$

above can always be worked out in closed form using the method of [23], although this can be somewhat tedious (in particular the case $p = 2$, which was worked out explicitly in section 3.8.) Theorem 4.8 of [18] also gives an interpretation of the coefficients when e is odd but this is more complicated.

4.3 Weight one

In weight 1, the L-series term is always holomorphic at $s = 0$. However, the zero-value

$$I(1, y, 0, 0) = -i\pi$$

being nonzero means that E_k still needs a correction term. Setting $s = 0$ in the real-analytic Eisenstein series gives

$$\begin{aligned} E_1^*(\tau, 0) = E_1(\tau) - \pi \frac{(-1)^{(2+b^- - b^+)/4}}{\sqrt{|\Lambda'/\Lambda|}} \frac{L(0, \chi_D)}{L(1, \chi_D)} \times \\ \times \sum_{\substack{\gamma \in \Lambda'/\Lambda \\ Q(\gamma) \in \mathbb{Z}}} \Big[\prod_{\text{bad}\, p} \lim_{s \to 0} (1 - p^{e/2-1-2s}) L_p(0, \gamma, e/2 + 2s) \Big] \mathfrak{e}_\gamma, \end{aligned}$$

where D is the discriminant $D = -4|\Lambda'/\Lambda|$ and the bad primes are the primes dividing D. In particular, E_1 may differ from the true modular form $E_1^*(\tau, 0)$ by a constant. (Of course, $E_1^*(\tau, 0)$ may be identically zero.)

For two-dimensional negative-definite lattices, the corrected Eisenstein series $E_1^*(\tau, 0)$ is often a multiple of the theta series. This leads to identities relating representation numbers of quadratic forms and divisor counts. Of course, such identities are well-known from the theory of modular forms of higher level. The vector-valued proofs tend to be shorter since

$M_k(\rho^*)$ is generally much smaller than the space of modular forms of higher-level in which the individual components lie, so there is less algebra (although computing the local factors takes some work). We give two examples here.

Example 57. Consider the quadratic form $Q(x,y) = -x^2 - xy - y^2$, with $|\Lambda'/\Lambda| = 3$. The L-function values are

$$L(0,\chi_{-12}) = \frac{2}{3}, \quad L(1,\chi_{-12}) = \frac{\pi\sqrt{3}}{6}$$

and the local L-series are

$$L_2(0,0,s) = \frac{1+2^{-s}}{1-2^{2-2s}}, \quad L_3(0,0,s) = \frac{1}{1-3^{1-s}}$$

with

$$\lim_{s\to 0}(1-2^{-2s})L_2(0,0,1+2s) = \frac{3}{4}, \quad \lim_{s\to 0}(1-3^{-2s})L_3(0,0,1+2s) = 1,$$

and therefore $E_1^*(\tau,0) = E_1(\tau) + \mathfrak{e}_0$. Since $M_1(\rho^*)$ is one-dimensional, comparing constant terms shows that

$$E_1(\tau) + \mathfrak{e}_0 = 2\vartheta.$$

Using remark 56, we find that the coefficient $c(n,0)$ of $q^n\mathfrak{e}_0$ in E_1 is

$$\begin{aligned} c(n,0) &= \frac{2\pi}{L(1,\chi_{-12})\cdot\sqrt{3}} \cdot \sigma_0(n,\chi_{-12}) \times \\ &\quad \times \underbrace{\begin{cases} 3/2: & v_2(n)\text{ even;} \\ 0: & v_2(n)\text{ odd;} \end{cases}}_{\text{local factor at 2}} \cdot \underbrace{\begin{cases} 2: & n \neq (3a+2)3^b \text{ for any } a,b \in \mathbb{N}_0; \\ 0: & n = (3a+2)3^b \text{ for some } a,b\in\mathbb{N}_0; \end{cases}}_{\text{local factor at 3}} \\ &= 12\Big[\sum_{d\mid n}\Big(\frac{-12}{d}\Big)\Big] \cdot \begin{cases} 1: & n \neq (3a+2)3^b; \\ 0: & n = (3a+2)3^b. \end{cases} \end{aligned}$$

This implies the identity

$$\begin{aligned} &\#\{(a,b)\in\mathbb{Z}^2 : a^2+ab+b^2 = n\} \\ &= 6\varepsilon \cdot \Big(\#\{\text{divisors } d = 6\ell+1 \text{ of } n\} - \#\{\text{divisors } d = 6\ell-1 \text{ of } n\}\Big), \end{aligned}$$

valid for $n \geq 1$, where $\varepsilon = 1$ unless n has the form $(3a+2)3^b$ for $a,b\in\mathbb{N}_0$, in which case $\varepsilon = 0$.

Example 58. Consider the quadratic form $Q(x,y) = -x^2 - y^2$, with $|\Lambda'/\Lambda| = 4$ and $\chi_{-16} = \chi_{-4}$. The L-function values are

$$L(0,\chi_{-4}) = \frac{1}{2}, \quad L(1,\chi_{-4}) = \frac{\pi}{4},$$

and the only bad prime is 2 with $L_2(0,0,s) = \frac{1}{1-2^{1-s}}$ and therefore

$$\lim_{s\to 0}(1 - 2^{e/2-1-2s})L_2(0,0,e/2+2s) = 1.$$

Therefore,

$$E_1^*(\tau,0) = E_1(\tau) + \mathfrak{e}_0.$$

Since $M_1(\rho^*)$ is one-dimensional, comparing constant terms gives $E_1(\tau) + \mathfrak{e}_0 = 2\vartheta(\tau)$.

By remark 56, the coefficient $c(n,0)$ of $q^n\mathfrak{e}_0$ in E_1 is

$$c(n,0) = \frac{2\pi}{L(1,\chi_{-4})\cdot 2}\cdot\sigma_0(n,\chi_{-4})\cdot\underbrace{\begin{cases} 2: & \left(\frac{-4}{n}\right) \neq -1; \\ 0: & \left(\frac{-4}{n}\right) = -1; \end{cases}}_{\text{local factor at 2}} = 8\sum_{d|n}\left(\frac{-4}{d}\right),$$

and therefore

$$\begin{aligned}\#\Big\{(a,b)\in\mathbb{Z}^2 :\, a^2+b^2 = n\Big\} &= 4\sum_{d|n}\left(\frac{-4}{d}\right) \\ &= 4\cdot\Big(\#\{\text{divisors}\, d = 4\ell+1 \,\text{of}\, n\} - \#\{\text{divisors}\, d = 4\ell+3\,\text{of}\, n\}\Big).\end{aligned}$$

Remark 59. Experimentally one often finds that the weight 1 Eisenstein series attached to a discriminant form equals a theta series even in cases where it is impossible to associate a weight 1 theta series to the discriminant form in a meaningful sense; such relations are almost certainly coincidence resulting from small cusp spaces in weight 1. For example, the indefinite lattice with Gram matrix

$$S = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$$

yields an Eisenstein series in which the component of $\mathfrak{e}_0$ is

$$E_1^*(\tau,0) = \frac{2}{3} + 4q + 4q^3 + 4q^4 + 8q^7 + 4q^9 + ...$$

i.e. $\frac{2}{3}$ times the theta series of the quadratic form $x^2 + xy + y^2$. However, the discriminant form of S has signature 2 mod 8 and is therefore not represented by a negative-definite lattice whose theta series has weight one.

On the other hand, replacing S by

$$-3S = \begin{pmatrix} -6 & 3 & 3 & 3 \\ 3 & -6 & 3 & 3 \\ 3 & 3 & -6 & 3 \\ 3 & 3 & 3 & -6 \end{pmatrix}$$

yields an Eisenstein series in which the component of $\mathfrak{e}_0$ is

$$E_1^*(\tau, 0) = \frac{34}{27} - \frac{4}{9}q + \frac{68}{9}q^3 - \frac{4}{9}q^4 - \frac{8}{9}q^7 + \frac{68}{9}q^9 \pm ...$$

with the surprising property that its coefficients have infinitely many sign changes; in particular, this example should make clear that $E_1^*(\tau, 0)$ is not simply a theta series for every lattice.

4.4 Weight $3/2$

In weight $3/2$, the L-series term is

$$\begin{aligned} &\tilde{L}(n, \gamma, 3/2 + e/2 + 2s) \\ =& \begin{cases} \frac{L(1+2s, \chi_{\mathcal{D}})}{\zeta(4s+2)} \prod_{\mathrm{bad}\, p} \frac{1 - p^{(e-3)/2-2s}}{1-p^{-2-4s}} L_p(n, \gamma, 1/2 + e/2 + 2s) : & n \neq 0; \\ \frac{\zeta(4s+1)}{\zeta(4s+2)} \prod_{\mathrm{bad}\, p} \frac{(1-p^{(e-3)/2-2s})(1-p^{-1-4s})}{1-p^{-2-4s}} L_p(n, \gamma, 1/2 + e/2 + 2s) : & n = 0; \end{cases} \end{aligned}$$

and it is holomorphic in $s = 0$ unless $n = 0$ or

$$\mathcal{D} = -2nd_\gamma^2 |\Lambda'/\Lambda| \prod_{\mathrm{bad}\, p} p^2$$

is a square. In these cases, $\tilde{L}(n, \gamma, 3/2 + e/2 + 2s)$ has a simple pole with residue

$$\frac{3}{\pi^2} \prod_{\mathrm{bad}\, p} \lim_{s \to 0} \frac{(1 - p^{e/2-3/2-2s})(1 - p^{-1})}{1 - p^{-2}} L_p(n, \gamma, 1/2 + e/2 + 2s)$$

if $n \neq 0$, and

$$\frac{3}{2\pi^2}\prod_{\mathrm{bad}\,p}\lim_{s\to 0}\frac{(1-p^{e/2-3/2-2s})(1-p^{-1})}{1-p^{-2}}L_p(n,\gamma,1/2+e/2+2s)$$

if $n = 0$.

This pole cancels with the zero of $I(k,y,n,s)$ at $s=0$, whose derivative there is

$$\frac{d}{ds}\Big|_{s=0} I(k,y,n,s) = -16\pi^2(1+i)y^{-1/2}\beta(4\pi|n|y), \quad \text{where } \beta(x) = \frac{1}{16\pi}\int_1^\infty u^{-3/2}e^{-xu}\,\mathrm{d}u,$$

as calculated in [38], section 2.2. This expression is also valid for $n=0$, where it reduces to

$$\frac{d}{ds}\Big|_{s=0} I(k,y,0,s) = 2\pi(-i)^{3/2}\frac{d}{ds}\Big|_{s=0} 2^{-1/2-2s}y^{-1/2-2s}\frac{\Gamma(2s+1/2)}{\Gamma(s)\Gamma(s+3/2)} = -\frac{2\pi}{\sqrt{y}}(1+i).$$

Therefore, $E^*_{3/2}(\tau,0)$ is a harmonic weak Maass form that is not generally holomorphic:

$$\begin{aligned} &E^*_{3/2}(\tau,0) \\ &= E_{3/2}(\tau) + \frac{3(-1)^{(3+b^+-b^-)/4}\sqrt{2}}{\pi\sqrt{y|\Lambda'/\Lambda|}}\Big(\sum_{\substack{\gamma\in\Lambda'/\Lambda \\ Q(\gamma)\in\mathbb{Z}}}\prod_{p|\#(\Lambda'/\Lambda)}\frac{1-p^{(e-3)/2}}{1+p^{-1}}L_p(0,\gamma,1/2+e/2)\mathfrak{e}_\gamma\Big)+ \\ &+\frac{48(-1)^{(3+b^+-b^-)/4}\sqrt{2}}{\sqrt{y|\Lambda'/\Lambda|}}\sum_{\substack{\gamma\in\Lambda'/\Lambda \\ n\in\mathbb{Z}-Q(\gamma) \\ -2n|\Lambda'/\Lambda|=\square}}\Big[\beta(4\pi|n|y)\prod_{\mathrm{bad}\,p}\frac{1-p^{(e-3)/2}}{1+p^{-1}}L_p(n,\gamma,1/2+e/2)\Big]q^n\mathfrak{e}_\gamma, \end{aligned}$$

where $-2n|\Lambda'/\Lambda| = \square$ means that $-2n|\Lambda'/\Lambda|$ should be a rational square. (In particular, the real-analytic correction involves only exponents $n \leq 0$.)

Example 60. Zagier's Eisenstein series [38] occurs as the Eisenstein series for the quadratic form $Q(x) = x^2$. The underlying harmonic weak Maass form is

$$\begin{aligned} &E_{3/2}(\tau) - \frac{3}{\pi\sqrt{y}}\mathfrak{e}_0 - \frac{48}{\sqrt{y}}\sum_{\gamma\in\Lambda'/\Lambda}\sum_{\substack{n\in\mathbb{Z}-Q(\gamma) \\ -n=\square}}\beta(4\pi|n|y)\underbrace{\prod_{\mathrm{bad}\,p}\frac{1-p^{-1}}{1+p^{-1}}L_p(n,\gamma,1)}_{=1}q^n\mathfrak{e}_\gamma \\ &= E_{3/2}(\tau) - \frac{24}{\sqrt{y}}\sum_{n=-\infty}^{\infty}\beta(4\pi(n/2)^2y)q^{-(n/2)^2}\mathfrak{e}_{n/2}. \end{aligned}$$

(Here the representation numbers in $L_p(n,\gamma,1)$ can be evaluated directly by Hensel's lemma.) The coefficient of $q^{n/4}$ in

$$E_{3/2}(\tau) = \Big(1 - 6q - 12q^2 - 16q^3 - ...\Big)\mathfrak{e}_0 + \Big(- 4q^{3/4} - 12q^{7/4} - 12q^{11/4} - ...\Big)\mathfrak{e}_{1/2}$$

is -12 times the Hurwitz class number $H(n)$. We obtain Zagier's Eisenstein series in its usual form by summing the components, replacing τ by 4τ and y by $4y$, and dividing by -12.

Remark 61. We can use essentially the same argument as Hirzebruch and Zagier [38] to derive the transformation law of the general $E_{3/2}$. Write $E^*_{3/2}(\tau,0)$ in the form

$$E^*_{3/2}(\tau,0) = E_{3/2} + \frac{1}{\sqrt{y}} \sum_{\gamma\in\Lambda'/\Lambda} \sum_{\substack{n\in\mathbb{Z}-Q(\gamma)\\ n\le 0}} a(n,\gamma)\beta(-4\pi ny)q^n\mathfrak{e}_\gamma$$

with coefficients $a(n,\gamma)$. Applying the ξ-operator $\xi = y^{3/2}\overline{\frac{\partial}{\partial\overline{\tau}}}$ of [16] to $E^*_{3/2}(\tau,0)$ and using

$$\frac{d}{dy}\Big[\frac{1}{\sqrt{y}}\beta(y)\Big] = \frac{1}{16\pi}\frac{d}{dy}\Big[\int_y^\infty v^{-3/2}e^{-v}\,\mathrm{d}v\Big] = -\frac{1}{16\pi}y^{-3/2}e^{-y}$$

shows that the "shadow"

$$\vartheta(\tau) = \sum_{\gamma,n} a(n,\gamma)q^{-n}\mathfrak{e}_\gamma$$

is a modular form of weight 1/2 for the representation ρ (not its dual!), and

$$\begin{aligned} E^*_{3/2}(\tau,0) - E_{3/2}(\tau) &= y^{-1/2}\sum_{\gamma\in\Lambda'/\Lambda}\sum_{n\in\mathbb{Z}-Q(\gamma)} a(n,\gamma)\beta(-4\pi ny)q^n\mathfrak{e}_\gamma \\ &= \frac{1}{16\pi}y^{-1/2}\int_1^\infty \sum_{\gamma,n} u^{-3/2}e^{-4\pi nuy}q^n\mathfrak{e}_\gamma\,\mathrm{d}u \\ &= \frac{1}{16\pi}y^{-1/2}\int_1^\infty u^{-3/2}\vartheta(2iuy-\tau)\,\mathrm{d}u \\ &= \frac{\sqrt{2i}}{16\pi}\int_{-x+iy}^{i\infty}(v+\tau)^{-3/2}\vartheta(v)\,\mathrm{d}v, \quad v = 2iuy-\tau. \end{aligned}$$

For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$, defining $\tilde{M} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ as in remark 6 and substituting $v = \tilde{M} \cdot t$ gives

$$\begin{aligned} E^*_{3/2}(M\cdot\tau, 0) - E_{3/2}(M\cdot\tau) &= \frac{\sqrt{2i}}{16\pi} \int_{-\overline{M\cdot\tau}}^{i\infty} \Big(\frac{a\tau+b}{c\tau+d} + v\Big)^{-3/2} \vartheta(v)\,\mathrm{d}v \\ &= \frac{\sqrt{2i}}{16\pi} \int_{-\overline{\tau}}^{d/c} \Big(\frac{a\tau+b}{c\tau+d} + \frac{at-b}{-ct+d}\Big)^{-3/2} \vartheta(\tilde{M}\cdot t)\,\frac{\mathrm{d}t}{(ct-d)^2} \\ &= \frac{\sqrt{2i}}{16\pi}(c\tau+d)^{3/2} \int_{-\overline{\tau}}^{d/c} (\tau+t)^{-3/2} \rho(\tilde{M})\vartheta(t)\,\mathrm{d}t \\ &= (c\tau+d)^{3/2}\rho^*(M)\Big[\frac{\sqrt{2i}}{16\pi} \int_{-\overline{\tau}}^{d/c} (\tau+t)^{-3/2}\vartheta(t)\,\mathrm{d}t\Big]. \end{aligned}$$

Since $E^*_{3/2}(M\cdot\tau,0) = (c\tau+d)^{3/2}\rho^*(M)E^*_{3/2}(\tau,0)$, we conclude that

$$E_{3/2}(M\cdot\tau) = (c\tau+d)^{3/2}\rho^*(M)\Big[E_{3/2}(\tau) + \frac{\sqrt{2i}}{16\pi}\int_{d/c}^{i\infty} (\tau+t)^{-3/2}\vartheta(t)\,\mathrm{d}t\Big]. \tag{4.4}$$

Remark 62. The transformation law above can be used to give an easier sufficient condition for when $E_{3/2}$ is actually a modular form. For example, one can show that $M_{1/2}(\rho) = 0$ for the quadratic form $Q(x,y,z) = -x^2 - y^2 - z^2$, which implies that the series ϑ defined above must be identically 0 and therefore

$$E_{3/2}(M\cdot\tau) = (c\tau+d)^{3/2}\rho^*(M)E_{3/2}(\tau),$$

so $E_{3/2}$ is a true modular form. (In this case, the local L-series $L_2(n,\gamma,2+2s)$ at $p=2$ is holomorphic at $s=0$, and therefore the factor $(1-2^{-2s})$ annihilates the L-series term $\tilde{L}(n,\gamma,3/2+e/2)$ in the shadow.) This must be the theta series because $M_{3/2}(\rho^*)$ is one-dimensional.

It may be worth pointing out that the coefficient formulas ([18], theorem 4.8) for this theta series and for the Zagier Eisenstein series are nearly identical, since the squarefree parts of their discriminant and the "bad primes" are the same: the only real difference between them is the local factor at 2. For odd integers n, the local factor at 2 is easily computed and in both cases depends only on the remainder of n mod 8, so the coefficients $r_3(n)$ of the theta series and $H(4n)$ of the Zagier Eisenstein series within these congruence classes are proportional. Specifically,

$$r_3(n) = 12H(4n),\ n\equiv 1,5\,(8);\quad r_3(n) = 6H(4n),\ n\equiv 3\,(8);\quad r_3(n) = 0,\ n\equiv 7\,(8).$$

These identities are well-known and were already proved by Gauss.

Example 63. Even when $M_{1/2}(\rho) \neq 0$, we can identify ϑ in $M_{1/2}(\rho)$ by computing finitely many coefficients. Consider the quadratic form $Q(x,y,z) = x^2 + y^2 - z^2$. The space $M_{1/2}(\rho)$ is always spanned by unary theta series embedded into $\mathbb{C}[\Lambda'/\Lambda]$ (as proven by Skoruppa [61]) and in this case one can find the basis

$$\begin{aligned} \vartheta_1(\tau) &= \Big(1 + 2q + 2q^4 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/2,0,1/2)}) + \\ &\quad + \Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(0,1/2,0)} + \mathfrak{e}_{(1/2,1/2,1/2)}), \\ \vartheta_2(\tau) &= \Big(1 + 2q + 2q^4 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(0,1/2,1/2)}) + \\ &\quad + \Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(1/2,0,0)} + \mathfrak{e}_{(1/2,1/2,1/2)}). \end{aligned}$$

The local L-series at the bad prime $p = 2$ for the constant term $n = 0$ are

$$(1 - 2^{-2s})L_p(0,0,2+2s) = \frac{1}{1 - 2^{-1-4s}} \quad \text{and} \quad (1 - 2^{-2s})L_p(0,\gamma,2+2s) = 1$$

for $\gamma \in \{(1/2,0,1/2),(0,1/2,1/2)\}$, which implies that

$$E^*_{3/2}(\tau,0) = E_{3/2}(\tau) - \frac{3}{2\pi\sqrt{y}}\Big(\frac{4}{3}\mathfrak{e}_{(0,0,0)} + \frac{2}{3}\mathfrak{e}_{(1/2,0,1/2)} + \frac{2}{3}\mathfrak{e}_{(0,1/2,1/2)}\Big) + ...$$

(where the ... involves only negative powers of q) and therefore that the shadow is

$$\vartheta(\tau) = -8\Big(\vartheta_1(\tau) + \vartheta_2(\tau)\Big).$$

In particular, the $\mathfrak{e}_0$-component $E_{3/2}(\tau)_0$ of $E_{3/2}(\tau)$ is a mock modular form of level 4 that transforms under $\Gamma(4)$ by

$$E_{3/2}(M \cdot \tau)_0 = (c\tau + d)^{3/2}\Big[E_{3/2}(\tau)_0 - \frac{\sqrt{2i}}{\pi}\int_{d/c}^{i\infty} (\tau + t)^{-3/2}\Theta(t)\,\mathrm{d}t\Big],$$

where $\Theta(t) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2 t)$ is the classical theta series. It was shown by Bringmann and Lovejoy [8] that the series

$$\overline{\mathcal{M}}(\tau + 1/2) = 1 - \sum_{n=1}^{\infty} |\overline{\alpha}(n)|q^n = 1 - 2q - 4q^2 - 8q^3 - 10q^4 - ...$$

of example 51, where $|\overline{\alpha}(n)|$ counts overpartition rank differences of n, has the same transformation behavior under the group $\Gamma_0(16)$, which implies that the difference between $\overline{\mathcal{M}}(\tau + 1/2)$ and the $\mathfrak{e}_0$-component of $E_{3/2}$ is a true modular form of level 16. We can verify that these are the same by comparing all Fourier coefficients up to the Sturm bound.

4.5 Weight two

In weight $k = 2$, the L-series term is

$$\tilde{L}(n,\gamma,2+e/2+2s) = \begin{cases} \frac{1}{L(1+2s,\chi_D)} \prod_{\mathrm{bad}\,p}(1-p^{e/2-2-2s})L_p(n,\gamma,1+e/2+2s) : & n \neq 0; \\ \\ \frac{L(1+2s,\chi_D)}{L(2+2s,\chi_D)} \prod_{\mathrm{bad}\,p}(1-p^{e/2-2-2s})L_p(n,\gamma,1+e/2+2s) : & n = 0. \end{cases}$$

Since $L(1,\chi)$ is never zero for any Dirichlet character, the only way a pole can occur at $s = 0$ is if $n = 0$ and $D = |\Lambda'/\Lambda|$ is square. (In particular, when $|\Lambda'/\Lambda|$ is not square, E_2 is a modular form.)

Assume that $|\Lambda'/\Lambda|$ is square. Then

$$L(1+2s,\chi_D) = \zeta(1+2s)\prod_{\mathrm{bad}\,p}(1-p^{-1-2s}),$$

and therefore $\tilde{L}(0,\gamma,2+e/2+2s)$, has a simple pole at $s=0$ with residue

$$\begin{aligned} &\mathrm{Res}\Big(\tilde{L}(0,\gamma,2+e/2+2s), s=0\Big) \\ &= \frac{1}{2L(2,\chi_D)}\prod_{\mathrm{bad}\,p}\Big[(1-p^{-1})\lim_{s\to 0}(1-p^{e/2-2-2s})L_p(0,\gamma,1+e/2+2s)\Big] \\ &= \frac{3}{\pi^2}\lim_{s\to 0}\prod_{\mathrm{bad}\,p}\frac{1-p^{e/2-2-2s}}{1+p^{-1}}L_p(0,\gamma,1+e/2+2s) \end{aligned}$$

for any $\gamma \in \Lambda'/\Lambda$ with $Q(\gamma) \in \mathbb{Z}$. This pole is canceled by the zero of $I(2,y,0,s)$ at $s=0$ which has derivative

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} I(2,y,0,s) &= -2\pi(2y)^{-1}\frac{d}{ds}\Big|_{s=0}(2y)^{-2s}\frac{\Gamma(2s+1)}{\Gamma(s)\Gamma(s+2)} \\ &= -\frac{\pi}{y}, \end{aligned}$$

so

$$E_2^*(\tau,0) = E_2(\tau) - \frac{3}{\pi y\sqrt{|\Lambda'/\Lambda|}}\lim_{s\to 0}\sum_{\substack{\gamma\in\Lambda'/\Lambda \\ Q(\gamma)\in\mathbb{Z}}}\prod_{\mathrm{bad}\,p}\frac{1-p^{e/2-2-2s}}{1+p^{-1}}L_p(0,\gamma,1+e/2+2s)\mathfrak{e}_\gamma. \quad (4.5)$$

Example 64. Let Λ be a unimodular lattice. The only bad prime is $p = 2$. Using the hyperbolic plane $Q(x, y) = xy$ to define Λ, the local L-function is

$$L_2(0, 0, s) = \frac{1 - 2^{-s}}{(1 - 2^{1-s})^2}$$

with $L_2(0, 0, 2) = 3$, so we obtain the well-known result

$$E_2^*(\tau, 0) = E_2(\tau) - \frac{3}{\pi y} \cdot \frac{1 - 1/2}{1 + 1/2} L_2(0, 0, 2) = E_2(\tau) - \frac{3}{\pi y}.$$

Remark 65. We can summarize the above by saying that

$$E_2^*(\tau, 0) = E_2(\tau) - \frac{1}{y} \sum_{\substack{\gamma \in \Lambda'/\Lambda \\ Q(\gamma) \in \mathbb{Z}}} A(\gamma) \mathfrak{e}_\gamma$$

is a Maass form for some constants $A(\gamma)$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$, since

$$E_2^*(M \cdot \tau, 0) = (c\tau + d)^2 \rho^*(M) E_2^*(\tau, 0),$$

we find the transformation law

$$\begin{aligned} E_2(M \cdot \tau) &= E_2^*(M \cdot \tau, 0) + \frac{|c\tau + d|^2}{y} \sum_{\substack{\gamma \in \Lambda'/\Lambda \\ Q(\gamma) \in \mathbb{Z}}} A(\gamma) \mathfrak{e}_\gamma \\ &= (c\tau + d)^2 \Big[\rho^*(M) E_2(\tau) - 2ic(c\tau + d) \sum_{Q(\gamma) \in \mathbb{Z}} A(\gamma) \rho^*(M) \mathfrak{e}_\gamma \Big]. \end{aligned}$$

Example 66. The weight-2 Eisenstein series for the quadratic form $Q(x, y) = x^2 + 3xy + y^2$ is a true modular form because the discriminant 5 of Q is not a square. In particular, the $\mathfrak{e}_0$-component

$$1 - 30q - 20q^2 - 40q^3 - 90q^4 - 130q^5 - 60q^6 - 120q^7 - 100q^8 - 210q^9 - ...$$

is a modular form of weight 2 for the congruence subgroup $\Gamma_1(5)$. Using remark 11, we see that the coefficient $c(n)$ of q^n for n coprime to 10 is

$$c(n) = \begin{cases} -30 \sum_{d|n} \left(\frac{5}{n/d}\right) d : & n \equiv \pm 1 \bmod 10; \\ -20 \sum_{d|n} \left(\frac{5}{n/d}\right) d : & n \equiv \pm 3 \bmod 10; \end{cases}$$

with a more complicated expression for other n involving the local factors at 2 and 5.

Example 67. The weight-2 Eisenstein series for the quadratic form $Q(x,y) = 2xy$ is

$$\begin{aligned} E_2(\tau) &= \Big(1 - 8q - 40q^2 - 32q^3 - 104q^4 - ...\Big)\mathfrak{e}_{(0,0)} \\ &\quad + \Big(- 16q - 32q^2 - 64q^3 - 64q^4 - 96q^5 - ...\Big)(\mathfrak{e}_{(0,1/2)} + \mathfrak{e}_{(1/2,0)}) \\ &\quad + \Big(- 8q^{1/2} - 32q^{3/2} - 48q^{5/2} - 64q^{7/2} - 104q^{9/2} - ...\Big)\mathfrak{e}_{(1/2,1/2)} \\ &= \Big(1 - 8\sum_{n=1}^{\infty}\Big[\sum_{d|2n}(-1)^d d\Big]q^n\Big)\mathfrak{e}_{(0,0)} \\ &\quad + \Big(- 8\sum_{n=1}^{\infty}\Big[\sum_{d|n}(1-(-1)^{n/d})d\Big]q^n\Big)(\mathfrak{e}_{(0,1/2)} + \mathfrak{e}_{(1/2,0)}) \\ &\quad + \Big(- 8\sum_{n=0}^{\infty}\sigma_1(2n+1)q^{n+1/2}\Big)\mathfrak{e}_{(1/2,1/2)}. \end{aligned}$$

It is not a modular form. On the other hand, the real-analytic correction (7) only involves the components $\mathfrak{e}_\gamma$ for which $Q(\gamma) \in \mathbb{Z}$, i.e. $\mathfrak{e}_{(0,0)}, \mathfrak{e}_{(0,1/2)}, \mathfrak{e}_{(1/2,0)}$, so the components

$$1 - 8\sum_{n=1}^{\infty}\Big[\sum_{d|2n}(-1)^d d\Big]q^n, \;\; \sum_{n=1}^{\infty}\Big[\sum_{d|n}(1-(-1)^{n/d})d\Big]q^n$$

are only quasimodular forms of level 4, while $\sum_{n=0}^{\infty}\sigma_1(2n+1)q^{2n+1}$ is a true modular form.

Example 68. Although the discriminant group of the quadratic form $Q(x_1,x_2,x_3,x_4) = -x_1^2 - x_2^2 - x_3^2 - x_4^2$ has square order 16, the correction term still vanishes in this case. This is because the local L-functions for $p = 2$,

$$L_2(0,\gamma,3+s) = \begin{cases} \frac{2+2^{-s}}{2-2^{-s}} : & \gamma = 0; \\ 1 : & \gamma = (1/2,1/2,1/2,1/2); \end{cases}$$

are both holomorphic at $s = 0$ and therefore annihilated by the term $(1 - p^{4/2-2-2s})$ at $s = 0$. (Another way to see this is that $\sum_{\substack{\gamma\in\Lambda'/\Lambda \\ Q(\gamma)\in\mathbb{Z}}} A(\gamma)\mathfrak{e}_\gamma$ is invariant under ρ due to the transformation law of E_2, but there are no nonzero invariants of ρ in this case.) In fact, the Eisenstein series E_2 for this lattice is exactly the theta series as one can see by calculating the first few coefficients. Comparing coefficients of the $\mathfrak{e}_0$-component leads immediately to

Jacobi's formula:

$$\begin{aligned}
\#\{(a,b,c,d)\in\mathbb{Z}^4:\ a^2+b^2+c^2+d^2=n\} &= \frac{(2\pi)^2}{L(2,\chi_{64})\cdot 4}\cdot\sigma_1(n,\chi_{64})\cdot L_2(n,0,3)\\
&= 8\cdot\Big[\sum_{d|n}\Big(\frac{4}{n/d}\Big)d\Big]\cdot\begin{cases} 1: & n\,\text{odd};\\ 3\cdot 2^{-v_2(n)}: & n\,\text{even};\end{cases}\\
&= \begin{cases} 8\sum_{d|n} d: & n\,\text{odd};\\ \\ 24\sum_{\substack{d|n\\ d\,\text{odd}}} d: & n\,\text{even};\end{cases}
\end{aligned}$$

for all $n\in\mathbb{N}$.

4.6 Remarks on weight $1/2$

The Fourier expansion defining $E_k(\tau)$ is no longer valid in weight $k=1/2$; in fact, the L-series factor in this case is

$$\tilde{L}\Big(n,\gamma,\frac{e+1}{2}+2s\Big)=\begin{cases}\frac{L(2s,\mathcal{D})}{\zeta(4s)}\prod_{\text{bad}\,p}\frac{1-p^{(e-1)/2-2s}}{1-p^{-4s}}L_p\Big(n,\gamma,\frac{e-1}{2}+2s\Big): & n\neq 0;\\ \\ \frac{\zeta(4s-1)}{\zeta(4s)}\prod_{\text{bad}\,p}\frac{(1-p^{(e-1)/2-2s})(1-p^{1-4s})}{1-p^{-4s}}L_p\Big(n,\gamma,\frac{e-1}{2}+2s\Big): & n=0;\end{cases}$$

which generally has a singularity at $s=0$, so our approach fails in weight $1/2$.

Despite this, the weight $1/2$ Eisenstein series $E^*_{1/2}(\tau,s)$ should extend analytically to $s=0$. One way to study $E^*_{1/2}(\tau,s)$ is by applying the Bruinier-Funke operator $\xi_{3/2}$ to the weight $3/2$ series $E^*_{3/2}(\tau,s)$ for the dual representation (i.e. the same lattice with negated quadratic form); from $\xi_{3/2}y^s=-\frac{si}{2}y^{s+1/2}$ one obtains $\xi_{3/2}E^*_{3/2}(\tau,s)=-\frac{si}{2}E^*_{1/2}(\tau,s+1/2)$ for all large enough s. Carrying over the arguments from the scalar-valued case (e.g. [28], section 4.10) should imply that $E^*_{1/2}(\tau,s)$ will satisfy some functional equation relating $E^*_{1/2}(\tau,s+1/2)$ to $E^*_{1/2}(\tau,-s)$ (or more likely some combination of $E^*_{1/2,\beta}(\tau,-s)$ as β runs through elements of Λ'/Λ with $Q(\beta)\in\mathbb{Z}$ in general) although in the half-integer case this seems less straightforward. Assuming this, for large enough $\mathrm{Re}[s]$ it follows that $E^*_{1/2}(\tau,-s)$ should be a linear combination of $\xi_{3/2}E^*_{3/2,\beta}(\tau,s)$ with coefficients depending on s but independent of τ; we might even expect this to hold for arbitrary s and therefore conjecture:

Conjecture 69. *The zero-value $E^*_{1/2}(\tau,0)$ for a discriminant form $(\Lambda'/\Lambda,Q)$ is a holomorphic modular form of weight $1/2$; moreover it is a linear combination of the shadows of mock Eisenstein series $E_{3/2,\beta}(\tau)$ for $(\Lambda'/\Lambda,-Q)$.*

Unfortunately, if this is true then from our point of view there is little motivation to consider $E^*_{1/2}(\tau,0)$ further: modular forms of weight $1/2$ are spanned by what are essentially unary theta series and any resulting identities among coefficients will be uninteresting. There may be interest in higher terms of the Taylor expansion of $E^*_{1/2}(\tau,s)$ in the variable s which might be used to generate mock modular forms of weight $1/2$ and higher depth, but this is outside the scope of this chapter.

There is one class of examples where this conjecture can be verified directly. In dimension $e=1$, where the quadratic form is $Q(x)=-mx^2$ for some $m\in\mathbb{N}$, we can make sense of the coefficient formula because the terms $1-p^{1-4s}$ are canceled by the numerators at $s=0$, and the Fourier series then provides the analytic continuation of $E^*_{1/2}(\tau,s)$ to $s=0$. The L-series factor in this case is

$$\tilde{L}(n,\gamma,1+2s)=\begin{cases}\frac{L(2s,\chi_{\mathcal{D}})}{\zeta(4s)}\prod_{\mathrm{bad}\,p}\frac{L_p(n,\gamma,2s)}{1+p^{-2s}}: & n\neq 0;\\ \\ \frac{\zeta(4s-1)}{\zeta(4s)}\prod_{\mathrm{bad}\,p}\frac{(1-p^{1-4s})L_p(n,\gamma,2s)}{1+p^{-2s}}: & n=0.\end{cases}$$

Here, $\mathcal{D}$ is the discriminant

$$\mathcal{D}=2d_\gamma^2 n|\Lambda'/\Lambda|\prod_{\mathrm{bad}\,p}p^2=4mnd_\gamma^2\prod_{\mathrm{bad}\,p}p^2.$$

Suppose for simplicity that m is squarefree (and in particular, $\beta=0$ is the only element of Λ'/Λ with $Q(\beta)\in\mathbb{Z}$). The local L-factors can be calculated by elementary means (for example, with Hensel's lemma), and the result in this case is that $E_{1/2}(\tau)=1\cdot\mathfrak{e}_0+\sum_{\gamma\in\Lambda'/\Lambda}\sum_{n\in\mathbb{Z}-Q(\gamma)}c(n,\gamma)q^n\mathfrak{e}_\gamma$ with

$$c(n,\gamma)=2\cdot(1/2)^{\varepsilon},\;\;\varepsilon=\#\{\text{primes }p\neq 2\text{ dividing }d_\gamma\}+\begin{cases}1: & 4|d_\gamma\\ 0: & \text{otherwise.}\end{cases}$$

Here, d_γ is the denominator of γ; that is, the smallest number for which $d_\gamma \gamma \in \Lambda$.

The shadow of the mock Eisenstein series $E_{3/2}(\tau)$ attached to mx^2 can be computed directly as well, although this is more difficult. On the other hand, one can use the following trick: via the theta decomposition, the nonholomorphic weight 3/2 Eisenstein series $E^*_{3/2}(\tau,0)$ corresponds to a nonholomorphic, scalar Jacobi Eisenstein series $E^*_{2,m}(\tau,z,0)$ of index m. The argument of chapter 4 of [31] still applies to this situation and in particular $E^*_{2,m}(\tau,z,0) = \frac{1}{\sigma_1(m)} E^*_{2,1}(\tau,z,0)|V_m$ for the Hecke-type operator

$$\Phi|V_m(\tau,z) = m\sum_M (c\tau+d)^{-2}\mathbf{e}\Big(-\frac{cmz^2}{c\tau+d}\Big)\Phi\Big(\frac{a\tau+b}{c\tau+d},\frac{mz}{c\tau+d}\Big),$$

the sum taken over cosets of determinant-m integral matrices M by $SL_2(\mathbb{Z})$. (Here we must assume that m is squarefree). However, $E^*_{2,1}(\tau,z,0)$ arises through the theta decomposition from the Zagier Eisenstein series and so its coefficients are well-known. In this way one can compute that $E^*_{3/2}(\tau,0)$ is

$$-\frac{12}{\sigma_1(m)}\sum_{\gamma\in\Lambda'/\Lambda}\sum_{n\in\mathbb{Z}-Q(\gamma)}\sum_{a|m} aH(4mn/a^2)q^n\mathfrak{e}_\gamma + \frac{1}{\sqrt{y}}\sum_{\gamma\in\Lambda'/\Lambda}\sum_{n\in\mathbb{Z}+Q(\gamma)} a(n,\gamma)q^{-n}\mathfrak{e}_\gamma,$$

where $H(n)$ is the Hurwitz class number (and $H(n)=0$ if n is noninteger) and the coefficients of the shadow are

$$a(n,\gamma) = \begin{cases} -24\sqrt{m}\frac{\sigma_0(m)}{\sigma_1(m)} : & n=0; \\ -48\sqrt{m}\frac{\sigma_0(\gcd(m,n))}{\sigma_1(m)} : & mn=\square,\ mn\neq 0; \\ 0 : & \text{otherwise}, \end{cases}$$

and where we use the convention $\gcd(m,n) = \prod_{v_p(m),v_p(n)\geq 0} p^{\min(v_p(m),v_p(n))}$ (so for example $\gcd(30,3/4)=3$). Unraveling this, we see that $E_{1/2}(\tau)$ differs from the shadow of $E_{3/2}(\tau)$ by the factor $-24\sqrt{m}\frac{\sigma_0(m)}{\sigma_1(m)}$.

Chapter 5

Poincaré square series of small weight

This chapter is taken from the paper [69].

5.1 Introduction

The purpose of this chapter is to extend the previous construction of Poincaré square series to weights $k = 3/2$ and $k = 2$. Convergence issues make these cases more difficult. One immediate problem is that the Eisenstein series may fail to define a modular form; in fact, it is not hard to find lattices where $M_k(\rho^*) = 0$. For example, the space of scalar-valued modular forms of weight 2 is zero. These weights remain relevant to the problem that motivated [68] of computing spaces of obstructions for the existence of Borcherds products. Modular forms of weight $k = 3/2$ resp. $k = 2$ are obstructions to the existence of Borcherds products on Grassmannians $G(2,1)$ (which includes scalar modular forms) resp. $G(2,2)$ (which includes Hilbert modular forms) as explained in [5]. The construction can be summarized as follows:

Proposition 70. *(i) In weight $k = 3/2$, there are modular forms $Q_{k,m,\beta} \in M_k(\rho^*)$ with rational coefficients and the property that*

$$(f, Q_{k,m,\beta}) = 2\frac{\Gamma(k-1)}{(4m\pi)^{k-1}} \sum_{\lambda=1}^{\infty} \frac{c(m\lambda^2, \lambda\beta)}{\lambda^{2k+s-2}}\Big|_{s=0}$$

for all cusp forms f.

(ii) In weight $k = 2$, there are quasimodular forms $Q_{k,m,\beta}$ with rational coefficients such that

$Q_{k,m,\beta} - E_k$ is a cusp form satisfying

$$(f, Q_{k,m,\beta} - E_k) = 2\frac{\Gamma(k-1)}{(4m\pi)^{k-1}} \sum_{\lambda=1}^{\infty} \frac{c(m\lambda^2, \lambda\beta)}{\lambda^{2k+s-2}}\Big|_{s=0}$$

for all cusp forms f.

Here the notation $|_{s=0}$ may be understood as taking the value of an analytic continuation at $s = 0$, regardless of whether the series above actually converge at $s = 0$.

The failure of the Jacobi Eisenstein series of weight $k \le 5/2$ to define a Jacobi form is closely related to the failure of the usual Eisenstein series of weight $k - 1/2$ to define a modular form. In particular, $k = 2$ is the most difficult weight to treat because Eisenstein series of weight $3/2$ are often mock theta functions that require a real-analytic correction term to transform correctly under $\tilde{\Gamma}$.

Even in the cases where there are no cusp forms, the computation of $Q_{k,m,\beta}$ may be interesting; for example, in the simplest case where Λ is unimodular and $m = 1$, the equation $Q_{2,1,0} = E_2$ is equivalent to the Kronecker-Hurwitz class number relation

$$\sum_{r=-\infty}^{\infty} H(4n - r^2) = 2\sigma_1(n) - \sum_{d|n} \min(d, n/d) = \sum_{d|n} \max(d, n/d).$$

5.2 The real-analytic Jacobi Eisenstein series

Fix an even lattice Λ, an element $\beta \in \Lambda'/\Lambda$ and a positive number $m \in \mathbb{Z} - Q(\beta)$.

Definition 71. The **real-analytic Jacobi Eisenstein series** of weight k and index m twisted at β is

$$\begin{aligned} &E^*_{k,m,\beta}(\tau, z, s) \\ &= \frac{y^s}{2} \sum_{c,d} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} |c\tau + d|^{-2s} \mathbf{e}\Big(m\lambda^2 (M \cdot \tau) + \frac{2m\lambda z - cmz^2}{c\tau + d}\Big) \rho^*(M)^{-1} \mathfrak{e}_{\lambda\beta}. \end{aligned}$$

Here, c, d runs through all pairs of coprime integers, and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}$ is any element with bottom row (c, d). This series converges locally uniformly for $\mathrm{Re}[s] > \frac{3-k}{2}$.

After writing
$$E^*_{k,m,\beta}(\tau,z,s) = \sum_{(\zeta,M)\in\mathcal{J}_\infty\backslash\mathcal{J}} (y^s\mathfrak{e}_0)\Big|_{k,m,\rho^*_\beta}(\zeta,M),$$
it is clear that $E^*_{k,m,\beta}(\tau,z,s)$ transforms like a Jacobi form of weight k and index m:
$$E^*_{k,m,\beta}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d},s\Big) = (c\tau+d)^k\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\rho^*(M)E^*_{k,m,\beta}(\tau,z,s)$$
and
$$E^*_{k,m,\beta}(\tau,z+\lambda\tau+\mu) = \mathbf{e}\Big(-m\lambda^2\tau-2m\lambda z-m(\lambda\mu+t)\Big)\sigma^*_\beta(\zeta)E^*_{k,m,\beta}(\tau,z,s)$$
for any $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \tilde{\Gamma}$ and $\zeta = (\lambda,\mu,t)\in\mathcal{H}$. This series has an analytic continuation to $s\in\mathbb{C}$, for which one can reduce to the continuation of the usual Eisenstein series by the same argument as [2] uses in the scalar-valued case. (Another point of view is that the components are essentially expansions of a Jacobi Eisenstein series for a congruence subgroup at various cusps.)

Using the argument of [68], we see that $E^*_{k,m,\beta}(\tau,z,s)$ has the Fourier expansion
$$E^*_{k,m,\beta}(\tau,z,s) = \sum_{\lambda\in\mathbb{Z}} q^{m\lambda^2}\zeta^{2m\lambda}\mathfrak{e}_{\lambda\beta} + \sum_{\gamma\in\Lambda'/\Lambda}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}\sum_{n\in\mathbb{Z}-Q(\gamma)} c'(n,r,\gamma,s,y)q^n\zeta^r\mathfrak{e}_\gamma,$$
where $q=\mathbf{e}(\tau)$ and $\zeta=\mathbf{e}(z)$ and the coefficient $c'(n,r,\gamma,s,y)$ represents the contribution from all $M\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}$ other than the identity, given by
$$c'(n,r,\gamma,s,y) = \frac{\sqrt{i}^{\,b^--b^+-1}}{\sqrt{2m|\Lambda'/\Lambda|}}I(k-1/2,y,n-r^2/4m,s)\tilde{L}(n,r,\gamma,k+e/2+2s).$$
Here, $I(k,y,\omega,s)$ denotes the integral
$$I(k,y,\omega,s) = y^{1-k-s}e^{2\pi\omega y}\int_{-\infty}^{\infty}(t+i)^{-k}(t^2+1)^{-s}\mathbf{e}(-\omega yt)\,\mathrm{d}t,$$
and $\tilde{L}$ is the L-series
$$\tilde{L}(n,r,\gamma,s) = \zeta(s-e-1)^{-1}L(n,r,\gamma,s-1),$$
where
$$L(n,r,\gamma,s) = \prod_{p\,\mathrm{prime}}\Big(\sum_{\nu=0}^\infty \mathbf{N}(p^\nu)p^{-\nu s}\Big),$$

and $\mathbf{N}(p^\nu)$ is the number of zeros $(v,\lambda) \in \mathbb{Z}^{e+1}/p^\nu\mathbb{Z}^{e+1}$ of the polynomial $Q(v+\lambda\beta-\gamma)+m\lambda^2-r\lambda+n$.

Remark 72. Gross and Zagier consider in [37] the integral

$$V_s(\omega)=\int_{-\infty}^{\infty}(t+i)^{-k}(t^2+1)^{-s}\mathbf{e}(-\omega t)\,\mathrm{d}t,$$

(notice that k in that paper represents $\frac{k+1}{2}$ here), and they show that for $\omega \neq 0$, the completed integral

$$V_s^*(\omega)=(\pi|\omega|)^{-s-k}\Gamma(s+k)V_s(\omega)$$

is an entire function of s that satisfies the functional equation

$$V_s^*(\omega)=\mathrm{sgn}(\omega)V_{1-k-s}^*(\omega).$$

Since

$$I(k,y,\omega,s)=\frac{ye^{2\pi\omega y}(\pi|\omega|)^{s+k}}{\Gamma(s+k)}V_s^*(\omega y),$$

this extends $I(k,y,\omega,s)$ meromorphically to all $s\in\mathbb{C}$ and gives the functional equation

$$I(k,y,\omega,s)=\mathrm{sgn}(\omega)(\pi|\omega|)^{2s+k-1}\frac{\Gamma(1-s)}{\Gamma(s+k)}I(k,y,\omega,1-k-s),\ \ \omega\neq 0.$$

The integral for $\omega=0$ is

$$I(k,y,0,s)=\pi(-i)^k2^{1-s}(2y)^{1-k-s}\frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)}.$$

Remark 73. The local L-series

$$L_p(n,r,\gamma,s)=\sum_{\nu=0}^{\infty}\mathbf{N}(p^\nu)p^{-\nu s}$$

that occur in $L(n,r,\gamma,s)$ can be evaluated in the same way as the local L-series of [69]. Namely, for fixed $\gamma,\beta\in\Lambda'/\Lambda$ and $n\in\mathbb{Z}-Q(\gamma)$, $m\in\mathbb{Z}-Q(\beta)$, $r\in\mathbb{Z}-\langle\gamma,\beta\rangle$, we define discriminants

$$\mathcal{D}'=d_\beta^2d_\gamma^2(-1)^{e/2+1}(4mn-r^2)|\Lambda'/\Lambda|$$

if e is even and

$$D'=2md_\beta^2(-1)^{(e+1)/2}|\Lambda'/\Lambda|$$

if e is odd.

Define the "bad primes" to be $p = 2$ as well as all odd primes dividing $|\Lambda'/\Lambda|$ or md_β^2 or the numerator or denominator of $(n - r^2/4m)d_\beta^2 d_\gamma^2$, and set

$$\mathcal{D} = \mathcal{D}' \cdot \prod_{\mathrm{bad}\, p} p^2, \quad D = D' \cdot \prod_{\mathrm{bad}\, p} p^2.$$

If e is even, then for primes $p \nmid \mathcal{D}$,

$$L_p(n,r,\gamma,s) = \sum_{\nu=0}^\infty \mathbf{N}(p^\nu) p^{-\nu s} = \begin{cases} \frac{1}{1-p^{e-s}} \Big[1 + \Big(\frac{\mathcal{D}}{p} \Big) p^{e/2-s} \Big] : & r^2/4m - n \neq 0; \\ \\ \frac{1-p^{e-2s}}{(1-p^{e-s})(1-p^{1+e-2s})} : & r^2/4m - n = 0; \end{cases}$$

and if e is odd, then for primes $p \nmid D$,

$$L_p(n,r,\gamma,s) = \sum_{\nu=0}^\infty \mathbf{N}(p^\nu) p^{-\nu s} = \begin{cases} \frac{1}{1-p^{e-s}} \Big[1 - \Big(\frac{D}{p} \Big) p^{(e-1)/2-s} : & r^2/4m - n \neq 0; \\ \\ \dfrac{1 - \big(\frac{D}{p}\big) p^{(e-1)/2-s}}{(1-p^{e-s}) \Big[1 - \big(\frac{D}{p} \big) p(e+1)/2-s \Big]} : & r^2/4m - n = 0; \end{cases}$$

where $\Big(\frac{D}{p}\Big), \Big(\frac{\mathcal{D}}{p}\Big)$ denote the Legendre (quadratic reciprocity) symbol. This gives the meromorphic extensions

$$\tilde L(n,r,\gamma,s) = \begin{cases} \frac{L(s-1-e/2, \chi_{\mathcal{D}})}{\zeta(2s-2-e)} \prod_{\mathrm{bad}\, p} \frac{1-p^{e+1-s}}{1-p^{e+2-2s}} L_p(n,r,\gamma,s-1) : & r^2/4m - n \neq 0; \\ \\ \frac{\zeta(2s-3-e)}{\zeta(2s-2-e)} \prod_{\mathrm{bad}\, p} \frac{(1-p^{e+1-s})(1-p^{e-3-2s})}{1-p^{e-2-2s}} L_p(n,r,\gamma,s-1) : & r^2/4m - n = 0; \end{cases}$$

for even e, and

$$\tilde L(n,r,\gamma,s) = \begin{cases} \frac{1}{L(s-(e+1)/2, \chi_D)} \prod_{\mathrm{bad}\, p} \Big[(1-p^{e+1-s}) L_p(n,r,\gamma,s-1) \Big] : & r^2/4m - n \neq 0; \\ \\ \frac{L(s-(e+3)/2, \chi_D)}{L(s-(e+1)/2, \chi_D)} \prod_{\mathrm{bad}\, p} \Big[(1-p^{e+1-s}) L_p(n,r,\gamma,s-1) \Big] : & r^2/4m - n = 0; \end{cases}$$

for odd e.

Together, this gives the analytic continuation of the Fourier coefficients $c'(n, r, \gamma, s, y)$ of $E^*_{k,m,\beta}(\tau, z, s)$ to $s \in \mathbb{C}$ (possibly with poles) which must be the Fourier coefficients of the continuation of $E^*_{k,m,\beta}(\tau, z, s)$ away from $\mathrm{Re}[s] > \frac{3-k}{2}$.

Remark 74. We denote by $E_{k,m,\beta}(\tau, z)$ the series that results by naively evaluating the coefficient formula of [68] at $k = 3/2$ or $k = 2$ (without the weight 5/2 correction). In the derivation of this formula it was assumed that $I(k-1/2, y, n-r^2/4m, 0) = 0$ for $n-r^2/4m \leq 0$ and that $\tilde{L}(n, r, \gamma, s)$ is holomorphic at $s = 0$. These assumptions are not generally satisfied when $k \leq 5/2$, and $E_{k,m,\beta}(\tau, z)$ generally fails to be a Jacobi form in those cases. (In particular, $E_{k,m,\beta}(\tau, 0)$ generally fails to be a modular form.)

5.3 A Petersson scalar product

Recall that the Petersson scalar product on $S_k(\rho^*)$ is defined by

$$(f, g) = \int_{\tilde{\Gamma}\backslash\mathbb{H}} \langle f(\tau), g(\tau)\rangle y^{k-2}\, \mathrm{d}x\, \mathrm{d}y, \quad f, g \in S_k(\rho^*).$$

This is well-defined because cusp forms $f(\tau)$ satisfy the "trivial bound" $\|f(\tau)\| \leq C \cdot y^{-k/2}$ for some constant C (this is clear on the standard fundamental domain by continuity, and $\|f(\tau)\|y^{k/2}$ is invariant under $\tilde{\Gamma}$), and because $\langle f(\tau), g(\tau)\rangle y^{k-2}\, \mathrm{d}x\, \mathrm{d}y$ is invariant under $\tilde{\Gamma}$. More generally, we can define (f, g) for any functions f, g that transform like modular forms of weight k and for which the integral above makes sense. (This includes the case that $f, g \in M_k(\rho^*)$ and only one of f, g is a cusp form.)

In many cases it is useful to apply the following "unfolding argument" to evaluate $\langle f, g\rangle$, which is well-known. If $g(\tau)$ can be written in the form

$$g(\tau) = \sum_{M \in \tilde{\Gamma}_\infty\backslash\tilde{\Gamma}} u\Big|_{k,\rho^*} M$$

for some function $u(\tau)$ that decays sufficiently quickly as $y \to \infty$, then for any cusp form f,

$$\begin{aligned}(f, g) &= \int_{\tilde{\Gamma}\backslash\mathbb{H}} \sum_{M \in \tilde{\Gamma}_\infty\backslash\Gamma} \langle f, u|_{k,\rho^*} M\rangle y^{k-2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{-1/2}^{1/2} \int_0^\infty \langle f, u\rangle y^{k-2}\, \mathrm{d}y\, \mathrm{d}x.\end{aligned}$$

This is because there is a unique representative of every class $M \in \tilde{\Gamma}_\infty \backslash \Gamma$ that maps the strip $[-1/2, 1/2] \times [0, \infty)$ to itself, "unfolding" the fundamental domain of $\tilde{\Gamma} \backslash \mathbb{H}$ to the strip.

Example 75. Taking the Petersson scalar product with the real-analytic Eisenstein series

$$E_k^*(\tau, s) = \sum_{M \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} (y^s \mathfrak{e}_0)\Big|_{k,\rho^*} M$$

gives

$$\langle f, E_k^*(\tau, s) \rangle = \int_0^\infty \underbrace{\int_{-1/2}^{1/2} \langle f(\tau), \mathfrak{e}_0 \rangle \, \mathrm{d}x}_{=0} \, y^{k+s-2} \, \mathrm{d}y = 0$$

for all cusp forms f and sufficiently large $\mathrm{Re}[s]$ (and more generally by analytic continuation).

The more important example will be

$$g(\tau) = E_{k,m,\beta}^*(\tau, 0, 0) - E_k^*(\tau, 0) = \lim_{s \to 0} \sum_{M \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \Big(\sum_{\lambda \neq 0} y^s \mathbf{e}(m\lambda^2 \tau) \mathfrak{e}_{\lambda\beta} \Big)\Big|_{k,\rho^*} M.$$

Lemma 76. *For any cusp form* $f(\tau) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - Q(\gamma)} c(n, \gamma) q^n \mathfrak{e}_\gamma$,

$$(f, g) = 2 \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{\lambda=1}^\infty \frac{c(m\lambda^2, \lambda\beta)}{\lambda^{2k+s-2}}\Big|_{s=0}.$$

Proof. If $\mathrm{Re}[s]$ is large enough to guarantee that all series involved converge absolutely and locally uniformly, then the unfolding argument gives

$$\begin{aligned}(f, E_{k,m,\beta}^*(\tau, 0, s) - E_k^*(\tau, s)) &= \sum_{\lambda \neq 0} \int_{-1/2}^{1/2} \int_0^\infty \langle f(\tau), \mathbf{e}(m\lambda^2\tau)\mathfrak{e}_{\lambda\beta} \rangle y^{k+s-2} \, \mathrm{d}y \, \mathrm{d}x \\ &= 2 \cdot \sum_{\lambda=1}^\infty c(m\lambda^2, \lambda\beta) \int_0^\infty e^{-4\pi m \lambda^2 y} y^{k+s-2} \, \mathrm{d}y \\ &= 2 \cdot \Gamma(k+s-1) \sum_{\lambda=1}^\infty \frac{c(m\lambda^2, \lambda\beta)}{(4\pi m \lambda^2)^{k+s-1}}.\end{aligned}$$

Series of the form $\sum_{\lambda=1}^\infty \frac{c(m\lambda^2, \lambda\beta)}{\lambda^s}$ are closely related to symmetric square L-functions (see for example [59]) and have analytic continuations, for which one can reduce to the scalar case because the components of f are cusp forms of higher level. We take analytic continuations of both sides to $s = 0$. ☐

Remark 77. For $k \geq 5/2$, a simple Möbius inversion argument was used in [68] to show that if a cusp form $f(\tau) = \sum_{n,\gamma} c(n,\gamma) q^n \mathfrak{e}_\gamma$ satisfies $\sum_{\lambda=1}^\infty \frac{c(m\lambda^2, \lambda\beta)}{(4\pi m \lambda^2)^{k-1}} = 0$ for all $\beta \in \Lambda'/\Lambda$ and $m \in \mathbb{Z} - Q(\beta)$, $m > 0$, then $f = 0$ identically. In weight $k = 2$ one can use Deligne's bound $c(n,\gamma) = O(n^{1/2+\varepsilon})$, which implies that $s = 0$ is on the boundary of the region of absolute convergence, and apply essentially the same argument: taking the limit $s \to 0$ in the Möbius inversion argument gives the same result. On the other hand, in weight $k = 3/2$ this argument would require switching the order of a limit process and an analytic continuation, which seems difficult to justify.

In the following sections, we will construct modular forms $Q_{3/2,m,\beta}(\tau) \in M_{3/2}(\rho^*)$ resp. cusp forms $Q_{2,m,\beta} - E_2 \in M_2(\rho^*)$ with rational coefficients that satisfy

$$(f, Q_{k,m,\beta}) = 2\Gamma(k-1) \cdot \sum_{\lambda=1}^\infty \frac{c(m\lambda^2, \lambda\beta)}{(4\pi m\lambda^2)^{k+s-1}}\Big|_{s=0}.$$

For the above reason, the proof of chapter 3 that such forms contain $S_k(\rho^*)$ within their span is not rigorous when $k = 3/2$ and may well be false. For example, there is a Jacobi form of weight two and index 37 ([31], table 4) corresponding to a modular form of weight 3/2 for the quadratic form $Q(x) = 37x^2$. I suspect that it cannot be constructed from the forms $Q_{3/2,m,\beta}$ (all of which seem to be identically zero).

5.4 Weight 3/2

Proposition 78. *The value $E^*_{3/2,m,\beta}(\tau, z, 0)$ at $s = 0$ is a holomorphic Jacobi form of weight 3/2. It differs from the result $E_{3/2,m,\beta}(\tau, z)$ of the coefficient formula of [68] naively evaluated at $k = 3/2$ by a weight 1/2 theta series.*

Proof. The L-series term in this case is

$$\tilde{L}(n, r, \gamma, 3/2 + e/2 + 2s)$$
$$= \begin{cases} \frac{1}{L(2s,\chi_D)} \prod_{\mathrm{bad}\, p} (1 - p^{e/2-1-2s}) L_p(n, r, \gamma, 1/2 + e/2 + 2s) : & n - r^2/4m \neq 0; \\ \\ \frac{L(2s,\chi_D)}{L(1+2s,\chi_D)} \prod_{\mathrm{bad}\, p} (1 - p^{e/2-1-2s}) L_p(n, r, \gamma, 1/2 + e/2 + 2s) : & n - r^2/4m = 0. \end{cases}$$

This is holomorphic in $s = 0$ because the Dirichlet L-series $L(s, \chi_D)$ never has a pole at $s = 0$ or a zero at $s = 1$ and because the local L-factors $L_p(n, r, \gamma, 1/2 + e/2 + 2s)$ are rational functions of s with finitely many poles, while the dimension e can be made arbitrarily large without changing the underlying discriminant form (and therefore the value of $\tilde{L}$). Note that $L_p(n, r, \gamma, 1/2 + e/2 + 2s)$ may have a simple pole at 0 if $e = 2$, but this is canceled by the factor $1 - p^{e/2-1-2s}$; in this case, we will write

$$(1 - p^{e/2-1})L_p(n, r, \gamma, 1/2 + e/2) = \lim_{s\to 0}(1 - p^{e/2-1-2s})L_p(n, r, \gamma, 1/2 + e/2 + 2s)$$

by abuse of notation.

Despite this, the coefficient formula [68] still requires a correction because the zero-value $I(1, y, 0, 0) = -\pi i$ is nonzero. This is easiest to calculate as a Cauchy principal value:

$$I(1, y, 0, 0) = \lim_{s\to 0} \int_{-\infty}^{\infty} (t + i)^{-1}(t^2 + 1)^{-s}\, \mathrm{d}t = PV\Big[\int_{-\infty}^{\infty} (t + i)^{-1}\Big] = -\pi i.$$

The corrected series

$$\begin{aligned} E^*_{3/2,m,\beta}(\tau, z, 0) &= E_{3/2}(\tau, z) - \pi i \frac{\sqrt{i}^{b^- - b^+ - 1}}{\sqrt{2m|\Lambda'/\Lambda|}} \sum_{r^2=4mn} \tilde{L}(n, r, \gamma, 3/2 + e/2)q^n\zeta^r \\ &= E_{3/2}(\tau, z) - \pi \frac{(-1)^{(1+b^- - b^+)/4}}{\sqrt{2m|\Lambda'/\Lambda|}} \sum_{r^2=4mn} \tilde{L}(n, r, \gamma, 3/2 + e/2)q^n\zeta^r \end{aligned}$$

is holomorphic in τ and therefore defines a Jacobi form. □

Remark 79. The exponent $(n, r) = (0, 0)$ occurs in this correction term and therefore $E^*_{3/2,m,\beta}(\tau, z, 0)$ will generally not have constant term $1 \cdot \mathfrak{e}_0$ and may even vanish identically. This is not surprising because there are many cases where no nonzero Jacobi forms of weight 3/2 exist at all.

Definition 80. We define $Q_{3/2,m,\beta}(\tau) = E^*_{3/2,m,\beta}(\tau, 0, 0)$. In particular, this differs from the computation of [68] by a weight 1/2 theta series.

These series produce modular forms which represent the functional through the Petersson inner product as claimed in proposition 70. There is a unique cusp form that represents the same functional, and its difference with $Q_{3/2,m,\beta}$ will lie in the Eisenstein subspace, but this subspace is more difficult to describe in weight 3/2; in particular, the difference will almost never be the Eisenstein series $E_{3/2}(\tau)$ of chapter 4 (which is often not a modular form at all).

Example 81. Let $\Lambda = \mathbb{Z}^3$ with quadratic form $Q(x,y,z) = 2xz + y^2$; then $M_{3/2}(\rho^*)$ is one-dimensional, spanned by

$$\begin{aligned} Q_{3/2,1,0}(\tau) &= \Big(\frac{1}{2} + 3q + 6q^2 + 4q^3 + ...\Big)(\mathfrak{e}_{(0,0,0)} - \mathfrak{e}_{(0,0,1/2)} - \mathfrak{e}_{(1/2,0,0)}) \\ &+ \Big(4q^{3/4} + 12q^{11/4} + ...\Big)(\mathfrak{e}_{(0,1/2,0)} - \mathfrak{e}_{(0,1/2,1/2)} - \mathfrak{e}_{(1/2,1/2,0)}) \\ &+ \Big(- 6q^{1/2} - 12q^{3/2} - 12q^{5/2} - ...\Big)\mathfrak{e}_{(1/2,0,1/2)} \\ &+ \Big(- 3q^{1/4} - 12q^{3/4} - 15q^{9/4} - ...\Big)\mathfrak{e}_{(1/2,1/2,1/2)}, \end{aligned}$$

with constant term $\frac{1}{2}\mathfrak{e}_{(0,0,0)} - \frac{1}{2}\mathfrak{e}_{(0,0,1/2)} - \frac{1}{2}\mathfrak{e}_{(1/2,0,0)}$. Unlike the case of weight $k \geq 5/2$, there is no way to produce a modular form with constant term $1 \cdot \mathfrak{e}_0$. (Following [5], the theta series in $M_{1/2}(\rho)$ act as obstructions to producing modular forms in $M_{3/2}(\rho^*)$ with arbitrary constant term.)

5.5 Weight two

The value $E^*_{2,m,\beta}(\tau, z, 0)$ at $s = 0$ is not generally holomorphic:

Proposition 82. *There are constants $A(n,r,\gamma)$, $\gamma \in \Lambda'/\Lambda$, $n \in \mathbb{Z} - Q(\gamma)$, $r \in \mathbb{Z} - \langle \gamma, \beta \rangle$ given by*

$$A(n,r,\gamma) = \frac{48(-1)^{(4+b^+-b^-)/4}}{\sqrt{m \cdot |\Lambda'/\Lambda|}} \prod_{\mathrm{bad}\, p} \frac{1 - p^{e/2-1}}{1 + p^{-1}} L_p(n,r,\gamma,1+e/2) \times \begin{cases} 1: & r^2 \neq 4mn; \\ 1/2: & r^2 = 4mn \end{cases}$$

such that

$$E^*_2(\tau,z,0) = E_2(\tau,z) + \frac{1}{\sqrt{y}} \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - Q(\gamma)} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} A(n,r,\gamma) \beta(\pi y(r^2/m - 4n)) q^n \zeta^r \mathfrak{e}_\gamma.$$

Here, $\beta(x)$ is a sort of incomplete Gamma function:

$$\beta(x) = \frac{1}{16\pi} \int_1^\infty u^{-3/2} e^{-xu} \, \mathrm{d}u,$$

and we abuse notation and write

$$(1-p^{e/2-1})L_p(n,r,\gamma,1+e/2) = \lim_{s\to 0}(1-p^{e/2-1-s})L_p(n,r,\gamma,1+e/2+s)$$

in the cases where L_p has a simple pole at $1+e/2$.

Proof. In weight $k=2$, the L-series term is

$$\begin{aligned}&\tilde{L}(n,r,\gamma,2+e/2+2s)\\ =&\begin{cases}\frac{L(2s+1,\chi_{\mathcal{D}})}{\zeta(4s+2)}\prod_{\mathrm{bad}\,p}\frac{1-p^{e/2-1-2s}}{1-p^{e/2-2s}}L_p(n,r,\gamma,1+e/2+2s): & n-r^2/4m\neq 0;\\ \\ \frac{\zeta(4s+1)}{\zeta(4s+2)}\prod_{\mathrm{bad}\,p}\frac{(1-p^{e/2-1-2s})(1-p^{-1-4s})}{1-p^{-2-4s}}L_p(n,r,\gamma,1+e/2+2s): & n-r^2/4m=0.\end{cases}\end{aligned}$$

Here, $\mathcal{D}$ denotes the discriminant

$$\mathcal{D} = (r^2-4mn)|\Lambda'/\Lambda|d_\beta^2 d_\gamma^2 \prod_{\mathrm{bad}\,p} p^2.$$

This L-series has a pole in $s=0$ when $n-r^2/4m=0$ or when $\mathcal{D}$ is a square, and in these cases the residue at $s=0$ is

$$\begin{aligned}&\mathrm{Res}\Big(\tilde{L}(n,r,\gamma,2+e/2+2s), s=0\Big)\\ =&\frac{3}{\pi^2}\Big[\prod_{\mathrm{bad}\,p}\frac{1-p^{e/2-1}}{1+p^{-1}}L_p(n,r,\gamma,1+e/2)\Big]\times\begin{cases}1: & n-r^2/4m\neq 0;\\ 1/2: & n-r^2/4m=0.\end{cases}\end{aligned}$$

The pole of $\tilde{L}$ cancels with the zero of $I(3/2,y,n-r^2/4m,s)$ at $s=0$, whose derivative there is

$$\frac{d}{ds}\Big|_{s=0} I(3/2,y,n-r^2/4m,s) = -16\pi^2(1+i)y^{-1/2}\beta(\pi|4n-r^2/m|y).$$

(This is essentially the same computation that arises when studying the weight 3/2 Eisenstein series in [69]). ∎

In particular, $E_2^*(\tau,0,0)$ is generally far from being a holomorphic modular form. Instead, we define a family of cusp forms $Q_{2,m,\beta}^*(\tau,s)\in S_2(\rho^*)$ by taking the orthogonal projection of $E_{2,m,\beta}^*(\tau,0,s)-E_2^*(\tau,s)$ to $S_2(\rho^*)$ with respect to the Petersson scalar product, i.e. by holomorphic projection of the zero-values of $E^*_{2,m,\beta}(\tau,0,s)$. Explicitly, if $e_1,...,e_n$ are an orthonormal basis of weight-2 cusp forms then

$$Q^*_{2,m,\beta}(\tau,s) = \sum_{j=1}^n \Big(E^*_{2,m,\beta}(\tau,0,s) - E^*_2(\tau,s), e_j(\tau)\Big) \cdot e_j(\tau).$$

From the definition it is clear that for large enough $\mathrm{Re}[s]$, $Q^*_{2,m,\beta}(\tau,s)$ is the cusp form satisfying

$$(f, Q^*_{2,m,\beta}(\tau,s)) = (f, E^*_{2,m,\beta}(\tau,0,s) - E^*_2(\tau,s)) = 2\cdot\Gamma(1+s)\sum_{\lambda=1}^\infty \frac{c(m\lambda^2,\lambda\beta)}{(4\pi m\lambda^2)^{1+s}}$$

for any cusp form $f(\tau) = \sum_{n,\gamma} c(n,\gamma)q^n \mathfrak{e}_\gamma$.

Remark 83. For any $\beta \in \Lambda'/\Lambda$ and $m \in \mathbb{Z} - Q(\beta)$, $m > 0$, the Poincaré series of weight 2 is defined by

$$P_{2,m,\beta}(\tau) = \sum_{M \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} \Big(\mathbf{e}(m\tau)\mathfrak{e}_\beta\Big)\Big|_{2,\rho^*} M = \frac{1}{2}\sum_{c,d} (c\tau+d)^{-2}\mathbf{e}\Big(m(M\cdot\tau)\Big)\rho^*(M)^{-1}\mathfrak{e}_\beta,$$

where c,d runs through all pairs of coprime integers and $M \in \tilde{\Gamma}$ is any element with bottom row (c,d). This series does not converge absolutely, but as shown in [46],

$$\lim_{s\to 0} \sum_{M\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}} \Big(y^s \mathbf{e}(m\tau)\mathfrak{e}_\beta\Big)\Big|_{2,\rho^*} M$$

is holomorphic in τ and therefore $P_{2,m,\beta}(\tau)$ defines a cusp form. The unfolding argument characterizes $P_{2,m,\beta}$ by

$$(f, P_{2,m,\beta}) = \frac{c(m,\beta)}{4\pi m} \text{ for any cusp form } f(\tau) = \sum_{n,\gamma} c(n,\gamma)q^n\mathfrak{e}_\gamma$$

as usual.

Remark 84. Writing $Q^*_{2,m,\beta}(\tau,s) = \sum_{\gamma\in\Lambda'/\Lambda}\sum_{n\in\mathbb{Z}-Q(\gamma)} b(n,\gamma,s)q^n\mathfrak{e}_\gamma$, the fact that $Q^*_{2,m,\beta}(\tau,s) - E^*_{k,m,\beta}(\tau,0,s)$ is orthogonal to all Poincaré series implies that

$$\begin{aligned} \frac{b(n,\gamma,s)}{4\pi n} &= \Big(Q^*_{2,m,\beta}(\tau,s), P_{2,n,\gamma}\Big) \\ &= \Big(E^*_{2,m,\beta}(\tau,0,s), P_{2,n,\gamma}\Big) \\ &= \int_0^\infty c(n,\gamma,y,s)e^{-4\pi n y}y^s\,\mathrm{d}y, \end{aligned}$$

where $c(n,\gamma,y,s)$ is the coefficient of $q^n\mathfrak{e}_\gamma$ in $E^*_{2,m,\beta}(\tau,0,s)$.

Definition 85. The **Poincaré square series** of weight 2 is the quasimodular form

$$Q_{2,m,\beta}(\tau) = E_2(\tau) + Q^*_{2,m,\beta}(\tau, 0).$$

It follows from the above remarks that $Q_{2,m,\beta}(\tau)$ differs from the computation of [68] as follows: we can write

$$Q_{2,m,\beta}(\tau) = E_2(\tau, 0) + \sum_{\gamma \in \Lambda'/\Lambda} \sum_{\substack{n \in \mathbb{Z} - Q(\gamma) \\ n > 0}} b(n, \gamma) q^n \mathfrak{e}_\gamma,$$

with coefficients that are determined by

$$\begin{aligned} \frac{b(n,\gamma)}{4\pi n} &= \lim_{s \to 0} \int_0^\infty c(n, \gamma, y, s) e^{-4\pi n y} y^s \, \mathrm{d}y \\ &= \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} A(n, r, \gamma) \int_0^\infty e^{-4\pi n y} \beta(\pi y (r^2/m - 4n)) y^{-1/2} \, \mathrm{d}y \\ &= \frac{1}{16\pi} \sum_r A(n, r, \gamma) \int_0^\infty \int_1^\infty u^{-3/2} y^{-1/2} e^{4\pi n y (u-1) - \pi r^2 y u/m} \, \mathrm{d}u \, \mathrm{d}y \\ &= \frac{1}{16\pi} \sum_r A(n, r, \gamma) \int_1^\infty u^{-3/2} \Big((r^2/m - 4n) u + 4n \Big)^{-1/2} \, \mathrm{d}u \\ &= \frac{1}{32\pi n \sqrt{m}} \sum_r A(n, r, \gamma) \Big(|r| - \sqrt{r^2 - 4mn} \Big), \end{aligned}$$

i.e.

$$b(n, \gamma) = \frac{1}{8\sqrt{m}} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} A(n, r, \gamma) \Big(|r| - \sqrt{r^2 - 4mn} \Big).$$

When $|\Lambda'/\Lambda|$ is square, it turns out that for fixed n and γ, the sum above is finite and can be calculated directly. Otherwise, this tends to be a truly infinite series and we will need some preparation to prove that $b(n, \gamma)$ are rational and to evaluate them with a finite computation.

5.6 A Pell-type equation

The condition

$$\mathcal{D} = d_\beta^2 d_\gamma^2 (r^2 - 4mn) |\Lambda'/\Lambda| \prod_{\text{bad } p} p^2 = \square$$

is equivalent to requiring $(a,b) = d_\gamma d_\beta(\sqrt{|\Lambda'/\Lambda|(r^2-4mn)}, r)$ to occur as an integer solution of the Pell-type equation

$$a^2 - |\Lambda'/\Lambda|b^2 = -4|\Lambda'/\Lambda|(d_\beta^2 m)(d_\gamma^2 n)$$

satisfying the congruence $b \equiv d_\beta d_\gamma \langle \gamma, \beta\rangle \mod d_\gamma d_\beta \mathbb{Z}$. We will study such equations in general.

Definition 86. A **Pell-type problem** is a problem of the form

$$\text{find all integer solutions } (a,b) \text{ of } a^2 - Db^2 = -4CD$$

for some $C, D \in \mathbb{N}$.

The behavior of solutions is quite different depending on whether or not D is square. If D is a square, then the equation can be factored as

$$(a - \sqrt{D}b)(a + \sqrt{D}b) = a^2 - Db^2 = -4CD,$$

from which it follows that there are only finitely many solutions and all are bounded by $|a|, \sqrt{D}|b| \le CD + 1$.

Assume from now on that D is nonsquare. In this case, the solutions of the Pell-type problem are closely related to the solutions of the true Pell equation

$$a^2 - Db^2 = 1.$$

It follows from Dirichlet's unit theorem that there are infinitely many solutions (a,b) of the Pell equation and all have the form

$$a + \sqrt{D}b = \pm\varepsilon_0^n, \quad n \in \mathbb{Z},$$

where $\varepsilon_0 \in \mathbb{Z}[\sqrt{D}]$ is the **fundamental solution** $\varepsilon_0 = a + \sqrt{D}b$, which is the minimal solution satisfying $\varepsilon_0 > 1$. The problem of determining ε_0 is well-studied; see for example [43] for an overview.

Lemma 87. *Assume that D is squarefree and let $K = \mathbb{Q}(\sqrt{D})$ with ring of integers $\mathcal{O}_K$. Then the solutions (a,b) of the Pell-type equation $a^2 - Db^2 = -4CD$ are in bijection with elements $\mu \in \mathcal{O}_K$ having norm C.*

Proof. Let (a, b) be any solution of the Pell-type equation and define $\mu = \frac{a+\sqrt{D}b}{2\sqrt{D}}$. This is an algebraic integer because its trace $\mu + \overline{\mu} = b$ and norm $\mu\overline{\mu} = C$ are both integers. Conversely, given any algebraic integer $\mu \in \mathcal{O}_K$ of norm C, we can define (a, b) by $a + \sqrt{D}b = 2\sqrt{D}\mu$. $\square$

Lemma 88. *Assume that D is squarefree. Then there are finitely many elements* $\mu_1, ..., \mu_n \in \mathcal{O}_K$*, all satisfying* $0 \le \mathrm{Tr}_{K/\mathbb{Q}}(\mu_i) \le 2\sqrt{C\varepsilon_1}$*, such that*

$$\Big\{\mu \in \mathcal{O}_K : \; N_{K/\mathbb{Q}}(\mu) = C\Big\} = \bigcup_{i=1}^n \mu_i \cdot \mathcal{O}_K^{\times,1}.$$

Here, ε_1 is the fundamental solution to $N_{K/\mathbb{Q}}(\varepsilon_1) = 1$. In other words ε_1 is either the fundamental unit or its square if the fundamental unit has norm -1. Also,

$$\mathcal{O}_K^{\times,1} = \{\varepsilon \in \mathcal{O}_K^{\times} : \; N_{K/\mathbb{Q}}(\varepsilon) = 1\}.$$

Proof. Suppose μ is any solution of $N_{K/\mathbb{Q}}(\mu) = C$, and choose $n \in \mathbb{Z}$ such that

$$|\log(\varepsilon_1^n \mu) - \log(\sqrt{C})|$$

is minimal. Then it follows that

$$|\log(\varepsilon_1^n \mu) - \log(\sqrt{C})| \le \frac{1}{2}\log(\varepsilon_1).$$

In particular, $\varepsilon_1^n \mu \le \sqrt{C\varepsilon_1}$ and $\varepsilon_1^{-n}\mu^{-1} \le \sqrt{\varepsilon_1/C}$. It follows that

$$\Big|\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon_1^n \mu)\Big| = \Big|\varepsilon_1^n + C\varepsilon_1^{-n}\mu^{-1}\Big| \le 2\sqrt{C\varepsilon_1}.$$

By replacing μ by $-\mu$ we may assume that $\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon_1^n \mu) \ge 0$.

In particular, μ lies in the same $\mathcal{O}_K^{\times,1}$-orbit as a root of one of finitely many polynomials $X^2 + \lambda X + C$ with $0 \le \lambda \le \lfloor 2\sqrt{C\varepsilon_1} \rfloor$, which also shows that there are finitely many orbits. $\square$

Example 89. Consider the Pell-type equation $a^2 - 33b^2 = -528$ with $D = 33$ and $C = 4$. There are three orbits of elements $\mu \in \mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{33})/2]$ with norm 4, represented by

$$\mu = 2, \; \mu = \frac{7 \pm \sqrt{33}}{2},$$

having traces 4 and 7. The bound in this case is $2\sqrt{C\varepsilon_1} \approx 28$. Note that elements μ that are conjugate by $\mathrm{Gal}(K/\mathbb{Q})$ result in the same solutions to the Pell equation.

Remark 90. Let $b_0, n \in \mathbb{N}$. Reducing modulo n shows that the set of solutions (a, b) to

$$a^2 - Db^2 = -4CD, \quad b \equiv b_0 \mod n$$

is also in bijection via $(a, b) \mapsto \mu = \frac{a+\sqrt{D}b}{2\sqrt{D}}$ to a union of finitely many orbits (possibly none):

$$\bigcup_{i=1}^{n} \mu_i \cdot \langle \varepsilon_{\mu_i} \rangle,$$

where the "congruent fundamental solution" ε_{μ_i} is the minimal power of the fundamental solution ε_1 such that $\mathrm{Tr}_{K/\mathbb{Q}}(\mu_i(1 - \varepsilon_{\mu_i})) \equiv 0 \,(\mathrm{mod}\, n)$.

When D is not squarefree, we can pull out the largest square factor of D to reduce the equation

$$a^2 - Db^2 = -4CD$$

to a squarefree Pell-type equation with congruence condition.

Lemma 91. *Fix $\gamma \in \Lambda'/\Lambda$ and $n \in \mathbb{Z} - Q(\gamma)$, $n > 0$. Then the value of*

$$A(n, r, \gamma) \times \begin{cases} 1: & r^2 \neq 4mn; \\ 2: & r^2 = 4mn; \end{cases}$$

depends only on the orbit of $d_\gamma d_\beta \sqrt{|\Lambda'/\Lambda|}(r + \sqrt{r^2 - 4mn})$ as a solution of the Pell-type equation

$$a^2 - Db^2 = -4CD, \quad D = |\Lambda'/\Lambda|, \; C = d_\beta^2 d_\gamma^2 mn,$$

with congruence condition $b \equiv d_\beta d_\gamma \langle \gamma, \beta \rangle$ mod $d_\gamma d_\beta \mathbb{Z}$.

Proof. Assume first that $\beta = 0$ and abbreviate $D = |\Lambda'/\Lambda|$. Multiplying $r + \sqrt{r^2 - 4mn}$ by the congruent fundamental solution $\varepsilon = a + b\sqrt{D}$ replaces r by

$$r\frac{\varepsilon + \varepsilon^{-1}}{2} + \sqrt{r^2 - 4mn}\frac{\varepsilon - \varepsilon^{-1}}{2} = ar + b\sqrt{D(r^2 - 4mn)},$$

and $r^2 - 4mn$ by

$$(r^2 - 4mn) + 2Db^2(r^2 - 4mn) + 4mnDb^2 + 2abr\sqrt{D(r^2 - 4mn)},$$

which is congruent to $r^2 - 4mn$ modulo the largest modulus whose square divides D.

Since $\beta = 0$, it follows that $E^*_{2,m,\beta}(\tau, z, s)$ arises from a weight-3/2 real-analytic Maass form (here the Eisenstein series $E^*_{3/2}(\tau, s)$) for the quadratic form

$$\tilde{Q}(v, \lambda) = Q(v) + m\lambda^2$$

through the theta decomposition; in other words, the coefficient of $q^n \zeta^r \mathfrak{e}_\gamma$ in $E^*_{2,m,\beta}(\tau, z, s)$ equals the coefficient of $q^{n-r^2/4m}\mathfrak{e}_{(\gamma,r/2m)}$ in $E^*_{3/2}(\tau, s)$. In particular, this equality also holds for the real-analytic parts. The coefficients $A(n, r, \gamma)$ in the real-analytic part of $E^*_{3/2}(\tau, 0)$ occur (up to a constant factor) as the coefficients of its shadow, which is a modular form of weight 1/2 for the quadratic form $-\tilde{Q}$. Using Skoruppa's strengthening of the Serre-Stark basis theorem ([61], Satz 5.1; see also (3.5) of [20]), it is known that for any Weil representation $\rho : \tilde{\Gamma} \to \mathrm{Aut}\, \mathbb{C}[\Lambda'/\Lambda]$, $M_{1/2}(\rho)$ is spanned by modular forms that are $\mathbb{C}[\Lambda'/\Lambda]$-linear combinations of the theta series

$$\vartheta_{\ell,b} = \sum_{\substack{v \in \mathbb{Z} \\ v \equiv b\,(2\ell)}} \mathbf{e}\Big(\frac{v^2}{4\ell}\tau\Big), \quad b \in \mathbb{Z},$$

where ℓ runs through divisors of $4N$ for which N/ℓ is squarefree (where N is the level of the discriminant form Λ'/Λ), in which the Fourier coefficient of q^n (multiplied by 1/2 if $n = 0$) depends only on whether ℓn is square and if so on the remainder of $\sqrt{4\ell n}$ modulo 2ℓ. The previous paragraph implies this congruence for $n - r^2/4m$ for all $r + \sqrt{r^2 - 4mn}$ in the same orbit.

For general β, we can embed the space of Jacobi forms for ρ^*_β of index m as "old" Jacobi forms of index md_β^2 for the trivial action of the Heisenberg group via the Hecke-type operator

$$U_\beta \Phi(\tau, z) = \Phi(\tau, d_\beta z)$$

and apply the argument for $\beta = 0$. □

Proposition 92. *The Poincaré square series $Q_{2,m,\beta}(\tau)$ has rational Fourier coefficients.*

Proof. The expression for the coefficients of $E_2(\tau, 0)$ in [68] consists of special values of Dirichlet L-functions and finitely many local L-series, and these remain rational in weight $k = 2$. Therefore, we need to show that the correction terms

$$b(n, \gamma) = \frac{1}{8\sqrt{m}} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} A(n, r, \gamma)\Big(|r| - \sqrt{r^2 - 4mn}\Big)$$

are rational.

This is easy to see when $|\Lambda'/\Lambda|$ is square, since $b(n,\gamma)$ is a finite sum of rational numbers. Assume that $|\Lambda'/\Lambda|$ is not square.

Suppose first that $\beta = 0$. By lemma 91, we can write

$$\sum_{r \in \mathbb{Z}} A(n,r,\gamma)\Big(|r| - \sqrt{r^2 - 4mn}\Big) = \sum_{i=1}^N A(n,r,\gamma) \sum_r \Big(|r| - \sqrt{r^2-4mn}\Big),$$

where for each i, the sum over r is taken over solutions

$$(a,b) = d_\gamma\Big(\sqrt{|\Lambda'/\Lambda|(r^2-4mn)}, r\Big)$$

of the Pell equation with congruence condition coming from the orbit of an element μ_i of norm C and minimal trace as in lemma 88. These solutions are given by

$$r + \sqrt{r^2-4mn} = \pm \frac{2\sqrt{|\Lambda'/\Lambda|}}{d_\gamma}\mu_i \varepsilon_i^n,$$

which runs through the solutions r twice if $\overline{\mu_i}/\mu_i \in \mathcal{O}_K$ and once otherwise. The minimality of $\mathrm{Tr}_{K/\mathbb{Q}}(\mu_i)$ implies that the terms in the series are

$$|r| - \sqrt{r^2-4mn} \in \{2\mu, 2\mu\varepsilon^{-n}, 2\overline{\mu}\varepsilon^{-n} : \ n \in \mathbb{N}\},$$

and

$$\begin{aligned} \sum_r \Big(|r| - \sqrt{r^2-4mn}\Big) &= \Big(\frac{\mu}{1-\varepsilon^{-1}} + \frac{\overline{\mu}\varepsilon^{-1}}{1-\varepsilon^{-1}}\Big) \times \begin{cases} 1: & \overline{\mu}/\mu \in \mathcal{O}_K; \\ 2: & \text{otherwise} \end{cases} \\ &= \frac{1}{N_{K/\mathbb{Q}}(1-\varepsilon)}\Big(\mu - \overline{\mu} + \overline{\mu\varepsilon} - \mu\varepsilon\Big) \times \begin{cases} 1: & \overline{\mu}/\mu \in \mathcal{O}_K; \\ 2: & \text{otherwise;} \end{cases} \end{aligned}$$

and we see that $\frac{1}{\sqrt{|\Lambda'/\Lambda|}}\sum_r \Big(|r| - \sqrt{r^2-4mn}\Big)$ is rational. Since

$$A(n,r,\gamma) = \frac{1}{\sqrt{m|\Lambda'/\Lambda|}} \cdot \Big(\text{rational number}\Big),$$

we see that $b(n,\gamma)$ is rational.

The argument for general β is essentially the same but slightly messier because $r + \sqrt{r^2-4mn}$ and $-r+\sqrt{r^2-4mn}$ generally occur as solutions of the Pell equation with different congruence conditions. In this case we can use the identity $b(n,\gamma) = b(n,-\gamma) = \frac{b(n,\gamma)+b(n,-\gamma)}{2}$ and consider both congruence conditions at once. $\square$

The formula above has been implemented in SAGE and is available on the author's university webpage.

5.7 Example: the class number relation

In the simplest case of a unimodular lattice Λ and index $m = 1$, the fact that

$$Q_{2,1,0}(\tau) = E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n = 1 - 24q - 72q^2 - 96q^3 - ...$$

(since the difference $Q_{2,1,0} - E_2$ is a scalar-valued cusp forms of weight 2 and level 1 so it vanishes) implies the Kronecker-Hurwitz class number relations. We explain this here.

The real-analytic Jacobi form $E^*_{2,1,0}(\tau, z, 0)$ arises from the real-analytic correction of Zagier's Eisenstein series (in the form of example 60 of chapter 4),

$$E^*_{3/2}(\tau, 0) = 1 - 12\sum_{n=1}^{\infty} H(n)q^{n/4}\mathfrak{e}_{n/2} - \frac{24}{\sqrt{y}}\sum_{n=-\infty}^{\infty} \beta(\pi y n^2)q^{-n^2/4}\mathfrak{e}_{n/2}$$

through the theta decomposition, where $H(n)$ is the Hurwitz class number of n. Therefore, $E^*_{2,1,0}(\tau, z, 0)$ is

$$1 - 12\sum_{n=1}^{\infty}\sum_{r=-\infty}^{\infty} H(4n - r^2)q^n\zeta^r + \frac{1}{\sqrt{y}}\sum_{n=-\infty}^{\infty}\sum_{r^2-4n=\square} A(n, r)\beta(\pi y(r^2 - 4n))q^n\zeta^r$$

where the correction constants are

$$A(n, r) = \begin{cases} -24 : & r^2 - 4n = 0; \\ -48 : & r^2 - 4n \neq 0. \end{cases}$$

It follows that

$$\begin{aligned} &Q_{2,1,0}(\tau) \\ &= 1 - 12\sum_{n=1}^{\infty}\sum_{r=-\infty}^{\infty} H(4n - r^2)q^n + \frac{1}{8}\sum_{n=1}^{\infty}\sum_{r^2-4n=\square} A(n, r)\Big(|r| - \sqrt{r^2 - 4n}\Big)q^n \\ &= 1 - 12\sum_{n=1}^{\infty}\sum_{r=-\infty}^{\infty} H(4n - r^2)q^n - 6\sum_{n=1}^{\infty}\sum_{r^2-4n=\square} \Big(|r| - \sqrt{r^2 - 4n}\Big)q^n + 12\sum_{n=1}^{\infty} nq^{n^2}. \end{aligned}$$

The identity $Q_{2,1,0} = E_2$ implies that for all $n \in \mathbb{N}$,

$$\sum_{r=-\infty}^{\infty} H(4n-r^2) = 2\sigma_1(n) - \frac{1}{2} \sum_{\substack{r \in \mathbb{Z} \\ r^2-4n=\square}} \Big(|r| - \sqrt{r^2-4n}\Big) + \begin{cases} \sqrt{n} : & n = \square; \\ 0 : & \text{otherwise.} \end{cases}$$

Here, $\frac{1}{2}\Big(|r| - \sqrt{r^2-4n}\Big)$ takes exactly the values $\min(d, n/d)$ as d runs through divisors of n (but counts $\sqrt{n}$ twice if n is square); so this can be rearranged to

$$\sum_{r=-\infty}^{\infty} H(4n-r^2) = 2\sigma_1(n) - \sum_{d|n} \min(d, n/d).$$

Remark 93. Mertens [47] has given other proofs of this and similar class number relations using mock modular forms. It seems likely that we can recover other class number relations (possibly some of the other relations of [47]) by studying the higher development coefficients (as defined in chapter 3 of [31]) of the real-analytic Jacobi Eisenstein series $E^*_{2,1,0}(\tau, z, s)$ in the same way that we have studied its zeroth development coefficient $E^*_{2,1,0}(\tau, 0, s)$, but we will not pursue that here.

5.8 Example: overpartition rank differences

Consider the lattice $\Lambda = \mathbb{Z}^2$ with quadratic form $Q(x, y) = x^2 - y^2$. There are no modular forms of weight 2 for the dual Weil representation, and the $\mathfrak{e}_{(0,0)}$-component of the quasimodular Eisenstein series is

$$E_2(\tau)_{(0,0)} = 1 - 16q - 24q^2 - 64q^3 - 72q^4 - 96q^5 - 96q^6 - 128q^7 - ...$$

This is a quasimodular form of level 4 and we can verify by computing a few coefficients that it is

$$E_2(\tau)_{(0,0)} = E_2(2\tau) - 16 \sum_{n\,\mathrm{odd}} \sigma_1(n) q^n = 1 - 16 \sum_{n\,\mathrm{odd}} \sigma_1(n) q^n - 24 \sum_{n\,\mathrm{even}} \sigma_1(n/2) q^n.$$

The real-analytic Jacobi Eisenstein series of index $(m, \beta) = (1, 0)$ corresponds to the real-analytic Eisenstein series for the lattice $\tilde{\Lambda} = \mathbb{Z}^3$ with quadratic form $Q'(x, y, z) = x^2 - y^2 + z^2$

under the theta decomposition. It was shown in example 63 of chapter 4 that the component of $\mathfrak{e}_{(0,0,0)}$ in the corresponding mock Eisenstein series is

$$1 - 2q - 4q^2 - 8q^3 - ... = \sum_{n=0}^{\infty} (-1)^n \overline{\alpha(n)} q^n,$$

where $\overline{\alpha(n)}$ is the difference between the number of even-rank and odd-rank overpartitions of n. (We refer to [7] for the definition of overpartition rank differences and their appearance in weight-3/2 mock modular forms.) We will also need to understand the component of $\mathfrak{e}_{(0,0,1/2)}$ in this mock Eisenstein series. A quick computation shows that this is the series

$$E_{3/2}(\tau)_{(0,0,1/2)} = -4q^{3/4} - 4q^{7/4} - 12q^{11/4} - 8q^{15/4} - 12q^{19/4} - 12q^{23/4} - 16q^{27/4} - ...$$

Lemma 94. *The coefficient of $q^{n/4}$ in the series $E_{3/2}(\tau)_{(0,0,1/2)}$ is*

$$\begin{cases} -12H(n) : & n \equiv 3 \bmod 8; \\ -4H(n) : & n \equiv 7 \bmod 8; \end{cases}$$

where $H(n)$ is the Hurwitz class number.

We remark without proof that this series appears to have an interesting closed form:

$$\begin{aligned} & -4q^{3/4} - 4q^{7/4} - 12q^{11/4} - 8q^{15/4} - 12q^{19/4} - ... \\ &= -12 \sum_{n \equiv 3\,(8)} H(n)q^{n/4} - 4 \sum_{n \equiv 7\,(8)} H(n) q^{n/4} \\ &= -4q^{-1/4} \Big(\prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} \Big) \Big(\frac{q}{1+q} - \frac{3q^4}{1+q^3} + \frac{5q^9}{1+q^5} - \frac{7q^{16}}{1+q^7} + \frac{9q^{25}}{1+q^9} - ... \Big). \end{aligned}$$

Proof. We can use the exact formula for the coefficients given by Bruinier and Kuss [18], theorem 4.8: for an odd-dimensional lattice of dimension e, the coefficient $c(n,\gamma)$ of $E_k(\tau)$ is given by

$$\frac{(2\pi)^k n^{k-1} (-1)^{b^+/2} L(k-1/2, \chi_{\mathcal{D}})}{\sqrt{|\Lambda'/\Lambda|}\Gamma(k)\zeta(2k-1)} \Big[\sum_{d|f} \mu(d) \chi_{\mathcal{D}}(d) d^{1/2-k} \sigma_{2-2k}(f/d) \Big] \times$$
$$\times \prod_{p|(2|\Lambda'/\Lambda|)} \Big[\frac{1-p^{e/2-k}}{1-p^{1-2k}} L_p(n,\gamma,k+e/2-1) \Big].$$

(We do not need the assumption that $k = e/2$ because this is only used in the computations of local factors L_p in [18]; here we are working with a different expression.) Here, $\mathcal{D}$ is a discriminant defined in theorem 4.5 of [18], and $\chi_{\mathcal{D}}(d) = \left(\frac{\mathcal{D}}{d}\right)$ is the Kronecker symbol, and f^2 is the largest square dividing n that is coprime to $2 \cdot |\Lambda'/\Lambda|$. For the lattice $\mathbb{Z}$ with quadratic form $Q(x) = x^2$ (where $E_{3/2}$ is Zagier's mock Eisenstein series), and $\gamma = 1/2$ and $n \in \mathbb{Z} - Q(\gamma)$, it is not hard to see that the local factor at $p = 2$ is

$$L_2(n,\gamma,s) = \begin{cases} 1: & 4n \equiv 3 \,(\mathrm{mod} 8); \\ (2^s+1)/(2^s-1): & 4n \equiv 7 \,(\mathrm{mod} 8); \end{cases}$$

since n always has valuation -2 modulo $p = 2$, resulting in the values

$$(1 - 2^{-1})L_2(n,\gamma,1) = \begin{cases} 1/2: & 4n \equiv 3 \bmod 8; \\ 3/2: & 4n \equiv 7 \bmod 8. \end{cases}$$

On the other hand, for the lattice $\mathbb{Z}^3$ with quadratic form $Q(x,y,z) = x^2 - y^2 + z^2$, the local factor is always

$$L_2(n,\gamma,s) = \frac{2^s}{2^s - 4}$$

with $\lim_{s\to 0}(1 - 2^{-2s})L_2(n,\gamma,2+2s) = 1$. Since all other terms in the formula are the same between the two lattices (other than an extra factor of $1/2$ from $\frac{1}{\sqrt{|\Lambda'/\Lambda|}}$), and the coefficient of $q^{n/4}$ in Zagier's mock Eisenstein series is $-12H(n)$, we get the claimed formula. □

In example 63 of section 4.4 we saw that the real-analytic correction of $E_{3/2}(\tau)$ for the lattice $\tilde{\Lambda}$ is

$$E^*_{3/2}(\tau,0) = E_{3/2}(\tau) + \frac{1}{\sqrt{y}} \sum_{\gamma \in \tilde{\Lambda}'/\tilde{\Lambda}} \sum_{\substack{n \in \mathbb{Z} - Q(\gamma) \\ n \leq 0}} a(n,\gamma)\beta(-4\pi n y) q^n \mathfrak{e}_\gamma$$

with shadow

$$\begin{aligned} &\sum_{\gamma,n} a(-n,\gamma) q^n \mathfrak{e}_\gamma \\ = &-8\Big(1 + 2q + 2q^4 + ...\Big)(2\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/2,1/2,0)} + \mathfrak{e}_{(0,1/2,1/2)}) - \\ &- 8\Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(2\mathfrak{e}_{(1/2,1/2,1/2)} + \mathfrak{e}_{(0,0,1/2)} + \mathfrak{e}_{(1/2,0,0)}). \end{aligned}$$

Therefore the constants of section 5 for square $r^2 - 4n$ are

$$A(n,r,0) = \begin{cases} -16: & r^2 - 4n \text{ odd or zero;} \\ -32: & r^2 - 4n \text{ even and nonzero.} \end{cases}$$

The formula for the Poincaré square series of index $(1,0)$ implies that the coefficient of $q^n \mathfrak{e}_{(0,0)}$ in $Q_{2,1,0}$ is

$$\sum_{r=-\infty}^{\infty} -|\overline{\alpha}(n-r^2)| - (8 - 4(-1)^n) \sum_{r \text{ odd}} H(4n - r^2) +$$
$$+ \frac{1}{8} \sum_{\substack{r \in \mathbb{Z} \\ r^2 - 4n = \square}} A(n,r,0)\Big(|r| - \sqrt{r^2 - 4n}\Big) + \begin{cases} 4: & n = \square; \\ 0: & \text{otherwise.} \end{cases}$$

The additional 4 at the end if n is square is due to the constant term in the mock Eisenstein series $E_{3/2}$ being 1 rather than -1:

$$E_{3/2}(\tau)_{(0,0,0)} = 1 - \sum_{n=1}^{\infty} |\overline{\alpha}(n)| q^n = -\sum_{n=0}^{\infty} |\overline{\alpha}(n)| + 2,$$

and because we use the convention $\overline{\alpha}(0) = 1$. As before, $\frac{|r| - \sqrt{r^2-4n}}{2}$ takes exactly the values $\min(d, n/d)$ for divisors d of n (but counts $\sqrt{n}$ twice if n is square); and one can show that if n is odd and $r^2 - 4n$ is square, then $r^2 - 4n$ is always even, while if n is even, then $r^2 - 4n$ is even exactly when the divisor $d = \frac{|r| - \sqrt{r^2-4n}}{2}$ and n/d are both even.

Denote $\lambda_1(n) = \frac{1}{2} \sum_{d|n} \min(d, n/d)$ as in [47]. Comparing coefficients with the Eisenstein series $E_2(\tau)_{(0,0)}$ gives the following formula:

Proposition 95. *If $n \in \mathbb{N}$ is odd, then $\sum_{r=-\infty}^{\infty} |\overline{\alpha}(n-r^2)|$ equals*

$$-16\lambda_1(n) + 16\sigma_1(n) - 12 \sum_{r \text{ odd}} H(4n - r^2) + \begin{cases} 4: & n = \square; \\ 0: & \text{otherwise.} \end{cases}$$

If $n \in \mathbb{N}$ is even, then $\sum_{r=-\infty}^{\infty} |\overline{\alpha}(n-r^2)|$ equals

$$-8\lambda_1(n) - 16\lambda_1(n/4) + 24\sigma_1(n/2) - 4 \sum_{r \text{ odd}} H(4n - r^2) + \begin{cases} 4: & n = \square; \\ 0: & \text{otherwise.} \end{cases}$$

Here, we set $\lambda_1(n/4) = 0$ if n is not divisible by 4, and $\overline{\alpha}(n) = H(n) = 0$ for $n < 0$. Note that this can also be expressed as a relation among Hurwitz class numbers since $|\overline{\alpha}(n)|$ itself can be written in terms of Hurwitz class numbers, as observed in corollary 1.2 of [8].

5.9 Example: computing an obstruction space for Borcherds products

The interpretation of certain Borcherds products for $O(2,2)$ as Hilbert modular forms is well-known and described in detail in Bruinier's lectures [14], in particular section 3.2. For a fundamental discriminant $m \equiv 1 \mod 4$, the relevant obstruction space for Hilbert modular forms for the field $\mathbb{Q}(\sqrt{m})$ consists of weight-two modular forms for the dual Weil representation attached to the Gram matrix $S = \begin{pmatrix} 2 & 1 \\ 1 & -\frac{m-1}{2} \end{pmatrix}$. The smallest example where this space contains cusp forms is $m = 21$.

It is not very difficult to compute a basis of this space by other means but one can also use the functions described here to do this. The Eisenstein series is a true modular form whose coefficients up to order $O(q^4)$ can be computed as

$$\begin{aligned}
E_2(\tau) = &(1 - 6q - 12q^2 - 40q^3 - ...)\mathfrak{e}_0 \\
&+ q^{1/21}(-1/2 - 5q - 22q^2 - 43/2q^3 - ...)\sum_{Q(\gamma)=-1/21+\mathbb{Z}} \mathfrak{e}_\gamma \\
&+ q^{4/21}(-3/2 - 31/2q - 11q^2 - 34q^3 - ...)\sum_{Q(\gamma)=-4/21+\mathbb{Z}} \mathfrak{e}_\gamma \\
&+ q^{1/3}(-4 - 12q - 25q^2 - 18q^3 - ...)(\mathfrak{e}_{(1/3,1/3)} + \mathfrak{e}_{(2/3,2/3)}) \\
&+ q^{3/7}(-5 - 6q - 36q^2 - 20q^3 - ...)(\mathfrak{e}_{(-1/7,2/7)} + \mathfrak{e}_{(1/7,-2/7)}) \\
&+ q^{5/7}(-12 - 15q - 18q^2 - 24q^3 - ...)(\mathfrak{e}_{(2/7,3/7)} + \mathfrak{e}_{(5/7,4/7)}) \\
&+ q^{16/21}(-11/2 - 19q - 14q^2 - 40q^3 - ...)\sum_{Q(\gamma)=-16/21+\mathbb{Z}} \mathfrak{e}_\gamma \\
&+ q^{18/21}(-4 - 12q - 36q^2 - 41q^3 - ...)(\mathfrak{e}_{(3/7,1/7)} + \mathfrak{e}_{(4/7,6/7)})
\end{aligned}$$

and the cusp space is represented by

$$\begin{aligned}
&7(Q_{2,16/21,(10/21,1/21)} - E_2) \\
= &q^{1/21}(1 - 14q + 12q^2 + ...)(\mathfrak{e}_{(8/21,-16/21)} + \mathfrak{e}_{(-8/21,16/21)} - \mathfrak{e}_{(-1/21,2/21)} - \mathfrak{e}_{(1/21,-2/21)}) \\
+ &q^{4/21}(5 - 5q + 14q^2 + ...)(\mathfrak{e}_{(-2/21,4/21)} + \mathfrak{e}_{(2/21,-4/21)} - \mathfrak{e}_{(16/21,10/21)} - \mathfrak{e}_{(5/21,11/21)}) \\
+ &q^{16/21}(11 + 6q - 28q^2 + ...)(\mathfrak{e}_{(10/21,1/21)} + \mathfrak{e}_{(11/21,-1/21)} - \mathfrak{e}_{(-4/21,8/21)} - \mathfrak{e}_{(4/21,-8/21)}).
\end{aligned}$$

Following [5], there exists a vector-valued nearly-holomorphic modular form of weight k for the Weil representation ρ with principal part $\sum_\gamma \sum_{n<0} a(n,\gamma)q^n \mathfrak{e}_\gamma + a(0)\mathfrak{e}_0$ if and only if $a(n,\gamma) = a(n,-\gamma)$ and $\sum_{n<0} a(n,\gamma)b(-n,\gamma) = 0$ for all (true) modular forms $\sum_{\gamma,n} b(n,\gamma)q^n \mathfrak{e}_\gamma$ of weight $2-k$ for the dual representation ρ^*. From this principle and the above computations, we find that the following principal parts extend to nearly-holomorphic modular forms f_i:

$$\begin{aligned}
f_1(\tau) &= q^{-1/21}(\mathfrak{e}_{(-1/21,2/21)} + \mathfrak{e}_{(1/21,-2/21)} + \mathfrak{e}_{(8/21,-16/21)} + \mathfrak{e}_{(-8/21,16/21)}) + 2\mathfrak{e}_{(0,0)} + ...\\
f_4(\tau) &= q^{-4/21}(\mathfrak{e}_{(-2/21,4/21)} + \mathfrak{e}_{(2/21,-4/21)} + \mathfrak{e}_{(-5/21,10/21)} + \mathfrak{e}_{(5/21,-10/21)}) + 6\mathfrak{e}_{(0,0)} + ...\\
f_7(\tau) &= q^{-7/21}(\mathfrak{e}_{(1/3,1/3)} + \mathfrak{e}_{(2/3,2/3)}) + 8\mathfrak{e}_{(0,0)} + ...\\
f_9(\tau) &= q^{-9/21}(\mathfrak{e}_{(-1/7,2/7)} + \mathfrak{e}_{(1/7,-2/7)}) + 10\mathfrak{e}_{(0,0)} + ...\\
f_{15}(\tau) &= q^{-15/21}(\mathfrak{e}_{(2/7,-4/7)} + \mathfrak{e}_{(-2/7,4/7)}) + 24\mathfrak{e}_{(0,0)} + ...\\
f_{16}(\tau) &= q^{-16/21}(\mathfrak{e}_{(10/21,-20/21)} + \mathfrak{e}_{(-10/21,20/21)} + \mathfrak{e}_{(-4/21,8/21)} + \mathfrak{e}_{(4/21,-8/21)}) + 22\mathfrak{e}_{(0,0)} + ...\\
f_{18}(\tau) &= q^{-18/21}(\mathfrak{e}_{(3/7,1/7)} + \mathfrak{e}_{(4/7,6/7)}) + 8\mathfrak{e}_{(0,0)} + ...\\
f_{21}(\tau) &= q^{-1}\mathfrak{e}_{(0,0)} + 6\mathfrak{e}_{(0,0)} + ...
\end{aligned}$$

These inputs produce holomorphic Borcherds products

$$\psi_1, \psi_3^{(1)}, \psi_3^{(2)}, \psi_4^{(1)}, \psi_4^{(2)}, \psi_5, \psi_{11}, \psi_{12}$$

as Hilbert modular forms for $\mathbb{Q}(\sqrt{21})$, each ψ_k having weight k and only simple zeros. The principal parts above are enough to determine the divisors and weights of these products, using theorem 13.3 of [4], and this is enough for some applications. To calculate the products explicitly, one needs to compute the coefficients of higher powers of q in the input functions $f_i(\tau)$. One way to do this algorithmically is by identifying $\Delta(\tau)f_i(\tau)$ in $M_{12}(\rho)$ using the algorithm of [68], where $\Delta(\tau)$ is the discriminant; this is a messy but straightforward computation.

Chapter 6

Vector-valued Hirzebruch-Zagier series and class number sums

This chapter is taken from the paper [72].

6.1 Introduction

The **Hurwitz class numbers** $H(n)$ are essentially the class numbers of imaginary quadratic fields. To be more specific, if $-D$ is a fundamental discriminant then

$$H(D) = \frac{2h(D)}{w(D)}$$

where $h(D)$ is the class number of $\mathbb{Q}(\sqrt{-D})$ and $w(D)$ is the number of units in its ring of integers (in particular, $w(D) = 2$ for $D \neq 3, 4$). More generally,

$$H(n) = \frac{2h(D)}{w(D)} \sum_{d|f} \mu(d) \left(\frac{D}{d}\right) \sigma_1(f/d)$$

if $-n = Df^2$, where D is the discriminant of $\mathbb{Q}(\sqrt{-n})$, and μ is the Möbius function, σ_1 is the divisor sum, and $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol; and by convention one sets $H(0) = -\frac{1}{12}$ and $H(n) = 0$ whenever $n \equiv 1, 2 \pmod 4$. Hurwitz class numbers have natural interpretations in terms of equivalence classes of binary quadratic forms or orders in imaginary quadratic fields. We refer to section 5.3 of [22] for more details.

Many identities are known to hold between Hurwitz class numbers, the prototypical identity being the Kronecker-Hurwitz relation

$$\sum_{r \in \mathbb{Z}} H(4n - r^2) = \sum_{d|n} \max(d, n/d),$$

where we set $H(n) = 0$ if $n < 0$. These and other identities have interpretations in the theory of modular forms. The most influential result in this area is probably Hirzebruch and Zagier's discovery [38] that for any prime $p \equiv 1 \, (\mathrm{mod}\, 4)$ the sums

$$H_p(n) = \sum_{4n - r^2 \equiv 0 \, (p)} H\left(\frac{4n - r^2}{p}\right)$$

can be corrected to the coefficients of a modular form of weight 2 and level $\Gamma_0(p)$ and Nebentypus $\chi(n) = \left(\frac{p}{n}\right)$, and that these corrected coefficients can be interpreted as intersection numbers of curves on Hilbert modular surfaces. (The construction of the modular form there also goes through when p is replaced by the discriminant of a real-quadratic number field.) The paper [38] is a pioneering use of what are now called mock modular forms. Related techniques have turned out to be effective at deriving other identities among class numbers (among many other things); the papers [6], [47], [48] are some examples of this.

As observed by Bruinier and Bundschuh [15], there are isomorphisms between the spaces of vector-valued modular forms that transform with the Weil representation attached to a lattice of prime discriminant p and a plus- or minus-subspace (depending on the signature of the lattice) of scalar modular forms of level $\Gamma_0(p)$ and Nebentypus. We prove here that up to a constant factor, the Hirzebruch-Zagier series of level p mentioned above corresponds to a Poincaré square series of index $1/p$ (in the sense of [69]; see also section 2) by computing the latter series directly. One feature of this construction is that p being prime or even a fundamental discriminant is irrelevant: the construction holds and produces a modular form attached to a quadratic form of discriminant m for arbitrary $m \equiv 0, 1 \, (\mathrm{mod}\, 4)$ whose coefficients are corrections of the class number sums $\sum_r H(4n - mr^2)$. (However, if m is a perfect square then it will produce a quasimodular form similar to the classical Eisenstein series of weight 2, rather than a true modular form.) It seems natural to call these vector-valued functions **Hirzebruch-Zagier series** as well.

Our construction starts with a nonholomorphic vector-valued Jacobi Eisenstein series $E^*_{2,1/m,\beta}(\tau, z, s; Q)$ of weight 2 and index $1/m$ whose Fourier coefficients involve the expressions $H(4n - mr^2)$. (Jacobi forms of fractional index are acceptable when the Heisenberg group also acts through a nontrivial representation.) The action through the Petersson scalar product of the value of the Jacobi Eisenstein series at $z = 0$ is straightforward to describe using the usual unfolding argument (e.g. [12], section 1.2.2) for large enough $\mathrm{Re}(s)$, and it follows for all s by analytic continuation. We construct the Hirzebruch-Zagier series by projecting the zero value $E^*_{2,1/m,\beta}(\tau, 0, 0; Q)$ orthogonally into the space of cusp forms and then adding the Eisenstein series; neither of these processes change the value of its Petersson scalar product with any cusp form, so this construction makes the behavior of the Hirzebruch-Zagier series with respect to the Petersson scalar product clear for arbitrary m. (In the case m is prime, this was left as a conjecture at the end of [38]). This method of constructing holomorphic modular forms from real-analytic forms is **holomorphic projection** and it remains valid for vector-valued modular forms (see also [42]).

For small values of m, there are several examples where the Hirzebruch-Zagier series equals Bruinier's Eisenstein series of weight 2. By comparing coefficients that are chosen to make the correction term in the Hirzebruch-Zagier series vanish, one can find several identities relating $\sum_r H(4n - mr^2)$ to a twisted divisor sum. A typical example is

$$\sum_{r \in \mathbb{Z}} H(4n - 3r^2) = \frac{5}{6} \sigma_1(n, \chi_{12}), \quad n \equiv 7 \,(\mathrm{mod}\, 12),$$

where $\chi_m(n) = \left(\frac{m}{n}\right)$ is the Kronecker symbol and where

$$\sigma_1(n, \chi_m) = \sum_{d|n} d\chi_m(n/d).$$

Additionally, by taking $m = d^2 \in \{4, 9, 25, 49\}$ we give another derivation for identities involving sums of the form $\sum_{r \equiv a\,(d)} H(4n - r^2)$ which were considered in [6], [11].

6.2 The case $m \equiv 1 \mod 4$

Fix any number $m \equiv 1 \mod 4$ and consider the quadratic form $Q(x, y) = x^2 + xy - \frac{m-1}{4} y^2$ of discriminant m. There is a unique pair of elements $\pm\beta \in A$ in the associated discriminant form with $Q(\beta) = 1 - \frac{1}{m}$; they are represented by $\pm(-1/m, 2/m)$. (In particular, the discriminant form is cyclic and these elements are generators, and Q takes values in $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$.)

We will also consider the ternary quadratic form

$$\mathbf{Q}(x,y,z) = Q(x,y) + 2xz + z^2,$$

which has discriminant

$$\mathrm{discr}(\mathbf{Q}) = \det \begin{pmatrix} 2 & 1 & 2 \\ 1 & -\frac{m-1}{2} & 0 \\ 2 & 0 & 2 \end{pmatrix} = -2.$$

Comparing the coefficient formulas for the Jacobi Eisenstein series of index $1/m$ ([69], section 3) and the usual Eisenstein series of [18], (for which the correction in weight 3/2 was worked out in [70]), we see that the coefficient of $q^n \zeta^r$ in $E^*_{2,1/m,\beta}(\tau,z,s;Q)$ equals the coefficient of $q^{n-mr^2/4}$ in $E^*_{3/2}(\tau,s;\mathbf{Q})$. To be more precise, we should consider the coefficients of $q^n\zeta^r\mathfrak{e}_\gamma$ for elements $\gamma \in A$ instead; however, the condition $r \in \mathbb{Z} - \langle \gamma,\beta\rangle$ determines $\langle\gamma,\beta\rangle \in \mathbb{Q}/\mathbb{Z}$, and due to our choice of β this determines γ uniquely. Both coefficient formulas involve zero-counts of quadratic polynomials modulo prime powers and $\mathbf{Q}$ is chosen to make these zero-counts equal; specifically, for all $\gamma \in A$ and $n \in \mathbb{Z} - Q(\gamma)$, $r \in \mathbb{Z} - \langle\gamma,\beta\rangle$,

$$\begin{aligned} &\#\{(v,\lambda) \in \mathbb{Z}^3 \bmod p^k : \; Q(v+\lambda\beta-\gamma) + \lambda^2/m - r\lambda + n \equiv 0\} \\ = &\#\{v \in \mathbb{Z}^3 \bmod p^k : \; \mathbf{Q}(v-\gamma_r) + (n - mr^2/4) \equiv 0\}, \end{aligned}$$

where $\gamma_r = \left(\gamma - \frac{rm}{2}\beta, \frac{rm}{2}\right) \in (\mathbb{Q}/\mathbb{Z})^3$ lies in the dual lattice and $n - \frac{mr^2}{4} \in \mathbb{Z} - \mathbf{Q}(\gamma_r)$. On the level of matrices, letting $S = \begin{pmatrix} 2 & 1 \\ 1 & -\frac{m-1}{2} \end{pmatrix}$ be the Gram matrix of Q, this follows because the Gram matrix of $\mathbf{Q}$ has block form

$$\mathrm{Gram}(\mathbf{Q}) = \begin{pmatrix} 2 & 1 & 2 \\ 1 & -\frac{m-1}{2} & 0 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} S & S\beta \\ \beta^T S & 2(\frac{1}{m} + \beta^T S\beta) \end{pmatrix}$$

for the representative $\beta = (\frac{m-1}{m}, \frac{2}{m}) \in \mathbb{Q}^2$. (See also remark 30 of chapter 3.)

Since $|\mathrm{discr}(\mathbf{Q})| = 2$ and $\mathrm{sig}(\mathbf{Q}) = 1$, the discriminant form of $\mathbf{Q}$ is isomorphic to that of x^2 and so the nonholomorphic weight 3/2 Eisenstein series attached to it is the Zagier Eisenstein series in which the coefficient of q^n is $H(4n)$. Evaluating at $s = 0$ and using the previous paragraph, we find

$$\begin{aligned}&E^*_{2,1/m,\beta}(\tau,z,0;Q)\\&=-12\sum_{\gamma\in A}\sum_{n\in\mathbb{Z}-Q(\gamma)}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}H(4n-mr^2)q^n\zeta^r\mathfrak{e}_\gamma+\\&\quad+\frac{1}{\sqrt{y}}\sum_{\gamma\in A}\sum_{n\in\mathbb{Z}-Q(\gamma)}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}A(n,r,\gamma)\beta(\pi y(mr^2-4n))q^n\zeta^r\mathfrak{e}_\gamma,\end{aligned}\tag{6.1}$$

where the coefficients $A(n,r,\gamma)$ are given by

$$A(n,r,\gamma)=\begin{cases}-24: & mr^2=4n;\\ -48: & mr^2-4n \text{ is a nonzero square};\\ 0: & \text{otherwise.}\end{cases}$$

The main result of chapter 5 is a coefficient formula for the Poincaré square series $Q_{2,d,\beta}(\tau)$ for any $\beta\in A$ and $d\in\mathbb{Z}-Q(\beta)$. If

$$\begin{aligned}E^*_{2,d,\beta}(\tau,0,0;Q)=&\sum_{\gamma\in A}\sum_{n\in\mathbb{Z}-Q(\gamma)}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}c(n,r,\gamma)q^n\zeta^r\mathfrak{e}_\gamma\\&+\frac{1}{\sqrt{y}}\sum_{\gamma\in A}\sum_{n\in\mathbb{Z}-Q(\gamma)}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}A(n,r,\gamma)\beta(\pi y(r^2/d-4n))q^n\zeta^r\mathfrak{e}_\gamma,\end{aligned}$$

then the coefficient $C(n,\gamma)$ of $q^n\mathfrak{e}_\gamma$ in $Q_{2,d,\beta}$ is

$$C(n,\gamma)=\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}c(n,r,\gamma)+\frac{1}{8\sqrt{d}}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}\Big[A(n,r,\gamma)\Big(|r|-\sqrt{r^2-4dn}\Big)\Big].$$

In our case where $d=1/m$, these coefficients are

$$\begin{aligned}&-12\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}H(4n-mr^2)-6\sqrt{m}\sum_{\substack{r\in\mathbb{Z}-\langle\gamma,\beta\rangle\\ mr^2-4n=\square}}\Big(|r|-\sqrt{r^2-4n/m}\Big)+\\&+\begin{cases}12\sqrt{n}: & \exists r\in\mathbb{Z}-\langle\gamma,\beta\rangle \text{ with } mr^2=4n;\\ 0: & \text{otherwise.}\end{cases}\end{aligned}\tag{6.2}$$

Note that $\frac{m}{2}\Big(|r|-\sqrt{r^2-4n/m}\Big)$ is always an algebraic integer when mr^2-4n is square: if m itself is square then this is clear, and otherwise its conjugate is $\frac{m}{2}\Big(|r|+\sqrt{r^2-4n/m}\Big)$, so its trace is $m|r| \in m\mathbb{Z} \pm m\langle\gamma,\beta\rangle \subseteq \mathbb{Z}$ and its norm is $mn \in m\mathbb{Z} - mQ(\gamma) \subseteq \mathbb{Z}$. Conversely, if m is squarefree then

$$\frac{m}{2}\Big(|r| - \sqrt{r^2 - 4n/m}\Big), \;\; r \in \mathbb{Z} \pm \langle\gamma,\beta\rangle, \; \gamma \in A$$

runs exactly through the values taken by $\min(\lambda,\overline{\lambda})$, where λ is a positive integer of $\mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{m})$ with positive conjugate $\overline{\lambda}$ that has norm mn; however it double-counts $\sqrt{mn}$ if $r = \pm\sqrt{4n/m}$ occur in the sum. This allows one to remove the additional term $12\sqrt{n}$ in the formula (6.2). In some sense this remains true for $m = 1$ (with trivial discriminant form): then $\frac{1}{2}(|r| - \sqrt{r^2 - 4n})$, $r \in \mathbb{Z}$ takes the values $\min(d, n/d)$ where d runs through divisors of n in $\mathbb{Z}$ but it double-counts $\sqrt{n}$ if n is square.)

In this way, we obtain for $m = 1$ an identity equivalent to the Kronecker-Hurwitz relations:

$$\begin{aligned} E_2(\tau) &= 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n \\ &= Q_{2,1,0}(\tau) = 1 - 12\sum_{n\in\mathbb{Z}} \Big(\sum_{d|n} \min(d, n/d) + \sum_{r\in\mathbb{Z}} H(4n - r^2)\Big) q^n, \end{aligned}$$

whereas if $m = p$ is a prime, we obtain a vector-valued form of the Hirzebruch-Zagier series:

$$Q_{2,1/p,\beta}(\tau) = 1 - 12\sum_{\gamma,n} \varepsilon_\gamma \Big(\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle} H(4n - pr^2) + \frac{1}{\sqrt{p}} \sum_{\substack{\lambda\in\mathcal{O}_K \\ \lambda\gg 0 \\ \lambda\overline{\lambda}=pn}} \min(\lambda,\overline{\lambda})\Big) q^n \mathfrak{e}_\gamma,$$

where $K = \mathbb{Q}(\sqrt{p})$, and $\overline{\lambda}$ is the conjugate of λ, and $\lambda \gg 0$ means that both $\lambda, \overline{\lambda}$ are positive, and finally we set $\varepsilon_\gamma = 1$ if $\gamma = 0$ and $\varepsilon_\gamma = 1/2$ otherwise. (The factors ε_γ come from the fact that relating the sum over r to a divisor sum requires both congruences $r \in \mathbb{Z} \pm \langle\gamma,\beta\rangle$; but the coefficients $c(n,\gamma)$ of any modular form satisfy $c(n,\gamma) = c(n,-\gamma)$ by our assumption $2k + \mathrm{sig}(Q) \equiv 0\,(4)$ on their weight.) As shown by Bruinier and Bundschuh [15], there is an identification between modular forms attached to quadratic forms of prime discriminant and the plus space of modular forms with Nebentypus which is given essentially given by summing together all components and replacing n by n/p in the coefficient formula. It is not difficult to see that the image of $Q_{2,1/p,\beta}$ under this identification is

$$-12\varphi_p = 1 - 12\sum_{n\in\mathbb{Z}} \Big[\sum_{\substack{r\in\mathbb{Z} \\ 4n-r^2\equiv 0\,(p)}} H\Big(\frac{4n - r^2}{p}\Big) + \frac{1}{\sqrt{p}} \sum_{\substack{\lambda\in\mathcal{O}_K \\ \lambda\gg 0 \\ \lambda\overline{\lambda}=n}} \min(\lambda,\overline{\lambda})\Big] q^n \in M_2(\Gamma_0(p),\chi),$$

where φ_p is the function of Hirzebruch and Zagier's paper [38].

6.3 The case $m \equiv 0 \mod 4$

Our procedure in this case is nearly the same, but we consider instead the quadratic form $Q(x,y) = x^2 - \frac{m}{4}y^2$ of discriminant m. There is again an element $\beta \in A$ with $Q(\beta) = 1 - \frac{1}{m}$; in this case, one can choose the representative $(0, 2/m)$. (Note that this discriminant form is not cyclic. Also, β is not necessarily unique; but any other choice of β will give a similar result.) We also consider the ternary quadratic form $\mathbf{Q}(x,y,z) = Q(x,y) - yz$ which has discriminant

$$\mathrm{discr}(\mathbf{Q}) = \det \begin{pmatrix} 2 & 0 & 0 \\ 0 & -m/2 & -1 \\ 0 & -1 & 0 \end{pmatrix} = -2.$$

Comparing coefficient formulas between $E_2^*(\tau, z, s; Q)$ and $E_{3/2}^*(\tau; \mathbf{Q})$ gives exactly the same formula as equation (6.1) in the previous section:

$$E^*_{2,1/m,\beta}(\tau, z, 0; Q) = -12 \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} H(4n - mr^2) q^n \zeta^r \mathfrak{e}_\gamma +$$
$$+ \frac{1}{\sqrt{y}} \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} A(n, r, \gamma) \beta(\pi y (mr^2 - 4n)) q^n \zeta^r \mathfrak{e}_\gamma,$$

and therefore the same coefficient formula from the previous section again produces a modular form for Q.

6.4 Formulas for class number sums

In this section we compute values of m where the relevant space of weight 2 modular forms is one-dimensional and therefore the Hirzebruch-Zagier series $Q_{2,1/m,\beta}$ equals the Eisenstein series. We obtain formulas for class number sums by considering those exponents n for which the corrective term

$$-6\sqrt{m} \sum_{mr^2 - 4n = \square} (|r| - \sqrt{r^2 - 4n/m})$$

above vanishes. First we will list the numbers m for which the cusp space $S_2(\rho^*)$ attached the quadratic forms we considered above vanishes.

Lemma 96. *(i) Suppose $m \equiv 1$ mod 4 and let $Q(x,y) = x^2 + xy - \frac{m-1}{4}y^2$. Then the cusp space $S_2(\rho^*)$ vanishes if and only if $m \leq 25$.*
(ii) Suppose $m \equiv 0$ mod 4 and let $Q(x,y) = x^2 - \frac{m}{4}y^2$. Then the cusp space $S_2(\rho^*)$ vanishes if and only if $m \leq 20$.

Proof. Table 7 of [20] lists the genus symbols of all discriminant forms of signature 0 mod 8 with at most four generators for which the space of weight two cusp forms vanishes. We only need to find the values of m for which the discriminant form of Q appears in their table. □

In particular, when $m \leq 21$ or $m = 25$, due to the lack of cusp forms the Hirzebruch-Zagier series $Q_{2,1/m,\beta}(\tau;Q)$ equals the Eisenstein series $E_2(\tau;Q)$ in which the coefficient of $q^n \mathfrak{e}_\gamma$ is a multiple of the twisted divisor sum

$$\sigma_1(nd_\gamma^2, \chi_m) = \sum_{d|nd_\gamma^2} d \cdot \left(\frac{m}{nd_\gamma^2/d}\right),$$

where d_γ is the denominator of γ (i.e. the smallest number such that $d_\gamma \gamma = 0$ in A) and $\chi_m = \left(\frac{m}{-}\right)$ is the quadratic character attached to $\mathbb{Q}(\sqrt{m})$; and these multiples are constant when n is restricted to certain congruence classes. This leads to numerous identities relating class number sums of the form $\sum_{r \in \mathbb{Z} - \langle \gamma,\beta\rangle} H(4n - mr^2)$ (even in some cases where n is not integral!) to twisted divisor sums.

The simplest identities arise by comparing the components of $\mathfrak{e}_0$ in both series and restricting to odd integers n for which in addition $\chi_m(n) = -1$ (which is never true for $m = 1, 4, 9, 16, 25$); in these cases, the "correction term" in $Q_{2,1/m,\beta}(\tau;Q)$ vanishes and its coefficient of $q^n \mathfrak{e}_0$ is $-12\sum_{r\in\mathbb{Z}} H(4n - mr^2)$. This is then a constant multiple of $\sigma_1(n, \chi_m)$ depending on the remainder of n mod m. The constant multiple can be computed by studying the formula of [18] carefully but it is easier to compute by plugging in just one value of n. We list the results one can obtain with this argument:

(1) $m = 5$: for $n \equiv 3, 7 \pmod{10}$,

$$\sum_{r\in\mathbb{Z}} H(4n - 5r^2) = \frac{5}{3}\sigma_1(n, \chi_5).$$

(2) $m = 8$: for $n \equiv 3, 5 \pmod 8$,

$$\sum_{r\in\mathbb{Z}} H(4n - 2r^2) = \sum_{r\in\mathbb{Z}} H(4n - 8r^2) = \frac{7}{6}\sigma_1(n, \chi_8).$$

(3) $m = 12$: for $n \equiv 5 \,(\mathrm{mod}\, 12)$,

$$\sum_{r \in \mathbb{Z}} H(4n - 3r^2) = \sum_{r \in \mathbb{Z}} H(4n - 12r^2) = \sigma_1(n, \chi_{12}),$$

and for $n \equiv 7 \,(\mathrm{mod}\, 12)$,

$$\sum_{r \in \mathbb{Z}} H(4n - 3r^2) = \sum_{r \in \mathbb{Z}} H(4n - 12r^2) = \frac{5}{6}\sigma_1(n, \chi_{12}).$$

(4) $m = 13$: for $n \equiv 5, 7, 11, 15, 19, 21 \,(\mathrm{mod}\, 26)$,

$$\sum_{r \in \mathbb{Z}} H(4n - 13r^2) = \sigma_1(n, \chi_{13}).$$

(5) $m = 17$: for $n \equiv 3, 5, 7, 11, 23, 27, 29, 31 \,(\mathrm{mod}\, 34)$,

$$\sum_{r \in \mathbb{Z}} H(4n - 17r^2) = \frac{2}{3}\sigma_1(n, \chi_{17}).$$

(6) $m = 20$: for $n \equiv 3, 7, 13, 17 \,(\mathrm{mod}\, 20)$,

$$\sum_{r \in \mathbb{Z}} H(4n - 20r^2) = \frac{2}{3}\sigma_1(n, \chi_{20}).$$

(7) $m = 21$: for $n \equiv 11, 23, 29 \,(\mathrm{mod}\, 42)$,

$$\sum_{r \in \mathbb{Z}} H(4n - 21r^2) = \sigma_1(n, \chi_{21})$$

and for $n \equiv 13, 19, 31 \,(\mathrm{mod}\, 42)$,

$$\sum_{r \in \mathbb{Z}} H(4n - 21r^2) = \frac{2}{3}\sigma_1(n, \chi_{21}).$$

Remark 97. There are some values of m where the Eisenstein series does not equal the Hirzebruch-Zagier series (one should not expect it to when $\dim S_2(\rho^*) > 0$) but where one can still obtain some information by comparing coefficients within arithmetic progressions, yielding more identities than those above. (Note that if $f(\tau) = \sum_{n \in \mathbb{Z}} c(n) q^n$ is a modular form of some level N, then restricting to an arithmetic progression produces a modular form $\sum_{n \equiv r \,(\mathrm{mod}\, d)} c(n) q^n$ of the same weight and level Nd^2; so one can always check whether the coefficients of two modular forms agree in an arithmetic progression by computing finitely many coefficients.) In particular, we do not claim that the list of m above where $\sum_{r\in\mathbb{Z}} H(4n-mr^2)$ can be related to $\sigma_1(n,\chi_m)$ is complete. The vector-valued setting is useful here because it lowers the Sturm bound considerably.

Some examples of this occur when $m = 24, 28, 32, 40$. For $m = 24$ it is not true that $E_2 = Q_{2,1/m,\beta}$; however, the $\mathfrak{e}_0$-components of these series have the same coefficients of q^n when $n \equiv 5, 7 \,(\mathrm{mod}\, 8)$. This can be proved by writing $f = E_2 - Q_{2,1/m,\beta}$ and considering the form $\sum_{k\in\mathbb{Z}/(24\cdot 8)\mathbb{Z}} f(\tau + k/8)\mathbf{e}(3k/8)$, which is a modular form for (a representation of) $\Gamma_1(64)$ all of whose coefficients vanish except for those of $q^n\mathfrak{e}_0$ with $n \equiv 5 \bmod 8$. To check that it vanishes identically we consider coefficients $n \equiv 5\,(8)$ up to the Sturm bound $\frac{2}{12}[SL_2(\mathbb{Z}) : \Gamma_1(64)] = 512$; this was done in SAGE. The case $n \equiv 7 \bmod 8$ is similar. Specializing to the n with $\left(\frac{24}{n}\right) = -1$, we obtain

$$\sum_{r\in\mathbb{Z}} H(4n-6r^2) = \sum_{r\in\mathbb{Z}} H(4n-24r^2) = \frac{1}{2}\sigma_1(n,\chi_{24})$$

for all $n \equiv 7, 13 \,(\mathrm{mod}\, 24)$.

For $m = 28$ the series have the same coefficients when $n \equiv 3, 5, 6 \,(\mathrm{mod}\, 7)$, and specializing to the n with $\left(\frac{28}{n}\right) = -1$ gives

$$\sum_{r\in\mathbb{Z}} H(4n-7r^2) = \sum_{r\in\mathbb{Z}} H(4n-28r^2) = \frac{1}{2}\sigma_1(n,\chi_{28})$$

for all $n \equiv 5, 13, 17 \,(\mathrm{mod}\, 28)$.

For $m = 32$ it is not true that $E_2 = Q_{2,1/m,\beta}$; however, the $\mathfrak{e}_0$-components are the same, and we find

$$\sum_{r\in\mathbb{Z}} H(4n-32r^2) = \frac{1}{2}\sigma_1(n,\chi_{32}),\ n \equiv 1 \,(\mathrm{mod}\, 4),$$
$$\sum_{r\in\mathbb{Z}} H(4n-32r^2) = \frac{2}{3}\sigma_1(n,\chi_{32}),\ n \equiv 3 \,(\mathrm{mod}\, 8)$$

by considering odd n for which $32r^2 - 4n = a^2$ is unsolvable in integers (a, r).

For $m = 40$ the series have the same coefficients when $n \equiv 3, 5 \,(\mathrm{mod}\, 8)$, and specializing to the n with $\left(\frac{40}{n}\right) = -1$ gives

$$\sum_{r\in\mathbb{Z}} H(4n-10r^2) = \sum_{r\in\mathbb{Z}} H(4n-40r^2) = \frac{1}{2}\sigma_1(n,\chi_{40})$$

for all $n \equiv 11, 19, 21, 29 \pmod{40}$.

However, there do not seem to be any such relations of this type for $m = 44$ (or indeed for "most" large enough m); in particular, $\sum_{r \in \mathbb{Z}} H(4n - 11r^2)$ is not obviously related to a twisted divisor sum within any congruence class mod 44.

6.5 Restricted sums of class numbers

Restrictions of the sums that occur in the Kronecker-Hurwitz relation to congruence classes, i.e. sums of the form

$$\sum_{r \equiv a\,(d)} H(4n - r^2),$$

have been evaluated in [6], [11] for $d = 2, 3, 5, 7$, where identities are obtained for all a and $d = 2, 3, 5$ and for some a when $d = 7$. These identities can be derived from the fact that the Hirzebruch-Zagier series equals the Eisenstein series when $m = 4, 9, 25$ and that some coefficients agree when $m = 49$, as we will show below. Here we need to compare coefficients of components $\mathfrak{e}_\gamma$ with $Q(\gamma) \in \mathbb{Z}$ but γ not necessarily zero. A somewhat stronger result in the case $m = 4$ was worked out in [71]; in the special case that n is odd, the result of [71] implies $\sum_{r \text{ odd}} H(4n - r^2) = \frac{2}{3}\sigma_1(n)$.

Therefore we will first consider the case $m = 9$ with quadratic form $Q(x, y) = x^2 + xy - 2y^2$. The elements $\gamma \in A$ of the associated discriminant group with $Q(\gamma) \in \mathbb{Z}$ are represented by

$$\gamma = (0, 0), (1/3, 1/3), (2/3, 2/3) \in (\mathbb{Q}/\mathbb{Z})^2$$

and their products with the element $\beta = (1/9, -2/9) \in A$ with $Q(\beta) = 1 - 1/9$ are respectively

$$\langle \gamma, \beta \rangle = 0, 1/3, 2/3 \bmod \mathbb{Z}.$$

We compute the Fourier coefficients $c(n, \gamma)$ of $E_2(\tau)$ using what is essentially theorem 4.8 of [18] in the form suggested in remark 56 of chapter 4: if $Q(\gamma) \in \mathbb{Z}$ then

$$\begin{aligned} c(n, \gamma) &= -\frac{4\pi^2}{3 \cdot L(2, \chi_{36})} \sigma_1(n, \chi_{36}) \cdot \prod_{p=2,3} \Big[(1 - p^{-1}) L_p(n, \gamma, 2) \Big] \\ &= 4 \Big(\sum_{\substack{d \mid n \\ 2,3 \nmid (n/d)}} d \Big) L_2(n, \gamma, 2) L_3(n, \gamma, 2). \end{aligned}$$

The local factor at 2 can be computed using the formula of section 3.8: letting $\kappa = v_2(n)$ we find

$$L_2(n,\gamma,s) = \frac{2^{(\kappa+1)s} + 2^{\kappa s} + 2^{(\kappa-1)s+1} + 2^{(\kappa-2)s+2} + ... + 2^{s+\kappa-1} - 2^{\kappa}}{2^{(\kappa+1)s} - 2^{\kappa s+1}}$$

and setting $s = 2$,

$$L_2(n,\gamma,2) = 3(1 - 2^{-\kappa-1}).$$

Similarly, we compute the local factor at $p = 3$ using proposition 33 of section 3.3: if $\gamma = (0,0)$ then we are always in part (iii) and so with $\kappa = v_3(n)$,

$$\zeta_{Ig}(f;3;s) = \frac{1}{\mathbf{p}_{(\kappa+1)}} 3^{-\kappa s} + \sum_{\nu=0}^{\kappa} \frac{1}{\mathbf{p}_{(\nu)}} I_{n/3^\nu}(\mathbf{r}_{(\nu)}, \mathbf{d}_{(\nu)}) 3^{-\nu s},$$

where $\mathrm{d}_{(\nu)} = (-1)^\nu$; where $\mathbf{r}_{(\nu)} = 0)$ if ν is even and $\mathbf{r}_{(\nu)} = (\nu - 1)$ if ν is odd; and altogether with some algebraic manipulation one obtains the following local L-functions: if $n \equiv 1$ mod 3 then $L_3(n,\gamma,s) = 1$; if $n \equiv 2$ mod 3 then $L_3(n,\gamma,s) = \frac{3^s+3}{3^s-3}$; and if $\kappa = v_3(n) > 0$ then

$$L_3(n,\gamma,s) = \frac{3^{(\kappa+1)s} + 2\cdot 3^{(\kappa-1)s+2} + 2\cdot 3^{(\kappa-2)s+3} + ... + 2\cdot 3^{s+\kappa} - 3^{\kappa+1}}{3^{(\kappa+1)s} - 3^{\kappa s+1}},$$

such that

$$L_3(n,\gamma,2) = \begin{cases} 1: & n \equiv 1\,(3); \\ 2: & n \equiv 2\,(3); \\ 2(1-3^{-\kappa}): & \kappa = v_3(n) > 0. \end{cases}$$

When $\gamma = \pm(1/3, 1/3))$ the local L-function at 3 is considerably simpler (since we are often in case (ii) which does not depend on the valuation $v_3(n)$): the result is

$$L_3(n,\gamma,s) = \begin{cases} 3^s/(3^s-3): & n \equiv 1\,(3); \\ 1: & n \equiv 2\,(3); \\ (3^s+3)/(3^s-3): & n \equiv 0\,(3); \end{cases}$$

and therefore $L_3(n,\gamma,2) = 3/2, 1, 2$ respectively. In this way we find

$$\begin{aligned} c(n,(0,0)) &= 4 \cdot \Big(\frac{2^{v_2(n)}}{1+2+...+2^{v_2(n)}} \cdot \frac{3^{v_3(n)}}{1+3+...+3^{v_3(n)}} \sigma_1(n) \Big) \times \\ &\quad \times 3(1-2^{-v_2(n)-1}) \cdot L_3(n,(0,0),2) \\ &= -4\sigma_1(n)(1-3^{-v_3(n)-1})^{-1} L_3(n,(0,0),2) \\ &= \begin{cases} -6\sigma_1(n) : & n \equiv 1\,(3); \\ -12\sigma_1(n) : & n \equiv 2\,(3); \\ -8\sigma_1(n) \cdot \frac{3^{\kappa+1}-3}{3^{\kappa+1}-1} : & \kappa = v_3(n) > 0; \end{cases} \end{aligned}$$

and if $\gamma = \pm(1/3, 1/3)$ then similarly

$$\begin{aligned} c(n,\gamma) &= -4\sigma_1(n)(1-3^{-v_3(n)-1})^{-1} L_3(n,\gamma,2) \\ &= \begin{cases} -9\sigma_1(n) : & n \equiv 1\,(3); \\ -6\sigma_1(n) : & n \equiv 2\,(3); \\ -8\sigma_1(n) \frac{3^{\kappa+1}}{3^{\kappa+1}-1} : & \kappa = v_3(n) > 0. \end{cases} \end{aligned}$$

Since there are no cusp forms, this equals the coefficient of the Hirzebruch-Zagier series:

$$\begin{aligned} &-12 \sum_{r \in \mathbb{Z} - \langle \gamma,\beta \rangle} H(4n-9r^2) - 18 \sum_{\substack{r \in \mathbb{Z} - \langle \gamma,\beta \rangle \\ 9r^2-4n=\square}} \Big(|r| - \sqrt{r^2-4n/9} \Big) + \\ &+ \begin{cases} 12\sqrt{n} : & \exists r \in \mathbb{Z} - \langle \gamma,\beta \rangle \text{ with } 9r^2 = 4n; \\ 0 : & \text{otherwise.} \end{cases} \end{aligned}$$

Here $3r$ runs through congruence classes mod 3; and $\frac{1}{2}(|r| - \sqrt{r^2-4n})$, $r \equiv \pm 3\langle \gamma,\beta \rangle$ mod 3 runs through the values $\min(d,n/d)$ for divisors $d|n$ with $|r| = d + n/d \equiv \pm 3\langle \gamma,\beta \rangle$ mod 3, but double-counts $\sqrt{n}$ if that occurs at all. Therefore, we can rewrite this as

$$-12 \sum_{r \equiv 3\langle \gamma,\beta \rangle\,(3)} H(4n-r^2) - 12\varepsilon_\gamma \sum_{\substack{d|n \\ d+n/d \equiv \pm 3\langle \gamma,\beta \rangle}} \min(d,n/d)$$

where $\varepsilon_\gamma = 1$ if $\gamma = 0$ and $\varepsilon_\gamma = 1/2$ otherwise. Comparing coefficients gives the formula:

Proposition 98. *(i) For any $n \in \mathbb{N}$,*

$$\sum_{r \equiv 0\,(3)} H(4n-r^2) + \sum_{\substack{d|n \\ d+n/d \equiv 0\,(3)}} \min(d, n/d) = \begin{cases} \frac{1}{2}\sigma_1(n): & n \equiv 1\,(3); \\ \sigma_1(n): & n \equiv 2\,(3); \\ \frac{3^\kappa - 1}{1+3+...+3^\kappa}\sigma_1(n): & \kappa = v_3(n) > 0. \end{cases}$$

(ii) For any $n \in \mathbb{N}$ and $a \in \{1, 2\}$,

$$\sum_{r \equiv a\,(3)} H(4n-r^2) + \frac{1}{2} \sum_{\substack{d|n \\ d+n/d \equiv \pm a\,(3)}} \min(d, n/d) = \begin{cases} \frac{3}{4}\sigma_1(n): & n \equiv 1\,(3); \\ \frac{1}{2}\sigma_1(n): & n \equiv 2\,(3); \\ \frac{3^\kappa}{1+3+...+3^\kappa}\sigma_1(n): & \kappa = v_3(n) > 0. \end{cases}$$

The computations for other m (and the results) are similar so we omit the details and state the identities that one obtains here. When $m = 16$ there are no cusp forms. However the fact that the quadratic form $Q(x, y) = x^2 - 8y^2$ has terms with larger valuation at 2 implies that the local factor at 2 is somewhat complicated.

Proposition 99. *(i) For any $n \in \mathbb{N}$,*

$$\sum_{r \equiv 0\,(4)} H(4n-r^2) + \sum_{\substack{d|n \\ d+n/d \equiv 0\,(4)}} \min(d, n/d) = \begin{cases} \frac{1}{2}\sigma_1(n): & n \equiv 1\,(4); \\ \frac{1}{3}\sigma_1(n): & n \equiv 2\,(4); \\ \frac{5}{6}\sigma_1(n): & n \equiv 3\,(4); \\ \frac{10}{21}\sigma_1(n): & n \equiv 4\,(8); \\ \frac{2^\kappa - 2}{1+2+...+2^\kappa}\sigma_1(n): & \kappa = v_2(n) \geq 3. \end{cases}$$

(ii) For any $n \in \mathbb{N}$ and $a \in \{1, 3\}$,

$$\sum_{r \equiv a\,(4)} H(4n-r^2) + \frac{1}{2} \sum_{\substack{d|n \\ d+n/d \equiv \pm a\,(4)}} \min(d, n/d) = \begin{cases} \frac{1}{3}\sigma_1(n): & n \equiv 1, 3\,(4); \\ \frac{2^\kappa}{1+2+...+2^\kappa}\sigma_1(n): & \kappa = v_2(n) > 0. \end{cases}$$

(iii) For any $n \in \mathbb{N}$,

$$\sum_{r \equiv 2\,(4)} H(4n - r^2) + \sum_{\substack{d \mid n \\ d + n/d \equiv 2\,(4)}} \min(d, n/d) = \begin{cases} \frac{5}{6}\sigma_1(n) : & n \equiv 1\,(4); \\ \frac{1}{3}\sigma_1(n) : & n \equiv 2\,(4); \\ \frac{1}{2}\sigma_1(n) : & n \equiv 3\,(4); \\ \frac{8}{21}\sigma_1(n) : & n \equiv 4\,(8); \\ \frac{2^\kappa}{1+2+...+2^\kappa}\sigma_1(n) : & \kappa = v_2(n) \geq 3. \end{cases}$$

Now we consider the case $m = 25$. There are again no cusp forms. Some of the details of this case appear in the published version [72].

Proposition 100. *(i) For any $n \in \mathbb{N}$,*

$$\sum_{r \equiv 0\,(5)} H(4n - r^2) + \sum_{\substack{d \mid n \\ d + n/d \equiv 0\,(5)}} \min(d, n/d) = \begin{cases} \frac{1}{2}\sigma_1(n) : & n \equiv \pm 1\,(5); \\ \frac{1}{3}\sigma_1(n) : & n \equiv \pm 2\,(5); \\ \frac{1}{2} \cdot \frac{5^\kappa - 1}{1+5+...+5^\kappa}\sigma_1(n) : & \kappa = v_5(n) > 0. \end{cases}$$

(ii) For any $n \in \mathbb{N}$ and $a \in \{1, 4\}$,

$$\sum_{r \equiv a\,(5)} H(4n - r^2) + \frac{1}{2} \sum_{\substack{d \mid n \\ d + n/d \equiv \pm a\,(5)}} \min(d, n/d) = \begin{cases} \frac{1}{3}\sigma_1(n) : & n \equiv 1, 2\,(5); \\ \frac{1}{2}\sigma_1(n) : & n \equiv 3\,(5); \\ \frac{5}{12}\sigma_1(n) : & n \equiv 4\,(5); \\ \frac{1}{2} \cdot \frac{5^\kappa}{1+5+...+5^\kappa}\sigma_1(n) : & \kappa = v_5(n) > 0. \end{cases}$$

(iii) For any $n \in \mathbb{N}$ and $a \in \{2, 3\}$,

$$\sum_{r \equiv a\,(5)} H(4n - r^2) + \frac{1}{2} \sum_{\substack{d \mid n \\ d + n/d \equiv \pm a\,(5)}} \min(d, n/d) = \begin{cases} \frac{5}{12}\sigma_1(n) : & n \equiv 1\,(5); \\ \frac{1}{2}\sigma_1(n) : & n \equiv 2\,(5); \\ \frac{1}{3}\sigma_1(n) : & n \equiv 3, 4\,(5); \\ \frac{1}{2} \cdot \frac{5^\kappa}{1+5+...+5^\kappa}\sigma_1(n) : & \kappa = v_5(n) > 0. \end{cases}$$

Finally we will consider $m = 36$. Here the difference between E_2 and $Q_{2,1/36,(0,1/18)}$ is a cusp form in which the components of $\mathfrak{e}_\gamma$ for $Q(\gamma) \in \mathbb{Z}$ are all multiples of the form

$$\eta(6\tau)^4 = q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} - 4q^{31} - 10q^{37} \pm ... \in M_2(\Gamma_0(36)).$$

In particular, the coefficients of these components of E_2 and $Q_{2,1/36,(0,1/18)}$ are equal unless $n \equiv 1 \bmod 6$. The identities we find in this case involve nontrivial local factors at both $p = 2$ and $p = 3$:

Proposition 101. *Let $a(n)$, $n \in \mathbb{N}$ denote the coefficient of q^n in $\eta(6\tau)^4$.*
(i) For any $n \in \mathbb{N}$,

$$\sum_{r \equiv 0\,(6)} H(4n - r^2) + \sum_{\substack{d|n \\ d+n/d \equiv 0\,(6)}} \min(d, n/d) - \frac{1}{6}a(n)$$

$$= \begin{cases} \frac{1}{3}\sigma_1(n) : & n \equiv 1\,(6); \\ \frac{2^{v_2(n)}-1}{1+2+...+2^{v_2(n)}}\sigma_1(n) : & n \equiv 2\,(6); \\ \frac{2}{3} \cdot \frac{3^{v_3(n)}-1}{1+3+...+3^{v_3(n)}}\sigma_1(n) : & n \equiv 3\,(6); \\ \frac{1}{2} \cdot \frac{2^{v_2(n)}-1}{1+2+...+2^{v_2(n)}}\sigma_1(n) : & n \equiv 4\,(6); \\ \frac{2}{3}\sigma_1(n) : & n \equiv 5\,(6); \\ \frac{(2^{v_2(n)}-1)(3^{v_3(n)}-1)}{(1+2+...+2^{v_2(n)})(1+3+...+3^{v_3(n)})}\sigma_1(n) : & n \equiv 0\,(6). \end{cases}$$

(ii) For any $n \in \mathbb{N}$ and $a \in \{1, 5\}$,

$$\sum_{r \equiv a\,(6)} H(4n - r^2) + \frac{1}{2} \sum_{\substack{d|n \\ d+n/d \equiv \pm a\,(6)}} \min(d, n/d) - \frac{1}{12}a(n)$$

$$= \begin{cases} \frac{1}{4}\sigma_1(n) : & n \equiv 1\,(6); \\ \frac{1}{2} \cdot \frac{2^{v_2(n)}}{1+2+...+2^{v_2(n)}}\sigma_1(n) : & n \equiv 2\,(6); \\ \frac{1}{3} \cdot \frac{3^{v_3(n)}}{1+3+...+3^{v_3(n)}}\sigma_1(n) : & n \equiv 3\,(6); \\ \frac{3}{2} \cdot \frac{2^{v_2(n)}}{1+2+...+2^{v_2(n)}}\sigma_1(n) : & n \equiv 4\,(6); \\ \frac{1}{6}\sigma_1(n) : & n \equiv 5\,(6); \\ \frac{2^{v_2(n)} \cdot 3^{v_3(n)}}{(1+2+...+2^{v_2(n)})(1+3+...+3^{v_3(n)})}\sigma_1(n) : & n \equiv 0\,(6). \end{cases}$$

(iii) For any $n \in \mathbb{N}$ and $a \in \{2,4\}$,

$$\sum_{r \equiv a\,(6)} H(4n-r^2) + \frac{1}{2} \sum_{\substack{d|n \\ d+n/d \equiv \pm a\,(6)}} \min(d, n/d) + \frac{1}{12} a(n)$$

$$= \begin{cases} \frac{1}{2}\sigma_1(n) : & n \equiv 1\,(6); \\ \frac{1}{2} \cdot \frac{2^{v_2(n)}-1}{1+2+\ldots+2^{v_2(n)}} : & n \equiv 2\,(6); \\ \frac{2}{3} \cdot \frac{3^{v_3(n)}}{1+3+\ldots+3^{v_3(n)}} : & n \equiv 3\,(6); \\ \frac{3}{4} \cdot \frac{2^{v_2(n)}-1}{1+2+\ldots+2^{v_2(n)}} : & n \equiv 4\,(6); \\ \frac{1}{3}\sigma_1(n) : & n \equiv 5\,(6); \\ \frac{(2^{v_2(n)}-1)\cdot 3^{v_3(n)}}{(1+2+\ldots+2^{v_2(n)})(1+3+\ldots+3^{v_3(n)})}\sigma_1(n) : & n \equiv 0\,(6). \end{cases}$$

(iv) For any $n \in \mathbb{N}$,

$$\sum_{r \equiv 3\,(6)} H(4n-r^2) + \sum_{\substack{d|n \\ d+n/d \equiv 3\,(6)}} \min(d, n/d) + \frac{1}{6} a(n)$$

$$= \begin{cases} \frac{1}{6}\sigma_1(n) : & n \equiv 1\,(6); \\ \frac{2^{v_2(n)}}{1+2+\ldots+2^{v_2(n)}}\sigma_1(n) : & n \equiv 2\,(6); \\ \frac{1}{3} \cdot \frac{3^{v_3(n)}-1}{1+3+\ldots+3^{v_3(n)}}\sigma_1(n) : & n \equiv 3\,(6); \\ \frac{1}{2} \cdot \frac{2^{v_2(n)}}{1+2+\ldots+2^{v_2(n)}}\sigma_1(n) : & n \equiv 4\,(6); \\ \frac{1}{3}\sigma_1(n) : & n \equiv 5\,(6); \\ \frac{2^{v_2(n)}\cdot(3^{v_3(n)}-1)}{(1+2+\ldots+2^{v_2(n)})(1+3+\ldots+3^{v_3(n)})}\sigma_1(n) : & n \equiv 0\,(6). \end{cases}$$

In general we always find some sort of expression relating

$$\sum_{r \equiv \pm a\,(N)} H(4n-r^2) + \sum_{\substack{d|n \\ d+n/d \equiv \pm a\,(N)}} \min(d, n/d)$$

to a multiple of the divisor sum $\sigma_1(n)$, with an error term coming from cusp forms.

Chapter 7

Rational Poincaré series in antisymmetric weights

7.1 Introduction

We retain the notation from previous chapters. The purpose of this chapter is to adapt the construction of modular forms with rational coefficients from chapter three to weights that satisfy $2k + b^+ - b^- \equiv 2 \bmod 4$. Since there does not seem to be a widely-used name for this case, we refer to such k as **antisymmetric weights** because the Fourier coefficients $c(n, \gamma)$ of any modular form $F(\tau) \in M_k(\rho^*)$ now satisfy $c(n, \gamma) = -c(n, -\gamma)$, as one can see by considering the action of $Z = (-I, i)$ on F.

The constructions of Eisenstein series $E_{k,0}$ and Poincaré square series $Q_{k,m,\beta}$ will be identically zero in these weights. However, it still makes sense to consider the usual Poincaré series $P_{k,m,\beta}$ as well as the Eisenstein series $E_{k,\beta}$ attached to nonzero elements $\beta \in A$ with $Q(\beta) \in \mathbb{Z}$ as in [12]. A computable exact formula for $E_{k,\beta}$ was given by Schwagenscheidt in [57]; however the Poincaré series $P_{k,m,\beta}$ are impractical to work with explicitly (as usual). Instead, infinite linear combinations of $P_{k,m,\beta}$ can again be used to produce bases with rational coefficients.

Modular forms of antisymmetric weight have received less attention in the literature, likely because they do not seem to be directly useful within the theory of theta lifts. On the other hand, one can produce obstructions to Borcherds products out of antisymmetric modular forms through simple operations such as Rankin-Cohen brackets. We can also mention that antisymmetric modular forms correspond to Jacobi forms (and skew-holomorphic Jacobi forms) via the usual theta decomposition ([31], chapter 5); as an example, we can give another construction of the Jacobi cusp form of weight 11 and index 2 (see the discussion in [31] after theorem 9.3).

7.2 Rational Poincaré series in antisymmetric weights

Suppose that $k \ge 7/2$ and let

$$P_{k,m,\beta}(\tau) = \sum_{M \in \Gamma_\infty \backslash \tilde{\Gamma}} (q^m \mathfrak{e}_\beta)\Big|_k M$$

be the Poincaré series of exponential type, which is characterized through the Petersson scalar product by

$$(f, P_{k,m,\beta}) = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} c(m,\beta) \text{ for } f(\tau) = \sum_{n,\gamma} c(n,\gamma) q^n \mathfrak{e}_\gamma \in S_k(\rho^*).$$

We denote by $R_{k,m,\beta}$ the series

$$R_{k,m,\beta} = \sum_{\lambda \in \mathbb{Z}} \lambda P_{k,\lambda^2 m, \lambda\beta}.$$

(This can be interpreted as the Poincaré averaging process applied to the weight 3/2 theta function $\sum_{\lambda \in \mathbb{Z}} \lambda q^{\lambda^2 m} \mathfrak{e}_{\lambda\beta}$.)

The convergence of $R_{k,m,\beta}$ follows from the same argument that we used for the Poincaré square series $Q_{k,m,\beta}$ in chapter 3: since $S_k(\rho^*)$ is finite-dimensional, it is enough to prove that $\sum_{\lambda \in \mathbb{Z}} \lambda P_{k,\lambda^2 m,\lambda\beta}$ converges weakly (i.e. as a functional through the inner product) and for any cusp form $f(\tau) = \sum_{n,\gamma} c(n,\gamma) q^n \mathfrak{e}_\gamma$ the convergence of

$$\sum_{\lambda \in \mathbb{Z}} (f, \lambda P_{k,\lambda^2 m,\lambda\beta}) = 2 \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{\lambda=1}^\infty \frac{\lambda c(\lambda^2 m, \lambda\beta)}{\lambda^{2k-2} m^{k-1}}$$

follows from known bounds on the coefficients of cusp forms when $k \ge 7/2$. Also, Möbius inversion implies $P_{k,m,\beta} = \frac{1}{2} \sum_{d=1}^\infty d\mu(d) R_{k,d^2 m, d\beta}$ with convergence by the same argument;

so all Poincaré series lie in the closure of

$$\mathrm{Span}(R_{k,m,\beta} :\ \beta \in A,\ m \in (\mathbb{Z} - Q(\beta))_{>0}).$$

By finite-dimensionality again, $\mathrm{Span}(R_{k,m,\beta})$ contains all Poincaré series and therefore all cusp forms.

By switching the order of summation (valid for large enough k), we see that $R_{k,m,\beta}$ arises from the Jacobi Eisenstein series $E_{k-1,m,\beta}$: differentiating with respect to z gives

$$\begin{aligned}
&\frac{\partial}{\partial z}\Big|_{z=0} E_{k-1,m,\beta}(\tau, z) \\
&= \frac{1}{2}\sum_{c,d}(c\tau+d)^{-k+1}\sum_{\lambda\in\mathbb{Z}}\frac{\partial}{\partial z}\Big|_{z=0}\mathbf{e}\Big(m\lambda^2(M\cdot\tau) + \frac{2m\lambda z - cmz^2}{c\tau+d}\Big)\rho^*(M)^{-1}\mathfrak{e}_{\lambda\beta} \\
&= \frac{4\pi mi}{2}\sum_{c,d}(c\tau+d)^{-k}\sum_{\lambda\in\mathbb{Z}}\lambda\mathbf{e}\Big(m\lambda^2(M\cdot\tau)\Big)\rho^*(M)^{-1}\mathfrak{e}_{\lambda\beta} \\
&= 4\pi mi R_{k,m,\beta}.
\end{aligned}$$

Such a manipulation is valid whenever $E_{k-1,m,\beta}$ converges locally uniformly as a triple series and in particular when $k > 4$. (The Hecke trick will show that it is also valid in weight $k = 4$.) In particular $R_{k,m,\beta}$ has rational coefficients that can be computed using the results of chapter 3: if $c(n, r, \gamma)$ are the Fourier coefficients of $E_{k-1,m,\beta}$ then

$$R_{k,m,\beta}(\tau) = \frac{1}{4\pi mi}\frac{\partial}{\partial z}\Big|_{z=0}E_{k-1,m,\beta}(\tau,z) = \frac{1}{2m}\sum_{\gamma\in A}\sum_{n\in\mathbb{Z}-Q(\gamma)}\Big(\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle} rc(n,r,\gamma)\Big)q^n\mathfrak{e}_\gamma.$$

For practical computations it is important to have a formula for $\dim M_k(\rho^*)$. A Riemann-Roch based formula that is also valid for antisymmetric weights appears as theorem 2.1 of [29]; with some modifications we can express it in a form similar to proposition 14 of chapter 2:

Proposition 102. *Suppose $2k + b^+ - b^- \equiv 2 \bmod 4$ and $k > 2$. Let d denote the number of pairs $\{\pm\gamma\}$, $\gamma \in A$ for which $\gamma \neq -\gamma$. Define*

$$B_1 = \sum_{\gamma\in A} B(Q(\gamma)),\quad B_2 = \sum_{\substack{\gamma\in A\\ 2\gamma=0}} B(Q(\gamma))$$

and

$$\tilde{\alpha}_4 = \#\{\gamma \in A : \ Q(\gamma) \in \mathbb{Z}, \ \gamma \neq -\gamma\}/ \pm I,$$

where $B(x)$ *is the auxiliary sawtooth function* $B(x) = x - \frac{\lfloor x \rfloor - \lfloor -x \rfloor}{2}$*. Then*

$$\begin{aligned} \dim M_k(\rho^*) = & \frac{d(k-1)}{12} \\ & + \frac{1}{4\sqrt{|A|}} \mathbf{e}\Big(\frac{2(k+1) + \mathrm{sig}(A)}{8}\Big) \mathrm{Im}[G(2,A)] \\ & - \frac{1}{3\sqrt{3|A|}} \mathrm{Re}\Big[\mathbf{e}\Big(\frac{4k + 3\mathrm{sig}(A) - 10}{24}\Big)(G(1,A) - G(-3,A))\Big] \\ & + \frac{\tilde{\alpha}_4 + B_1 - B_2}{2}, \end{aligned}$$

and $\dim S_k(\rho^*) = \dim M_k(\rho^*) - \tilde{\alpha}_4$*.*

To be clear, the differences between this formula and proposition 14 are a sign change in the third and fourth lines, the fact that k is replaced by $(k+1)$ in the second line, the imaginary part of the Gauss sum $G(2,A)$ being used instead of its real part, and the different definition of d.

Example 103. Classical Jacobi forms of index 2 correspond to modular forms for the quadratic form $Q(x) = 2x^2$. The discriminant group of Q is represented by $0, 1/2$ and $\pm 1/4$. The equation $P_{k,m,\beta} = -P_{k,m,-\beta}$ for antisymmetric weights k implies that $R_{k,m,\beta}$ is zero unless $\beta = \pm 1/4 \bmod \mathbb{Z}$. The dimension formula shows that the first nonzero instances of a form $R_{k,m,\beta}$ occur in weight $k = \frac{21}{2}$. In this weight we compute

$$R_{21/2,7/8,1/4}(\tau) = \frac{25920}{8831}\Big(q^{7/8} - 21q^{15/8} + 189q^{23/8} - 910q^{31/8} + 2205q^{39/8} \pm ...\Big)(\mathfrak{e}_{1/4} - \mathfrak{e}_{3/4}).$$

Via the theta decomposition this corresponds to a Jacobi cusp form of weight 11 with the same Fourier coefficients; it is essentially $E_{4,1}'E_{6,1} - E_{4,1}E_{6,1}'$. (Compare equation 13 of section 9 of [31] and with Table 3 (part d) at the end of [31]). Computing this in the form $R_{21/2,7/8,1/4}(\tau)$ may be more efficient since the coefficient of q^n is now a sum of only $O(\sqrt{n})$ coefficients of Eisenstein series.

Example 104. Modular forms for $Q(x) = -mx^2$, $m \in \mathbb{N}$ and weight k can be interpreted as skew-holomorphic Jacobi forms of index m and weight $k + 1/2$ (see [9],[62]), again via a theta decomposition that sends the skew-holomorphic Jacobi form

$$\varphi(\tau,z) = \sum_{\substack{n,r \\ 4mn-r^2\leq 0}} c(n,r)q^n\zeta^r e^{\frac{\pi}{m}(4mn-r^2)y}$$

to the modular form $\sum_{n,r} c(-n,r)q^{-n+r^2/4m}\mathfrak{e}_{r/2m}$. In that sense it is also straightforward to compute skew-holomorphic Jacobi forms of even weight using the formula here. When $m=2$ the dimension formula implies that we first find a nonzero cusp form in weight $k=11/2$ and the rational Poincaré series of index $(1/8,1/4)$ gives such a form:

$$R_{11/2,1/8,1/4}(\tau) = \Big(q^{1/8}+237q^{9/8}+1440q^{17/8}+245q^{25/8}-1440q^{33/8}\pm...\Big)(\mathfrak{e}_{1/4}-\mathfrak{e}_{3/4}).$$

7.3 Small weights

The extra terms that appear in $E^*_{k,m,\beta}(\tau,z,0)$ for $k\in\{3/2,2,5/2\}$ determine corrections to the coefficient formula for $R_{k,m,\beta}$ for $k\in\{5/2,3,7/2\}$ that can again be calculated using the holomorphic projection technique. Suppose first that (A,Q) has signature 3 mod 4.

Proposition 105. *Let*

$$E^*_{5/2,m,\beta}(\tau,z) = E_{5/2,m,\beta}(\tau,z)+\frac{1}{y}\vartheta(\tau,z)$$

denote the decomposition of the nonholomorphic Jacobi Eisenstein series into its holomorphic and real-analytic parts as in section 3.4. Then

$$R_{7/2,m,\beta}(\tau) = \frac{1}{4\pi mi}\frac{\partial}{\partial z}\Big|_{z=0}E_{5/2,m,\beta}(\tau,z)-\frac{1}{3\pi m}\frac{\partial^2}{\partial\tau\partial z}\Big|_{z=0}\vartheta(\tau,z).$$

Proof. $R_{7/2,m,\beta}$ is the projection of $\frac{1}{4\pi mi}\frac{\partial}{\partial z}\Big|_{z=0}E^*_{5/2,m,\beta}(\tau,z)$ to $S_{7/2}(\rho^*)$. Therefore its Fourier coefficients are determined as follows: writing $R_{7/2,m,\beta}(\tau)=\sum_{n,\gamma}b(n,\gamma)q^n\mathfrak{e}_\gamma$,

$$\begin{aligned} b(n,\gamma) &= \frac{(4\pi n)^{5/2}}{\Gamma(5/2)}(R_{7/2,m,\beta},P_{7/2,n,\gamma}) \\ &= \frac{64\pi^2}{3}\cdot\frac{n^{5/2}}{m}\Big(\frac{1}{2\pi i}\frac{\partial}{\partial z}\Big|_{z=0}E^*_{5/2,m,\beta}(\tau,z),P_{7/2,n,\gamma}\Big) \\ &= \frac{64\pi^2}{3}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}\frac{n^{5/2}r}{m}\int_0^\infty\Big(c(n,r,\gamma)+\frac{1}{y}A(n,r,\gamma)\Big)e^{-4\pi ny}y^{3/2}\,\mathrm{d}y \\ &= \frac{1}{2m}\sum_r rc(n,r,\gamma)+\frac{4\pi n}{3m}\sum_r rA(n,r,\gamma). \end{aligned}$$

In other words, writing $E^*_{5/2,m,\beta}(\tau,z) = E_{5/2,m,\beta}(\tau,z) + \frac{1}{y}\vartheta(\tau,z)$ where $E_{5/2,m,\beta}$ is holomorphic and ϑ is the weight 1/2 theta function from above,

$$R_{7/2,m,\beta}(\tau) = \frac{1}{4\pi mi}\frac{\partial}{\partial z}\Big|_{z=0} E_{5/2,m,\beta}(\tau,z) - \frac{1}{3\pi m}\frac{\partial^2}{\partial\tau\partial z}\Big|_{z=0}\vartheta(\tau,z)$$

as claimed. $\square$

Example 106. There is a unique skew-holomorphic Jacobi cusp form of weight 4 and index 3 up to scalar multiples. The corresponding modular form for $Q(x) = -3x^2$ is the (corrected) rational Poincaré series of weight 7/2 and index $(1/6, 1/12)$:

$$\begin{aligned} R_{7/2,1/6,1/12}(\tau) &= \Big(\frac{1}{3}q^{1/12} - 24q^{13/12} + \frac{19}{3}q^{25/12} + 24q^{37/12} \pm ...\Big)(\mathfrak{e}_{1/6} - \mathfrak{e}_{5/6}) \\ &\quad + \Big(-\frac{10}{3}q^{1/3} - \frac{28}{3}q^{4/3} - 48q^{7/3} + 96q^{10/3} \pm ...\Big)(\mathfrak{e}_{1/3} - \mathfrak{e}_{2/3}); \end{aligned}$$

but the coefficient formula of the previous section evaluated naively gives the series

$$\begin{aligned} &\Big(q^{1/12} - 24q^{13/12} - 77q^{25/12} + 24q^{37/12} \pm ...\Big)(\mathfrak{e}_{1/6} - \mathfrak{e}_{5/6}) \\ &+ \Big(2q^{1/3} - 52q^{4/3} - 48q^{7/3} + 96q^{10/3} \pm ...\Big)(\mathfrak{e}_{1/3} - \mathfrak{e}_{2/3}). \end{aligned}$$

These differ by what is essentially a theta series that has been differentiated 3/2 times. (In particular, the coefficients of q^n are unchanged unless $3n$ is a rational square.)

Now suppose (A,Q) has signature 0 mod 4.

Proposition 107. *Let*

$$E^*_{2,m,\beta}(\tau,z,0) = E_{2,m,\beta}(\tau,z) + \frac{1}{\sqrt{y}}\sum_{\gamma,n,r} A(n,r,\gamma)\beta(\pi y(r^2/m - 4n))q^n\zeta^r\mathfrak{e}_\gamma$$

be the splitting of $E^*_{2,m,\beta}(\tau,z,0)$ into its holomorphic and nonholomorphic parts, where $\beta(x) = \frac{1}{16\pi}\int_1^\infty u^{-3/2}e^{-xu}\,\mathrm{d}u$ and

$$\begin{aligned} A(n,r,\gamma) = \frac{48(-1)^{(4+b^+-b^-)/4}}{\sqrt{m|\Lambda'/\Lambda|}} \prod_{\mathrm{bad}\,p} \frac{1-p^{e/2-1}}{1+p^{-1}} L_p(n,r,\gamma,1+e/2)\times \\ \times \begin{cases} 1: & r^2 \neq 4mn; \\ 1/2: & r^2 = 4mn; \end{cases} \end{aligned}$$

if A arises from the even lattice Λ of dimension e and signature (b^+, b^-). Then

$$R_{3,m,\beta}(\tau) = \frac{1}{4\pi mi}\frac{\partial}{\partial z}\Big|_{z=0} E_2(\tau, z) + \frac{1}{32m^{3/2}}\sum_{\gamma,n,r}\mathrm{sgn}(r)A(n,r,\gamma)\Big(|r| - \sqrt{r^2-4mn}\Big)^2 q^n \mathfrak{e}_\gamma,$$

and all Fourier coefficients of $R_{3,m,\beta}$ are rational.

Proof. Holomorphic projection in this case gives $R_{3,m,\beta}(\tau) = \sum_{\gamma,n} b(n,\gamma)q^n\mathfrak{e}_\gamma$ where $b(n,\gamma)$ differs from $\frac{1}{4\pi mi}\sum_r rc(n,r,\gamma)$ by

$$\begin{aligned}
&\frac{(4\pi n)^2}{4\pi im\Gamma(2)}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}(2\pi ir)A(n,r,\gamma)\int_0^\infty e^{-4\pi ny}\beta(\pi y(r^2/m-4n))y^{1/2}\,\mathrm{d}y\\
=\,&\frac{\pi n^2}{4\pi im}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}(2\pi ir)A(n,r,\gamma)\int_0^\infty\int_1^\infty u^{-3/2}y^{1/2}e^{4\pi ny(u-1)-\pi r^2yu/m}\,\mathrm{d}u\,\mathrm{d}y\\
=\,&\frac{n^2}{4m}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}rA(n,r,\gamma)\int_1^\infty u^{-3/2}[(r^2/m-4n)u+4n]^{-3/2}\,\mathrm{d}u\\
=\,&\frac{1}{16m^{3/2}}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}\mathrm{sgn}(r)A(n,r,\gamma)\Big(r^2-2nm-|r|\sqrt{r^2-4nm}\Big)\\
=\,&\frac{1}{32m^{3/2}}\sum_{r\in\mathbb{Z}-\langle\gamma,\beta\rangle}\mathrm{sgn}(r)A(n,r,\gamma)\Big(|r|-\sqrt{r^2-4mn}\Big)^2.
\end{aligned}$$

As in section 5.6, if $|A|$ is square then this is a finite sum of rational numbers, and otherwise we can split it into a sum over finitely many orbits and calculate each as a geometric series. $\square$

Example 108. Consider the quadratic form $Q(x,y) = x^2 + xy - 2y^2$ of discriminant 9 and the real-analytic Jacobi Eisenstein series of weight 2 of index $(m,\beta) = (1/9, (1/9, -2/9))$:

$$-12\sum_{\gamma,n,r}H(4n-9r^2)q^n\zeta^r\mathfrak{e}_\gamma + \frac{1}{\sqrt{y}}\sum_{\gamma,n,r}A(n,r,\gamma)\beta(\pi y(9r^2-4n))q^n\zeta^r\mathfrak{e}_\gamma,$$

where $A(n,r,\gamma) = -24$ if $9r^2 = 4n$, and $A(n,r,\gamma) = -48$ if $9r^2 - 4n$ is a nonzero square, and $A(n,r,\gamma) = 0$ otherwise. The dimension formula implies $M_3(\rho^*) = 0$ and therefore $R_{3,m,\beta} = 0$. As in section 6.5 we can compute the component of $\mathfrak{e}_{(1/3,1/3)}$ in this series in terms of a divisor sum; the identity we find here is

$$\sum_{r\equiv 1\,(3)} rH(4n-r^2) = \varepsilon(n)\sum_{d|n}\Big(\frac{d}{3}\Big)\min(d,n/d)^2,$$

for all $n \in \mathbb{N}$, where $\varepsilon(n) = -1$ if $3|n$ and $\varepsilon(n) = 1/2$ otherwise. This is simpler than the formula for $\sum_{r \equiv 1\,(3)} H(4n - r^2)$ (see section 6.5) because there is no Eisenstein series $E_{3,0}$ in $M_3(\rho^*)$ to compare against.

Note in particular that the right hand side is zero when $n \equiv 2 \bmod 3$, since $d \mapsto n/d$ swaps the two sums; therefore $\sum_{r \equiv 1\,(3)} rH(4n - r^2) = 0$ for all $n \equiv 2 \bmod 3$. Also, by antisymmetry $\sum_{r \equiv 2\,(3)} rH(4n - r^2) = -\sum_{r \equiv 1\,(3)} rH(4n - r^2)$ (as one can see directly by swapping r with $-r$).

Finally, we consider weight $k = 5/2$ when (A, Q) has signature 1 mod 4. Here the corrected Jacobi Eisenstein series $E^*_{3/2,m,\beta}(\tau, z, 0)$ is holomorphic so it is easy to correct its derivative to $R_{5/2,m,\beta}$. However, unlike the case of weight $k \geq 3$ it seems likely that the series $R_{5/2,m,\beta}$ do not generally span $S_{5/2}(\rho^*)$. One example where I suspect it does not span is the case of the quadratic form $Q(x) = 13x^2$. There is a nonzero Jacobi cusp form of weight 3 and index 13, (unique up to scalar multiple), namely

$$\begin{aligned}\phi_3(\tau, z) = \Big(&9(\zeta - \zeta^{-1}) + 7(\zeta^2 - \zeta^{-2}) - 17(\zeta^3 - \zeta^{-3}) + 4(\zeta^4 - \zeta^{-4}) + \\ &+ 7(\zeta^5 - \zeta^{-5}) - 5(\zeta^6 - \zeta^{-6}) + (\zeta^7 - \zeta^{-7})\Big) q + O(q^2),\end{aligned}$$

which corresponds to a nonzero form in $S_{5/2}(\rho^*)$. It seems likely that all $R_{5/2,m,\beta}$ are identically zero. This implies a lot of vanishing of certain special values of L-functions attached to $\phi_3(\tau, z)$ but it is not a contradiction to the spanning claim of the previous section (which does not apply in small weights). The situation here is similar to the Jacobi form of weight 2 and index 37 as in the paragraph after remark 77.

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Appendix A

Paramodular forms

In this chapter we will apply the algorithm of chapter 3 to some spaces of paramodular forms of small level. A paramodular form of level $t \in \mathbb{N}$ is a holomorphic function on the Siegel upper half-space $\mathbb{H}_2$ of genus two that satisfies the usual functional equation (possibly with a character) for the group

$$\Gamma_t = \Big\{ M \in Sp_4(\mathbb{Q}) : \begin{pmatrix} I & 0 \\ 0 & t \end{pmatrix}^{-1} M \begin{pmatrix} I & 0 \\ 0 & t \end{pmatrix} \in \mathbb{Z}^{4\times 4} \Big\}$$

(where I above is the (3×3) identity matrix) instead of $Sp_4(\mathbb{Z})$. Paramodular forms can also be interpreted as orthogonal modular forms for the lattice $A_1(-t) \oplus II_{2,2}$ of signature $(2, 3)$ so there is a version of the Borcherds lift. The relevant obstruction space consists of weight $5/2$ modular forms for the dual Weil representation attached to $Q(x) = -tx^2$. A more detailed reference for product expansions of paramodular forms is section 2 of [35]. We use the formulation of theorem 7.1.1 of [44] here. For $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbb{H}_2$, write $q = \mathbf{e}(z_{11})$, $r = \mathbf{e}(z_{12})$, $s = \mathbf{e}(z_{22})$. Then the paramodular form of the Borcherds product with input function $F(\tau) = \sum_{n,\gamma} c(n, \gamma) q^n \mathfrak{e}_\gamma$ is

$$\Psi_F(Z) = q^{\rho_1} r^{2t\rho_2} s^{t\rho_3} \prod_{(m,n,l)>0} (1 - q^n r^l s^{tm})^{c(mn - l^2/4t, l/2t)}$$

where

$$\rho_1 = \frac{1}{24} \sum_{l \in \mathbb{Z}} c(-l^2/4t, l/2t), \ \rho_2 = \frac{1}{4t} \sum_{l=1}^{\infty} lc(-l^2/4t, l/2t), \ \rho_3 = \frac{1}{4t} \sum_{l \in \mathbb{Z}} l^2 c(-l^2/4t, l/2t)$$

(such that $\rho = (\rho_1, \rho_2, \rho_3)$ is the Weyl vector of [4]) and $(m, n, l) > 0$ means either $m > 0$; or $m = 0$ and $n > 0$; or $m = n = 0$ and $l < 0$.

The character group of Γ_t was determined by Gritsenko and Hulek [33]. If $t_1 = \gcd(t, 12)$ and $t_2 = \gcd(2t, 12)$ then the characters are exactly $\chi_{a,b}$ where $a, b \in \mathbb{Z}/t_2\mathbb{Z}$ satisfy $a - b \equiv t_2/t_1 \bmod t_2$ and they are determined by

$$\chi_{a,b}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{e}(a/t_2), \quad \chi_{a,b}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{e}(b/t_2).$$

For example, when $t = 1$, the character group consists of the trivial character $\chi_{0,0}$ and the Siegel character $\chi_{1,1}$. In particular the character of Ψ_F can be read off its Weyl vector: since

$$\Psi_F\Big(Z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Big) = \mathbf{e}(\rho_1)\Psi_F(Z), \quad \Psi_F\Big(Z + \begin{pmatrix} 0 & 0 \\ 0 & t^{-1} \end{pmatrix}\Big) = \mathbf{e}(\rho_3)\Psi_F(Z),$$

the character of Ψ_F is $\chi_{t_2\rho_1, t_2\rho_3}$. We also remark that Borcherds products transform under certain maps V_d which are not contained in Γ_t. The most important is the involution V_t : $\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \mapsto \begin{pmatrix} tz_{22} & z_{12} \\ z_{12} & t^{-1}z_{11} \end{pmatrix}$. For $t > 1$, the characters of the group $\Gamma_t^+ = \langle \Gamma_t, V_t \rangle$ are generated by the character $\chi_{t_2} = \chi_{1,1}$ and a character μ defined by $\mu(\Gamma_t) = 1$, $\mu(V_t) = -1$. Whether μ appears in the character for Ψ_F can be determined by

$$\Psi_F(V_t \cdot Z) = (-1)^D \Psi_F(Z) = (-1)^k \chi_{\Psi_F}(V_t)\Psi_F(Z), \quad D = \sum_{\substack{n \in \mathbb{N} \\ l \in \mathbb{Z}}} \sigma_1(n) c(-n - l^2/4t, l/2t).$$

(When $t = 1$ this degenerates to $\mu = \chi_2$.) In particular the Weyl vector (ρ_1, ρ_2, ρ_3) always satisfies $\rho_1 - \rho_3 \in \mathbb{Z}$, since the character under Γ_t is always a multiple of $\chi_{1,1}$.

Computations of Borcherds products for some small levels (and their applications to finding generators of rings of paramodular forms) have appeared in the literature. In [32], Freitag gave a short proof of Igusa's [40] computation of generators for $t = 1$, using a reduction process based on the product ψ_5 of ten theta-constants. Gritsenko-Nikulin [35] found paramodular forms of level $t = 2, 3$ (and $t = 4$) with similarly simple divisors, which were used to compute the graded rings by Ibukiyama-Onodera [39] and Dern [25], respectively.

Results of this type do not seem to be known for higher levels; from our point of view, the existence of cusp forms in the obstruction spaces makes the divisors of products of small weights more complicated and makes such a reduction process unlikely. One could try to find exponents for which the coefficients of all cusp forms are zero (which is sometimes indicated by those coefficients of the Eisenstein series being integral) and use these to produce Borcherds products with simple divisors, but it seems difficult to use these products to determine graded rings.

All computations were done in SAGE [64]. In all cases the obstruction space was computed in less than a second. The problem of finding holomorphic principal parts was interpreted here as finding integer points in a polytope described by an equation for every modular form in the obstruction space and an inequality for every Heegner divisor, the polytope being compact because of the obstruction by the Eisenstein series $E_{5/2}$. This becomes harder in large weights as the polytope dimension increases and the number of solutions grows rapidly. A recent preprint [52] describes a method to solve a similar problem (computing paramodular Borcherds products that are cusp forms) which does not rely on the obstruction principle; I do not know how it compares to this in practice.

Explanation: The tables below work out a basis of the obstruction space and the principal parts of the input functions for the holomorphic Borcherds products of smallest weights (from which one can read off the divisor) as well as their Weyl vectors and characters. (In half-integer weight there is a multiplier system rather than a character.) We use the Gram matrix $\mathbf{S} = (-2t)$ so the discriminant form is represented by $\gamma = a/2t$, $0 \le a \le 2t-1$. We omit any products that can be constructed as products or quotients of previous entries in the table. ψ_k denotes a product of weight k. The Weyl vector $\rho = (\rho_1, \rho_2, \rho_3)$ uses the convention of Borcherds [4]; in other works (e.g. [25],[44]) one sometimes sees the exponents $(A, B, C) = (\rho_1, 2t\rho_2, t\rho_3)$ used instead. As above, the character χ_d of Γ_t of order $d \in \mathbb{N}$ is characterized by

$$\chi_d\left(Z \mapsto Z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \chi_d\left(Z \mapsto Z + \begin{pmatrix} 0 & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = e^{2\pi i/d},$$

and the character μ of $\langle \Gamma_t, V_t \rangle$ indicates that the product is antisymmetric.

A.1 Level $t = 1$

Paramodular forms of level 1 are Siegel modular forms of genus 2.

Table A.1: Obstruction space, level $t = 1$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - 70q - 120q^2 - 240q^3 - 550q^4 - 528q^5 + O(q^6)$
$\mathfrak{e}_{1/2}$	$q^{1/4}(-10 - 48q - 250q^2 - 240q^3 - 480q^4 - 480q^5 + O(q^6))$

Table A.2: Holomorphic products of weight less than 100

	Principal part	Weyl vector	Character
ψ_5	$10\mathfrak{e}_0 + q^{-1/4}\mathfrak{e}_{1/2}$	$(1/2, 1/4, 1/2)$	$\chi_2\,(=\mu)$
ψ_{24}	$48\mathfrak{e}_0 + q^{-5/4}\mathfrak{e}_{1/2}$	$(2, 0, 0)$	–
ψ_{30}	$60\mathfrak{e}_0 - q^{-1/4}\mathfrak{e}_{1/2} + q^{-1}\mathfrak{e}_0$	$(5/2, 1/4, 3/2)$	$\chi_2\,(=\mu)$
ψ_{60}	$120\mathfrak{e}_0 + q^{-2}\mathfrak{e}_0$	$(5, 0, 0)$	–

A.2 Level $t = 2$

Table A.3: Obstruction space, level $t = 2$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - 24q - 166q^2 - 144q^3 - 312q^4 - 336q^5 + O(q^6)$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{1/8}(-2 - 50q - 96q^2 - 242q^3 - 288q^4 - 384q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/2}(-22 - 48q - 144q^2 - 192q^3 - 550q^4 - 336q^5 + O(q^6))$

Table A.4: Holomorphic products of weight less than 50

	Principal part	Weyl vector	Character
ψ_2	$4\mathfrak{e}_0 + q^{-1/8}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4})$	$(1/4, 1/8, 1/4)$	χ_4
ψ_9	$18\mathfrak{e}_0 - q^{-1/8}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4}) + q^{-1/2}\mathfrak{e}_{1/2}$	$(3/4, 1/8, 3/4)$	$\chi_4^3\mu$
ψ_{12}	$24\mathfrak{e}_0 + q^{-1}\mathfrak{e}_0$	$(1, 0, 0)$	μ

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ψ_{24}	$48\mathfrak{e}_0 + q^{-3/2}\mathfrak{e}_{1/2}$	$(2,0,0)$	μ
ψ_{48}	$96\mathfrak{e}_0 - q^{-1/8}(\mathfrak{e}_{1/4}+\mathfrak{e}_{3/4}) + q^{-9/8}(\mathfrak{e}_{1/4}+\mathfrak{e}_{3/4})$	$(4,1/4,2)$	$-$

A.3 Level $t = 3$

Table A.5: Obstruction space, level $t = 3$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - 24q - 72q^2 - 238q^3 - 216q^4 - 288q^5 + O(q^6)$
$\mathfrak{e}_{1/6}, \mathfrak{e}_{5/6}$	$q^{1/12}(-1 - 24q - 121q^2 - 120q^3 - 337q^4 - 264q^5 + O(q^6))$
$\mathfrak{e}_{1/3}, \mathfrak{e}_{2/3}$	$q^{1/3}(-7 - 55q - 96q^2 - 168q^3 - 264q^4 - 439q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{3/4}(-34 - 48q - 144q^2 - 192q^3 - 336q^4 - 288q^5 + O(q^6))$

Table A.6: Holomorphic products of weight less than 50

	Principal part	Weyl vector	Character
ψ_1	$2\mathfrak{e}_0 + q^{-1/12}(\mathfrak{e}_{1/6}+\mathfrak{e}_{5/6})$	$(1/6,1/12,1/6)$	$\chi_6\mu$
ψ_6	$12\mathfrak{e}_0 - q^{-1/12}(\mathfrak{e}_{1/6}+\mathfrak{e}_{5/6}) + q^{-1/3}(\mathfrak{e}_{1/3}+\mathfrak{e}_{2/3})$	$(1/2,1/12,1/2)$	χ_6^3
ψ_{12}	$24\mathfrak{e}_0 + q^{-1}\mathfrak{e}_0$	$(1,0,0)$	μ
ψ_{16}	$32\mathfrak{e}_0 - q^{-1/12}(\mathfrak{e}_{1/6}+\mathfrak{e}_{5/6}) + q^{-3/4}\mathfrak{e}_{1/2}$	$(4/3,1/6,4/3)$	χ_6^2
$\psi_{24}^{(1)}$	$48\mathfrak{e}_0 + q^{-7/4}\mathfrak{e}_{1/2}$	$(2,0,0)$	μ
$\psi_{24}^{(2)}$	$48\mathfrak{e}_0 + q^{-13/12}(\mathfrak{e}_{1/6}+\mathfrak{e}_{5/6})$	$(2,0,0)$	$-$
ψ_{36}	$72\mathfrak{e}_0 + q^{-2}\mathfrak{e}_0$	$(3,0,0)$	μ
ψ_{48}	$96\mathfrak{e}_0 - q^{-1/3}(\mathfrak{e}_{1/3}+\mathfrak{e}_{2/3}) + q^{4/3}(\mathfrak{e}_{1/3}+\mathfrak{e}_{2/3})$	$(4,1/6,2)$	$-$

A.4 Level $t = 4$

Paramodular forms of level 4 are Siegel modular forms of genus 2 for a congruence subgroup.

Table A.7: Obstruction space, level $t = 4$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - 24q - 72q^2 - 96q^3 - 358q^4 - 192q^5 + O(q^6)$
$\mathfrak{e}_{1/8}, \mathfrak{e}_{7/8}$	$q^{1/16}(-\frac{1}{2} - 24q - 72q^2 - \frac{337}{2}q^3 - 192q^4 - \frac{673}{2}q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{1/4}(-5 - 24q - 125q^2 - 120q^3 - 240q^4 - 240q^5 + O(q^6))$
$\mathfrak{e}_{3/8}, \mathfrak{e}_{5/8}$	$q^{9/16}(-\frac{25}{2} - \frac{121}{2}q - 96q^2 - 168q^3 - 264q^4 - 312q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$0 - 46q - 48q^2 - 144q^3 - 192q^4 - 336q^5 + O(q^6)$

Table A.8: Holomorphic products of weight less than 50

	Principal part	Weyl vector	Character
$\psi_{1/2}$	$\mathfrak{e}_0 + q^{-1/16}(\mathfrak{e}_{1/8} + \mathfrak{e}_{7/8})$	$(1/8, 1/16, 1/8)$	$\sqrt{\chi_4\mu}$
$\psi_{9/2}$	$9\mathfrak{e}_0 - q^{-1/16}(\mathfrak{e}_{1/8} + \mathfrak{e}_{7/8}) + q^{-1/4}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4})$	$(3/8, 1/16, 3/8)$	$\sqrt{\chi_4^3\mu}$
$\psi_{12}^{(1)}$	$24\mathfrak{e}_0 - q^{-1/16}(\mathfrak{e}_{1/8} + \mathfrak{e}_{7/8}) + q^{-9/16}(\mathfrak{e}_{3/8} + \mathfrak{e}_{5/8})$	$(1, 1/8, 1)$	$-$
$\psi_{12}^{(2)}$	$24\mathfrak{e}_0 + q^{-1}\mathfrak{e}_0$	$(1, 0, 0)$	μ
ψ_{18}	$36\mathfrak{e}_0 - q^{-1/4}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4}) + q^{-1}\mathfrak{e}_{1/2}$	$(3/2, 1/8, 3/2)$	χ_4^2
$\psi_{24}^{(1)}$	$48\mathfrak{e}_0 + q^{-2}\mathfrak{e}_{1/2}$	$(2, 0, 0)$	μ
$\psi_{24}^{(2)}$	$48\mathfrak{e}_0 + q^{-5/4}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4})$	$(2, 0, 0)$	$-$
$\psi_{24}^{(3)}$	$48\mathfrak{e}_0 + q^{-17/16}(\mathfrak{e}_{1/8} + \mathfrak{e}_{7/8})$	$(2, 0, 0)$	$-$
ψ_{36}	$72\mathfrak{e}_0 + q^{-2}\mathfrak{e}_0$	$(3, 0, 0)$	μ
ψ_{48}	$96\mathfrak{e}_0 + q^{-3}\mathfrak{e}_0$	$(4, 0, 0)$	$-$

A.5 Level $t = 5$

Table A.9: Obstruction space, level $t = 5$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - \frac{264}{13}q - \frac{840}{13}q^2 - \frac{1440}{13}q^3 - 168q^4 - \frac{5110}{13}q^5 + O(q^6)$

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$\mathfrak{e}_{1/10}, \mathfrak{e}_{9/10}$	$q^{1/20}(-\frac{5}{13} - \frac{240}{13}q - \frac{960}{13}q^2 - \frac{1320}{13}q^3 - \frac{3365}{13}q^4 - \frac{2280}{13}q^5 + O(q^6))$
$\mathfrak{e}_{1/5}, \mathfrak{e}_{4/5}$	$q^{1/5}(-\frac{35}{13} - \frac{360}{13}q - \frac{840}{13}q^2 - \frac{2195}{13}q^3 - \frac{2640}{13}q^4 - \frac{3000}{13}q^5 + O(q^6))$
$\mathfrak{e}_{3/10}, \mathfrak{e}_{7/10}$	$q^{9/20}(-\frac{125}{13} - \frac{360}{13}q - \frac{1685}{13}q^2 - \frac{1440}{13}q^3 - 240q^4 - \frac{3240}{13}q^5 + O(q^6))$
$\mathfrak{e}_{2/5}, \mathfrak{e}_{3/5}$	$q^{4/5}(-\frac{275}{13} - \frac{875}{13}q - \frac{1200}{13}q^2 - \frac{2280}{13}q^3 - \frac{3240}{13}q^4 - \frac{3960}{13}q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/4}(-\frac{24}{13} - \frac{730}{13}q - \frac{744}{13}q^2 - \frac{1920}{13}q^3 - \frac{2160}{13}q^4 - \frac{4320}{13}q^5 + O(q^6))$
	$\frac{65}{12}(E_{5/2} - Q_{5/2,1/20,1/10})$
$\mathfrak{e}_0$	$0 + 20q + 40q^2 - 80q^3 + 0q^4 - 60q^5 + O(q^6)$
$\mathfrak{e}_{1/10}, \mathfrak{e}_{9/10}$	$q^{1/20}(-1 + 30q - 10q^2 - 30q^3 + 29q^4 - 40q^5 + O(q^6))$
$\mathfrak{e}_{1/5}, \mathfrak{e}_{4/5}$	$q^{1/5}(6 - 20q + 40q^2 + 16q^3 - 60q^4 - 80q^5 + O(q^6))$
$\mathfrak{e}_{3/10}, \mathfrak{e}_{7/10}$	$q^{9/20}(1 - 20q + q^2 + 50q^3 + 0q^4 - 50q^5 + O(q^6))$
$\mathfrak{e}_{2/5}, \mathfrak{e}_{3/5}$	$q^{4/5}(-16 - 6q + 20q^2 - 40q^3 + 80q^4 + 40q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/4}(-10 + 10q - 50q^2 - 20q^3 + 140q^4 + 20q^5 + O(q^6))$

Table A.10: Holomorphic products of weight less than 50

	Principal part	Weyl vector	Character
ψ_4	$8\mathfrak{e}_0 + q^{-1/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) +$ $+ q^{-1/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5}) + q^{-1/4}\mathfrak{e}_{1/2}$	$(1/2, 3/20, 1/2)$	χ_2
ψ_5	$10\mathfrak{e}_0 + 6q^{-1/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) + q^{-1/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5})$	$(1, 2/5, 1)$	μ
ψ_{10}	$20\mathfrak{e}_0 + q^{-1/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) + q^{-9/20}(\mathfrak{e}_{3/10} + \mathfrak{e}_{7/10})$	$(1, 1/5, 1)$	$-$
ψ_{12}	$24\mathfrak{e}_0 + 2q^{-1/4}\mathfrak{e}_{1/2} + q^{-1}\mathfrak{e}_0$	$(1, 0, 0)$	μ
ψ_{24}	$48\mathfrak{e}_0 + 6q^{-1/4}\mathfrak{e}_{1/2} + q^{-21/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10})$	$(2, 0, 0)$	$-$
$\psi_{29}^{(1)}$	$58\mathfrak{e}_0 + q^{-1/4}\mathfrak{e}_{1/2} + q^{-5/4}\mathfrak{e}_{1/2}$	$(5/2, 1/4, 5/2)$	$\chi_2\mu$
$\psi_{29}^{(2)}$	$58\mathfrak{e}_0 - 3q^{1/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) + 3q^{-1/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5}) +$ $+ q^{-1/4}\mathfrak{e}_{1/2} + q^{-4/5}(\mathfrak{e}_{2/5} + \mathfrak{e}_{3/5})$	$(5/2, 7/20, 5/2)$	$\chi_2\mu$
$\psi_{35}^{(1)}$	$70\mathfrak{e}_0 - 2q^{-1/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) +$ $+ 3q^{-1/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5}) - q^{-1/4}\mathfrak{e}_{1/2} + q^{-9/4}\mathfrak{e}_{1/2}$	$(3, 1/5, 1)$	$-$
$\psi_{35}^{(2)}$	$70\mathfrak{e}_0 - 2q^{-1/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) +$ $+ 3q^{-1/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5}) + q^{-29/20}(\mathfrak{e}_{3/10} + \mathfrak{e}_{7/10})$	$(3, 1/5, 1)$	μ

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$\psi_{35}^{(3)}$	$70\mathfrak{e}_0 - 2q^{-1/20}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) +$ $+ 3q^{-1/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5}) + q^{-6/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5})$	$(3, 1/5, 1)$	μ
ψ_{36}	$72\mathfrak{e}_0 + 4q^{-1/4}\mathfrak{e}_{1/2} + q^{-2}\mathfrak{e}_0$	$(3, 0, 0)$	μ

Borcherds products for paramodular forms of level 5 were studied in detail in [44], especially chapter 7. The analogous tables on pages 90 and 121-122 of [44] have a number of entries such as the product $\psi_4^4\psi_5^{-1}$ of weight 11 which are omitted here. The products of weight 35 and 36 here do not appear in [44], possibly because their input functions have poles of higher order in ∞.

A.6 Level $t = 6$

Table A.11: Obstruction space, level $t = 6$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - \frac{72}{5}q - \frac{312}{5}q^2 - 96q^3 - \frac{936}{5}q^4 - \frac{816}{5}q^5 + O(q^6)$
$\mathfrak{e}_{1/12}, \mathfrak{e}_{5/2}, \mathfrak{e}_{7/2}, \mathfrak{e}_{11/12}$	$q^{1/24}(-\frac{1}{5} - \frac{121}{5}q - \frac{337}{5}q^2 - \frac{528}{5}q^3 - \frac{816}{5}q^4 - \frac{1321}{5}q^5 + O(q^6))$
$\mathfrak{e}_{1/6}, \mathfrak{e}_{5/6}$	$q^{1/6}(-\frac{11}{5} - \frac{96}{5}q - 72q^2 - \frac{456}{5}q^3 - \frac{1331}{5}q^4 - 192q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{3/8}(-\frac{34}{5} - \frac{144}{5}q - \frac{336}{5}q^2 - \frac{898}{5}q^3 - \frac{864}{5}q^4 - 240q^5 + O(q^6))$
$\mathfrak{e}_{1/3}, \mathfrak{e}_{2/3}$	$q^{2/3}(-\frac{83}{5} - \frac{168}{5}q - \frac{659}{5}q^2 - \frac{552}{5}q^3 - \frac{1248}{5}q^4 - \frac{1104}{5}q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/2}(-\frac{24}{5} - \frac{374}{5}q - \frac{288}{5}q^2 - 144q^3 - \frac{888}{5}q^4 - \frac{1584}{5}q^5 + O(q^6))$
	$5 \cdot (E_{5/2} - Q_{5/2,1/24,1/12})$
$\mathfrak{e}_0$	$0 + 48q + 48q^2 + 0q^3 - 96q^4 - 96q^5 + O(q^6)$
$\mathfrak{e}_{1/12}, \mathfrak{e}_{5/12}, \mathfrak{e}_{7/12}, \mathfrak{e}_{11/12}$	$q^{1/24}(-1 - q + 23q^2 - 48q^3 + 24q^4 - q^5 + O(q^6))$
$\mathfrak{e}_{1/6}, \mathfrak{e}_{5/6}$	$q^{1/6}(4 + 24q + 0q^2 + 24q^3 + 4q^4 - 120q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{3/8}(6 - 24q + 24q^2 - 18q^3 + 96q^4 - 120q^5 + O(q^6))$
$\mathfrak{e}_{1/3}, \mathfrak{e}_{2/3}$	$q^{2/3}(-8 - 48q + 16q^2 + 48q^3 - 48q^4 + 96q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/2}(-24 - 24q - 48q^2 + 0q^3 + 72q^4 + 96q^5 + O(q^6))$

Table A.12: Holomorphic products of weight less than 25

	Principal part	Weyl vector	Character
$\psi_3^{(1)}$	$6\mathfrak{e}_0 - q^{-1/24}(\mathfrak{e}_{1/12} + \mathfrak{e}_{11/12}) +$ $+ 5q^{-1/24}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12}) + q^{-1/6}(\mathfrak{e}_{1/6} + \mathfrak{e}_{5/6})$	$(1/4, 1/24, 1/4)$	$\chi_{12}^3\mu$
$\psi_3^{(2)}$	$6\mathfrak{e}_0 + 4q^{-1/24}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12}) + q^{-1/6}(\mathfrak{e}_{1/6} + \mathfrak{e}_{5/6})$	$(1/3, 1/12, 1/3)$	$\chi_{12}^4\mu$
ψ_6	$12\mathfrak{e}_0 - 2q^{1/24}(\mathfrak{e}_{1/12} + \mathfrak{e}_{5/12} + \mathfrak{e}_{7/12} + \mathfrak{e}_{11/12}) +$ $+ 2q^{-1/6}(\mathfrak{e}_{1/6} + \mathfrak{e}_{5/6}) + q^{-1/2}\mathfrak{e}_{1/2}$	$(1/2, 1/12, 1/2)$	χ_{12}^6
ψ_8	$16\mathfrak{e}_0 - q^{-1/24}(\mathfrak{e}_{1/12} + \mathfrak{e}_{11/12}) +$ $+ 7q^{-1/24}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12}) + q^{-3/8}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4})$	$(2/3, 1/12, 2/3)$	χ_{12}^8
ψ_{12}	$24\mathfrak{e}_0 + 24q^{-1/24}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12}) + q^{-1}\mathfrak{e}_0$	$(1, 0, 0)$	μ
ψ_{18}	$36\mathfrak{e}_0 - 2q^{-1/24}(\mathfrak{e}_{1/12} + \mathfrak{e}_{5/12} + \mathfrak{e}_{7/12} + \mathfrak{e}_{11/12}) +$ $+ q^{-1/6}(\mathfrak{e}_{1/6} + \mathfrak{e}_{5/6}) + q^{-2/3}(\mathfrak{e}_{1/3} + \mathfrak{e}_{2/3})$	$(3/2, 1/6, 3/2)$	–
$\psi_{24}^{(1)}$	$48\mathfrak{e}_0 - q^{-1/24}(\mathfrak{e}_{1/12} + \mathfrak{e}_{11/12}) +$ $+ q^{-25/24}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12})$	$(2, \frac{1}{6}, 2)$	–
$\psi_{24}^{(2)}$	$48\mathfrak{e}_0 - q^{-1/24}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12}) +$ $+ q^{-25/24}(\mathfrak{e}_{1/12} + \mathfrak{e}_{11/12})$	$(2, 0, 0)$	–
$\psi_{24}^{(3)}$	$48\mathfrak{e}_0 + 24q^{-1/24}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12}) +$ $+ q^{-7/6}(\mathfrak{e}_{1/6} + \mathfrak{e}_{5/6})$	$(2, 0, 0)$	–

A.7 Level $t = 7$

Table A.13: Obstruction space, level $t = 7$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - \frac{96}{5}q - 48q^2 - \frac{528}{5}q^3 - \frac{864}{5}q^4 - \frac{912}{5}q^5 + O(q^6)$
$\mathfrak{e}_{1/14}, \mathfrak{e}_{13/14}$	$q^{1/28}(-\frac{1}{5} - \frac{72}{5}q - \frac{336}{5}q^2 - \frac{432}{5}q^3 - \frac{864}{5}q^4 - \frac{864}{5}q^5 + O(q^6))$
$\mathfrak{e}_{1/7}, \mathfrak{e}_{6/7}$	$q^{1/7}(-\frac{7}{5} - \frac{108}{5}q - \frac{288}{5}q^2 - \frac{552}{5}q^3 - \frac{792}{5}q^4 - 275q^5 + O(q^6))$
$\mathfrak{e}_{3/14}, \mathfrak{e}_{11/4}$	$q^{9/28}(-5 - 24q - \frac{384}{5}q^2 - \frac{432}{5}q^3 - \frac{1321}{5}q^4 - 168q^5 + O(q^6))$
$\mathfrak{e}_{2/7}, \mathfrak{e}_{5/7}$	$q^{4/7}(-11 - \frac{168}{5}q - \frac{372}{5}q^2 - \frac{847}{5}q^3 - \frac{876}{5}q^4 - \frac{1248}{5}q^5 + O(q^6))$
$\mathfrak{e}_{5/14}, \mathfrak{e}_{9/14}$	$q^{25/28}(-\frac{121}{5} - \frac{168}{5}q - \frac{673}{5}q^2 - \frac{648}{5}q^3 - \frac{1152}{5}q^4 - \frac{1056}{5}q^5 + O(q^6))$

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$\mathfrak{e}_{3/7}, \mathfrak{e}_{4/7}$	$q^{2/7}(-\frac{12}{5} - 35q - \frac{439}{5}q^2 - 96q^3 - \frac{816}{5}q^4 - 264q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{3/4}(-\frac{48}{5} - \frac{386}{5}q - \frac{288}{5}q^2 - \frac{864}{5}q^3 - \frac{816}{5}q^4 - \frac{1536}{5}q^5 + O(q^6))$

	$\frac{35}{12}(E_{5/2} - Q_{5/2,1/28,1/14})$
$\mathfrak{e}_0$	$0 + 14q + 70q^2 - 28q^3 - 14q^4 - 112q^5 + O(q^6)$
$\mathfrak{e}_{1/14}, \mathfrak{e}_{13/14}$	$q^{1/28}(-1 + 28q + 14q^2 + 28q^3 - 14q^4 - 84q^5 + O(q^6))$
$\mathfrak{e}_{1/7}, \mathfrak{e}_{6/7}$	$q^{1/7}(3 + 7q + 42q^2 - 42q^3 + 28q^4 - 25q^5 + O(q^6))$
$\mathfrak{e}_{3/14}, \mathfrak{e}_{11/4}$	$q^{9/28}(5 + 0q - 14q^2 + 28q^3 + 19q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{2/7}, \mathfrak{e}_{5/7}$	$q^{4/7}(5 - 28q - 7q^2 + 33q^3 + 49q^4 - 98q^5 + O(q^6))$
$\mathfrak{e}_{5/14}, \mathfrak{e}_{9/14}$	$q^{25/28}(-11 - 28q + 17q^2 - 28q^3 + 28q^4 + 84q^5 + O(q^6))$
$\mathfrak{e}_{3/7}, \mathfrak{e}_{4/7}$	$q^{2/7}(-7 - 15q - 29q^2 + 0q^3 + 14q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{3/4}(-28 + 14q - 28q^2 - 84q^3 + 84q^4 + 84q^5 + O(q^6))$

Table A.14: Holomorphic products of weight less than 25

	Principal part	Weyl vector	Character
ψ_2	$4\mathfrak{e}_0 + 3q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) +$ $+ q^{-1/7}(\mathfrak{e}_{1/7} + \mathfrak{e}_{6/7})$	$(\frac{1}{2}, \frac{5}{28}, \frac{1}{2})$	χ_2
ψ_5	$10\mathfrak{e}_0 - q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) +$ $+ 2q^{-1/7}(\mathfrak{e}_{1/7} + \mathfrak{e}_{6/7}) + q^{-2/7}(\mathfrak{e}_{3/7} + \mathfrak{e}_{4/7})$	$(\frac{1}{2}, \frac{3}{28}, \frac{1}{2})$	$\chi_2\mu$
ψ_6	$12\mathfrak{e}_0 + 5q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) +$ $+ q^{-9/28}(\mathfrak{e}_{3/14} + \mathfrak{e}_{11/14})$	$(1, \frac{2}{7}, 1)$	$-$
$\psi_{10}^{(1)}$	$20\mathfrak{e}_0 - 2q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) +$ $+ 4q^{-1/7}(\mathfrak{e}_{1/7} + \mathfrak{e}_{6/7}) + q^{-3/4}\mathfrak{e}_{1/2}$	$(1, \frac{3}{14}, 1)$	$-$
$\psi_{10}^{(2)}$	$20\mathfrak{e}_0 + 2q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) -$ $- q^{-1/7}(\mathfrak{e}_{1/7} + \mathfrak{e}_{6/7}) + q^{-4/7}(\mathfrak{e}_{2/7} + \mathfrak{e}_{5/7})$	$(1, \frac{1}{7}, 1)$	$-$
ψ_{11}	$22\mathfrak{e}_0 + 7q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) + q^{-1}\mathfrak{e}_0$	$(\frac{3}{2}, \frac{1}{4}, \frac{1}{2})$	χ_2
ψ_{20}	$40\mathfrak{e}_0 + 28q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) +$ $+ q^{-29/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14})$	$(4, 1, 2)$	$-$

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ψ_{22}	$44\mathfrak{e}_0 + 14q^{-1/28}(\mathfrak{e}_{1/14} + \mathfrak{e}_{13/14}) - q^{-2/7}(\mathfrak{e}_{3/7} + \mathfrak{e}_{4/7}) + q^{-8/7}(\mathfrak{e}_{1/7} + \mathfrak{e}_{6/7})$	$(3, \frac{1}{2}, 1)$	–
ψ_{24}	$48\mathfrak{e}_0 + q^{-37/28}(\mathfrak{e}_{3/14} + \mathfrak{e}_{11/14})$	$(2, 0, 0)$	–

Gritsenko and Nikulin have constructed a holomorphic product of weight 12 and character μ and Weyl vector $(1, 0, 0)$ for every level $t > 1$ by restricting the singular-weight product on the Grassmannian of $II_{26,2}$ ([35], remark 4.4). It is easy to find in the previous tables. It does not appear in this table because for $t = 7$ it can be factored into products of smaller weight: $\psi_{11}\psi_5\psi_2^{-2}$. We leave it as an exercise to find this form in the tables below.

A.8 Level $t = 8$

Table A.15: Obstruction space, level $t = 8$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - 12q - 72q^2 - 72q^3 - 168q^4 - 168q^5 + O(q^6)$
$\mathfrak{e}_{1/16}, \mathfrak{e}_{15/16}$	$q^{1/32}(-\frac{1}{8} - 18q - 48q^2 - 102q^3 - 150q^4 - 192q^5 + O(q^6))$
$\mathfrak{e}_{1/8}, \mathfrak{e}_{7/8}$	$q^{1/8}(-\frac{5}{4} - \frac{75}{4}q - 60q^2 - \frac{363}{4}q^3 - 180q^4 - 144q^5 + O(q^6))$
$\mathfrak{e}_{3/16}, \mathfrak{e}_{13/16}$	$q^{9/32}(-\frac{25}{8} - 24q - 66q^2 - 108q^3 - 144q^4 - \frac{2185}{8}q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{1/2}(-11 - 24q - 72q^2 - 96q^3 - 275q^4 - 168q^5 + O(q^6))$
$\mathfrak{e}_{3/8}, \mathfrak{e}_{5/8}$	$q^{1/8}(-\frac{3}{4} - \frac{125}{4}q - 36q^2 - \frac{605}{4}q^3 - 108q^4 - 240q^5 + O(q^6))$
$\mathfrak{e}_{7/16}, \mathfrak{e}_{9/16}$	$q^{17/32}(-6 - \frac{337}{8}q - \frac{673}{8}q^2 - 108q^3 - 192q^4 - 234q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$0 - 12q - 94q^2 - 72q^3 - 144q^4 - 168q^5 + O(q^6)$

	$\frac{8}{3}(E_{5/2} - Q_{5/2,1/32,1/16})$
$\mathfrak{e}_0$	$0 + 32q + 0q^2 + 64q^3 + 0q^4 - 64q^5 + O(q^6)$
$\mathfrak{e}_{1/16}, \mathfrak{e}_{15/16}$	$q^{1/32}(-1 + 16q + 64q^2 - 16q^3 + 48q^4 - 128q^5 + O(q^6))$
$\mathfrak{e}_{1/8}, \mathfrak{e}_{7/8}$	$q^{1/8}(2 + 14q + 32q^2 + 14q^3 - 32q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{3/16}, \mathfrak{e}_{13/16}$	$q^{9/32}(7 + 0q + 16q^2 - 32q^3 + 64q^4 - 9q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/2}$	0
$\mathfrak{e}_{5/16}, \mathfrak{e}_{11/16}$	$q^{25/32}(7 - 48q - 16q^2 + 55q^3 + 16q^4 - 32q^5 + O(q^6))$

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$\mathfrak{e}_{3/8}, \mathfrak{e}_{5/8}$	$q^{1/8}(-2 - 14q - 32q^2 - 14q^3 + 32q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{7/16}, \mathfrak{e}_{9/16}$	$q^{17/32}(-16 - 17q - q^2 - 32q^3 - 64q^4 + 80q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$0 - 32q + 0q^2 - 64q^3 + 0q^4 + 64q^5 + O(q^6)$

Table A.16: Holomorphic products of weight less than 20

	Principal part	Weyl vector	Character
$\psi_{3/2}$	$3\mathfrak{e}_0 + 2q^{-1/32}(\mathfrak{e}_{1/16} + \mathfrak{e}_{15/16}) +$ $+ q^{-1/8}(\mathfrak{e}_{1/8} + \mathfrak{e}_{7/8})$	$(3/8, 1/8, 3/8)$	$\sqrt{\chi_4^3\mu}$
ψ_2	$4\mathfrak{e}_0 + q^{-1/8}(\mathfrak{e}_{1/8} + \mathfrak{e}_{3/8} + \mathfrak{e}_{5/8} + \mathfrak{e}_{7/8})$	$(1/4, 1/16, 1/4)$	χ_4
ψ_4	$8\mathfrak{e}_0 + 7q^{-1/32}(\mathfrak{e}_{1/16} + \mathfrak{e}_{15/16}) +$ $+ q^{-9/32}(\mathfrak{e}_{3/16} + \mathfrak{e}_{13/16})$	$(1, 5/16, 1)$	$-$
ψ_8	$16\mathfrak{e}_0 + 16q^{-1/32}(\mathfrak{e}_{1/16} + \mathfrak{e}_{15/16}) + q^{-1}\mathfrak{e}_0$	$(2, 1/2, 1)$	μ
ψ_9	$18\mathfrak{e}_0 - q^{-1/8}(\mathfrak{e}_{1/8} + \mathfrak{e}_{3/8} + \mathfrak{e}_{5/8} + \mathfrak{e}_{7/8}) +$ $+ q^{-1/2}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4})$	$(3/4, 1/16, 3/4)$	$\chi_4^3\mu$
$\psi_{12}^{(1)}$	$24\mathfrak{e}_0 - 2q^{-1/32}(\mathfrak{e}_{1/16} + \mathfrak{e}_{15/16}) +$ $+ 2q^{-9/32}(\mathfrak{e}_{3/16} + \mathfrak{e}_{13/16}) + q^{-1}\mathfrak{e}_{1/2}$	$(1, 1/8, 1)$	$-$
$\psi_{12}^{(2)}$	$24\mathfrak{e}_0 - 2q^{-1/32}(\mathfrak{e}_{1/16} + \mathfrak{e}_{15/16}) + 2q^{-9/32}(\mathfrak{e}_{3/16} +$ $+ \mathfrak{e}_{13/16}) + q^{-17/32}(\mathfrak{e}_{7/16} + \mathfrak{e}_{9/16})$	$(1, 1/8, 1)$	$-$
ψ_{16}	$32\mathfrak{e}_0 + 7q^{-1/32}(\mathfrak{e}_{1/16} + \mathfrak{e}_{15/16}) +$ $+ q^{-25/32}(\mathfrak{e}_{5/16} + \mathfrak{e}_{11/16})$	$(2, 3/8, 2)$	$-$

A.9 Level $t = 9$

Table A.17: Obstruction space, level $t = 9$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - \frac{56}{3}q - 40q^2 - 96q^3 - \frac{440}{3}q^4 - 176q^5 + O(q^6)$
$\mathfrak{e}_{1/18}, \mathfrak{e}_{17/18}$	$q^{1/36}(-\frac{1}{9} - \frac{40}{3}q - \frac{176}{3}q^2 - 72q^3 - \frac{512}{3}q^4 - 152q^5 + O(q^6))$
$\mathfrak{e}_{1/9}, \mathfrak{e}_{8/9}$	$q^{1/9}(-\frac{7}{9} - \frac{56}{3}q - \frac{152}{3}q^2 - 96q^3 - \frac{440}{3}q^4 - \frac{592}{3}q^5 + O(q^6))$

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$\mathfrak{e}_{1/6}, \mathfrak{e}_{5/6}$	$q^{1/4}(-\frac{11}{3} - 16q - 72q^2 - 88q^3 - 160q^4 - 144q^5 + O(q^6))$
$\mathfrak{e}_{2/9}, \mathfrak{e}_{7/9}$	$q^{4/9}(-\frac{55}{9} - \frac{88}{3}q - \frac{184}{3}q^2 - \frac{320}{3}q^3 - 168q^4 - \frac{2359}{9}q^5 + O(q^6))$
$\mathfrak{e}_{5/18}, \mathfrak{e}_{13/18}$	$q^{25/36}(-\frac{121}{9} - \frac{88}{3}q - \frac{272}{3}q^2 - \frac{272}{3}q^3 - \frac{2185}{9}q^4 - \frac{544}{3}q^5 + O(q^6))$
$\mathfrak{e}_{1/3}, \mathfrak{e}_{2/3}$	$0 - \frac{77}{3}q - 40q^2 - 72q^3 - \frac{605}{3}q^4 - 176q^5 + O(q^6)$
$\mathfrak{e}_{7/18}, \mathfrak{e}_{11/18}$	$q^{13/36}(-\frac{8}{3} - \frac{337}{9}q - 48q^2 - \frac{1321}{9}q^3 - \frac{344}{3}q^4 - \frac{784}{3}q^5 + O(q^6))$
$\mathfrak{e}_{4/9}, \mathfrak{e}_{5/9}$	$q^{7/9}(-\frac{32}{3} - \frac{439}{9}q - \frac{847}{9}q^2 - \frac{368}{3}q^3 - 168q^4 - \frac{728}{3}q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/4}(-\frac{8}{3} - 16q - 106q^2 - 64q^3 - 160q^4 - 192q^5 + O(q^6))$

	$\frac{9}{4}(E_{5/2} - Q_{5/2,1/36,1/18})$
$\mathfrak{e}_0$	$0 + 12q + 72q^2 + 0q^3 + 48q^4 - 72q^5 + O(q^6)$
$\mathfrak{e}_{1/18}, \mathfrak{e}_{17/18}$	$q^{1/36}(-1 + 24q + 30q^2 + 54q^3 - 6q^4 - 18q^5 + O(q^6))$
$\mathfrak{e}_{1/9}, \mathfrak{e}_{8/9}$	$q^{1/9}(2 + 12q + 48q^2 + 0q^3 + 48q^4 - 120q^5 + O(q^6))$
$\mathfrak{e}_{1/6}, \mathfrak{e}_{5/6}$	$q^{1/4}(3 + 18q + 0q^2 + 18q^3 + 18q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{2/9}, \mathfrak{e}_{7/9}$	$q^{4/9}(8 - 12q + 24q^2 - 24q^3 + 0q^4 + 26q^5 + O(q^6))$
$\mathfrak{e}_{5/18}, \mathfrak{e}_{13/18}$	$q^{25/36}(5 - 12q - 42q^2 + 12q^3 + 83q^4 - 30q^5 + O(q^6))$
$\mathfrak{e}_{1/3}, \mathfrak{e}_{2/3}$	$0 - 6q - 36q^2 + 0q^3 - 24q^4 + 36q^5 + O(q^6)$
$\mathfrak{e}_{7/18}, \mathfrak{e}_{11/18}$	$q^{13/36}(-6 - 13q - 54q^2 + 11q^3 + 12q^4 - 48q^5 + O(q^6))$
$\mathfrak{e}_{4/9}, \mathfrak{e}_{5/9}$	$q^{7/9}(-24 - 16q - 10q^2 - 60q^3 + 48q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/4}(-6 - 36q + 0q^2 - 36q^3 - 36q^4 + 0q^5 + O(q^6))$

Table A.18: Holomorphic products of weight less than 15

	Principal part	Weyl vector	Character
ψ_1	$2\mathfrak{e}_0 + 2q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) +$ $+ q^{-1/9}(\mathfrak{e}_{1/9} + \mathfrak{e}_{8/9})$	$(1/3, 1/9, 1/3)$	$\chi_6^2 \mu$
ψ_2	$4\mathfrak{e}_0 - q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) +$ $+ q^{-1/9}(\mathfrak{e}_{1/9} + \mathfrak{e}_{8/9}) + q^{-1/4}\mathfrak{e}_{1/2}$	$(1/6, 1/36, 1/6)$	χ_6
$\psi_4^{(1)}$	$8\mathfrak{e}_0 - 2q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) +$ $+ 2q^{-1/9}(\mathfrak{e}_{1/9} + \mathfrak{e}_{8/9}) + q^{-13/16}(\mathfrak{e}_{7/18} + \mathfrak{e}_{11/18})$	$(1/3, 1/18, 1/3)$	χ_6^2

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$\psi_4^{(2)}$	$8\mathfrak{e}_0 + 3q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) +$ $+ q^{-1/4}(\mathfrak{e}_{1/6} + \mathfrak{e}_{5/6})$	$(2/3, 1/6, 2/3)$	χ_6^4
ψ_6	$12\mathfrak{e}_0 + 6q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) -$ $- q^{-1/9}(\mathfrak{e}_{1/9} + \mathfrak{e}_{8/9}) + q^{-4/9}(\mathfrak{e}_{2/9} + \mathfrak{e}_{7/9})$	$(1, 2/9, 1)$	$-$
ψ_9	$18\mathfrak{e}_0 + 9q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) -$ $- q^{-1/4}\mathfrak{e}_{1/2} + q^{-1}\mathfrak{e}_0$	$(3/2, 1/4, 1/2)$	χ_6^3
ψ_{12}	$24\mathfrak{e}_0 - 6q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) +$ $+ 6q^{-1/9}(\mathfrak{e}_{1/9} + \mathfrak{e}_{8/9}) + q^{-5/4}\mathfrak{e}_{1/2}$	$(1, 1/6, 1)$	$-$
ψ_{14}	$28\mathfrak{e}_0 + 5q^{-1/36}(\mathfrak{e}_{1/18} + \mathfrak{e}_{17/18}) +$ $+ q^{-25/36}(\mathfrak{e}_{5/18} + \mathfrak{e}_{13/18})$	$(5/3, 5/18, 5/3)$	χ_6^4

A.10 Level $t = 10$

Table A.19: Obstruction space, level $t = 10$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - \frac{168}{13}q - \frac{648}{13}q^2 - \frac{816}{13}q^3 - 168q^4 - 144q^5 + O(q^6)$
$\mathfrak{e}_{1/20}, \mathfrak{e}_{19/20}$	$q^{1/40}(-\frac{1}{13} - \frac{192}{13}q - \frac{673}{13}q^2 - \frac{1321}{13}q^3 - \frac{1536}{13}q^4 - \frac{2352}{13}q^5 + O(q^6))$
$\mathfrak{e}_{1/10}, \mathfrak{e}_{9/10}$	$q^{1/10}(-\frac{11}{13} - \frac{168}{13}q - \frac{720}{13}q^2 - \frac{960}{13}q^3 - \frac{2112}{13}q^4 - 144q^5 + O(q^6))$
$\mathfrak{e}_{3/20}, \mathfrak{e}_{17/20}$	$q^{9/40}(-\frac{25}{13} - \frac{337}{13}q - 48q^2 - \frac{1200}{13}q^3 - \frac{2185}{13}q^4 - \frac{2256}{13}q^5 + O(q^6))$
$\mathfrak{e}_{1/5}, \mathfrak{e}_{4/5}$	$q^{2/5}(-\frac{83}{13} - \frac{240}{13}q - 72q^2 - \frac{1104}{13}q^3 - 168q^4 - \frac{2016}{13}q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{5/8}(-\frac{146}{13} - \frac{384}{13}q - \frac{864}{13}q^2 - \frac{1536}{13}q^3 - \frac{1824}{13}q^4 - \frac{3650}{13}q^5 + O(q^6))$
$\mathfrak{e}_{3/10}, \mathfrak{e}_{7/10}$	$q^{9/10}(-\frac{275}{13} - \frac{456}{13}q - \frac{1080}{13}q^2 - 96q^3 - \frac{3707}{13}q^4 - \frac{2040}{13}q^5 + O(q^6))$
$\mathfrak{e}_{7/20}, \mathfrak{e}_{13/20}$	$q^{9/40}(-\frac{25}{13} - \frac{337}{13}q - 48q^2 - \frac{1200}{13}q^3 - \frac{2185}{13}q^4 - \frac{2256}{13}q^5 + O(q^6))$
$\mathfrak{e}_{2/5}, \mathfrak{e}_{3/5}$	$q^{3/5}(-\frac{72}{13} - \frac{659}{13}q - \frac{600}{13}q^2 - \frac{2075}{13}q^3 - \frac{1776}{13}q^4 - 240q^5 + O(q^6))$
$\mathfrak{e}_{9/20}, \mathfrak{e}_{11/20}$	$q^{1/40}(-\frac{1}{13} - \frac{192}{13}q - \frac{673}{13}q^2 - \frac{1321}{13}q^3 - \frac{1536}{13}q^4 - \frac{2352}{13}q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/2}(-\frac{72}{13} - \frac{288}{13}q - \frac{1606}{13}q^2 - \frac{912}{13}q^3 - \frac{2232}{13}q^4 - \frac{2208}{13}q^5 + O(q^6))$

	$65(E_{5/2} - Q_{5/2,1/40,1/20})$
$\mathfrak{e}_0$	$0 + 720q + 1440q^2 + 2160q^3 + 0q^4 + 0q^5 + O(q^6)$
$\mathfrak{e}_{1/20}, \mathfrak{e}_{19/20}$	$q^{1/40}(-31 + 600q + 1315q^2 - 365q^3 + 3240q^4 - 2400q^5 + O(q^6))$

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$\mathfrak{e}_{1/10}, \mathfrak{e}_{9/10}$	$q^{1/10}(36 + 720q + 1080q^2 + 1440q^3 + 360q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{3/20}, \mathfrak{e}_{17/20}$	$q^{9/40}(161 - 125q + 1560q^2 + 240q^3 - 5q^4 - 1920q^5 + O(q^6))$
$\mathfrak{e}_{1/5}, \mathfrak{e}_{4/5}$	$q^{2/5}(144 + 360q + 0q^2 + 720q^3 + 0q^4 - 720q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{5/8}(180 - 360q + 360q^2 - 1440q^3 + 1800q^4 - 180q^5 + O(q^6))$
$\mathfrak{e}_{3/10}, \mathfrak{e}_{7/10}$	$q^{9/10}(-36 - 720q - 720q^2 - 36q^4 + 720q^5 + O(q^6))$
$\mathfrak{e}_{7/20}, \mathfrak{e}_{13/20}$	$q^{9/40}(-125 + 161q - 1560q^2 - 1320q^3 + 1841q^4 + 1200q^5 + O(q^6))$
$\mathfrak{e}_{2/5}, \mathfrak{e}_{3/5}$	$q^{3/5}(-360 - 864q - 1440q^2 - 144q^3 - 1080q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{9/20}, \mathfrak{e}_{11/20}$	$q^{1/40}(-5 - 960q - 271q^2 - 391q^3 - 1440q^4 - 840q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{1/2}(-360 - 1440q - 360q^2 - 1440q^3 - 1800q^4 + 1440q^5 + O(q^6))$
	$\frac{5}{2}(Q_{5/2,1/40,1/20} - Q_{5/2,1/40,9/20})$
$\mathfrak{e}_{1/20}, \mathfrak{e}_{19/20}$	$q^{1/40}(1 - 60q - 61q^2 - q^3 - 180q^4 + 60q^5 + O(q^6))$
$\mathfrak{e}_{3/20}, \mathfrak{e}_{17/20}$	$q^{9/40}(-11 + 11q - 120q^2 - 60q^3 + 71q^4 + 120q^5 + O(q^6))$
$\mathfrak{e}_{7/20}, \mathfrak{e}_{13/20}$	$q^{9/40}(11 - 11q + 120q^2 + 60q^3 - 71q^4 - 120q^5 + O(q^6))$
$\mathfrak{e}_{9/20}, \mathfrak{e}_{11/20}$	$q^{1/40}(-1 + 60q + 61q^2 + q^3 + 180q^4 - 60q^5 + O(q^6))$
other components	0

Table A.20: Holomorphic products of weight less than 15

	Principal part	Weyl vector	Character
ψ_1	$2\mathfrak{e}_0 + q^{-1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{9/20} + \mathfrak{e}_{11/20} + \mathfrak{e}_{19/20})+$ $+ q^{-1/10}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10})$	$(\frac{1}{4}, \frac{3}{40}, \frac{1}{4})$	$\chi_4\mu$
ψ_4	$8\mathfrak{e}_0 + q^{-1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{9/20} + \mathfrak{e}_{11/20} + \mathfrak{e}_{19/20})+$ $+ q^{-9/40}(\mathfrak{e}_{3/20} + \mathfrak{e}_{7/20} + \mathfrak{e}_{13/20} + \mathfrak{e}_{17/20})$	$(\frac{1}{2}, \frac{1}{10}, \frac{1}{2})$	χ_4^2
$\psi_5^{(1)}$	$10\mathfrak{e}_0 - 2q^{-1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{9/20} + \mathfrak{e}_{11/20} + \mathfrak{e}_{19/20})+$ $3q^{-1/10}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) + q^{-1/2}\mathfrak{e}_{1/2}$	$(\frac{1}{2}, \frac{1}{10}, \frac{1}{2})$	$\chi_4^2\mu$
$\psi_5^{(2)}$	$10\mathfrak{e}_0 - 2q^{-1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{19/20})+$ $+ 9q^{-1/40}(\mathfrak{e}_{9/20} + \mathfrak{e}_{11/20})+$ $+ 3q^{-1/10}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) + q^{-9/40}(\mathfrak{e}_{7/20} + \mathfrak{e}_{13/20})$	$(\frac{1}{2}, \frac{1}{10}, \frac{1}{2})$	$\chi_4^2\mu$

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ψ_6	$12\mathfrak{e}_0 + 3q^{-1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{9/20} + \mathfrak{e}_{11/20} + \mathfrak{e}_{19/20}) - {} - q^{-1/10}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) + q^{-2/5}(\mathfrak{e}_{1/5} + \mathfrak{e}_{4/5})$	$(\frac{3}{4}, \frac{1}{8}, \frac{3}{4})$	χ_4^3
ψ_8	$16\mathfrak{e}_0 + 10q^{-1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{9/20} + \mathfrak{e}_{11/20} + \mathfrak{e}_{19/20}) + {} + q^{-1}\mathfrak{e}_0$	$(\frac{3}{2}, \frac{1}{4}, \frac{1}{2})$	$\chi_4^2\mu$
ψ_9	$18\mathfrak{e}_0 - 5q^{-1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{9/20} + \mathfrak{e}_{11/20} + \mathfrak{e}_{19/20}) + {} + 5q^{-1/10}(\mathfrak{e}_{1/10} + \mathfrak{e}_{9/10}) + q^{-3/5}(\mathfrak{e}_{2/5} + \mathfrak{e}_{3/5})$	$(\frac{3}{4}, \frac{1}{8}, \frac{3}{4})$	$\chi_4^3\mu$
ψ_{12}	$24\mathfrak{e}_0 + 5q^{1/40}(\mathfrak{e}_{1/20} + \mathfrak{e}_{9/20} + \mathfrak{e}_{11/20} + \mathfrak{e}_{19/20}) + {} + q^{-5/8}(\mathfrak{e}_{1/4} + \mathfrak{e}_{3/4})$	$(\frac{3}{2}, \frac{1}{4}, \frac{3}{2})$	χ_4^2

A.11 Level $t = 11$

Table A.21: Obstruction space, level $t = 11$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - \frac{840}{61}q - \frac{2760}{61}q^2 - \frac{5040}{61}q^3 - \frac{7560}{61}q^4 - \frac{11040}{61}q^5 + O(q^6)$
$\mathfrak{e}_{1/22}, \mathfrak{e}_{21/22}$	$q^{1/44}(-\frac{5}{61} - \frac{744}{61}q - \frac{3120}{61}q^2 - \frac{4080}{61}q^3 - \frac{9360}{61}q^4 - \frac{7680}{61}q^5 + O(q^6))$
$\mathfrak{e}_{1/11}, \mathfrak{e}_{10/11}$	$q^{1/11}(-\frac{35}{61} - \frac{1080}{61}q - \frac{2400}{61}q^2 - \frac{5520}{61}q^3 - \frac{8184}{61}q^4 - \frac{10800}{61}q^5 + O(q^6))$
$\mathfrak{e}_{3/22}, \mathfrak{e}_{19/22}$	$q^{9/44}(-\frac{125}{61} - \frac{840}{61}q - \frac{4080}{61}q^2 - \frac{4320}{61}q^3 - \frac{9120}{61}q^4 - \frac{9720}{61}q^5 + O(q^6))$
$\mathfrak{e}_{2/11}, \mathfrak{e}_{9/11}$	$q^{4/11}(-\frac{275}{61} - \frac{1440}{61}q - \frac{3000}{61}q^2 - \frac{6600}{61}q^3 - \frac{8760}{61}q^4 - \frac{10200}{61}q^5 + O(q^6))$
$\mathfrak{e}_{5/22}, \mathfrak{e}_{17/22}$	$q^{25/44}(-\frac{605}{61} - \frac{1440}{61}q - \frac{4320}{61}q^2 - \frac{5160}{61}q^3 - \frac{11760}{61}q^4 - \frac{8424}{61}q^5 + O(q^6))$
$\mathfrak{e}_{3/11}, \mathfrak{e}_{8/11}$	$q^{9/11}(-\frac{875}{61} - \frac{2184}{61}q - \frac{4800}{61}q^2 - \frac{6480}{61}q^3 - \frac{9240}{61}q^4 - \frac{17555}{61}q^5 + O(q^6))$
$\mathfrak{e}_{7/22}, \mathfrak{e}_{15/22}$	$q^{5/44}(-\frac{24}{61} - \frac{1685}{61}q - \frac{2160}{61}q^2 - \frac{5760}{61}q^3 - \frac{6840}{61}q^4 - \frac{15125}{61}q^5 + O(q^6))$
$\mathfrak{e}_{4/11}, \mathfrak{e}_{7/11}$	$q^{5/11}(-\frac{264}{61} - \frac{2195}{61}q - \frac{3360}{61}q^2 - \frac{4920}{61}q^3 - \frac{11795}{61}q^4 - \frac{12960}{61}q^5 + O(q^6))$
$\mathfrak{e}_{9/22}, \mathfrak{e}_{13/22}$	$q^{37/44}(-\frac{600}{61} - \frac{3365}{61}q - \frac{3024}{61}q^2 - \frac{10925}{61}q^3 - \frac{7200}{61}q^4 - \frac{14400}{61}q^5 + O(q^6))$
$\mathfrak{e}_{5/11}, \mathfrak{e}_{6/11}$	$q^{3/11}(-\frac{120}{61} - \frac{1200}{61}q - \frac{4235}{61}q^2 - \frac{6875}{61}q^3 - \frac{6720}{61}q^4 - \frac{11880}{61}q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{3/4}(-\frac{720}{61} - \frac{1440}{61}q - \frac{7210}{61}q^2 - \frac{5280}{61}q^3 - \frac{11280}{61}q^4 - \frac{10800}{61}q^5 + O(q^6))$
	$\frac{671}{24}(E_{5/2} - Q_{5/2,1/44,1/22})$
$\mathfrak{e}_0$	$0 + 286q + 748q^2 + 374q^3 + 1232q^4 - 1034q^5 + O(q^6)$
$\mathfrak{e}_{1/22}, \mathfrak{e}_{21/22}$	$q^{1/44}(-15 + 330q + 583q^2 + 814q^3 + 407q^4 + 506q^5 + O(q^6))$
$\mathfrak{e}_{1/11}, \mathfrak{e}_{10/11}$	$q^{1/11}(17 + 176q + 913q^2 + 154q^3 + 946q^4 - 924q^5 + O(q^6))$

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$\mathfrak{e}_{3/22}, \mathfrak{e}_{19/22}$	$q^{9/44}(52 + 286q + 143q^2 + 704q^3 + 517q^4 - 429q^5 + O(q^6))$
$\mathfrak{e}_{2/11}, \mathfrak{e}_{9/11}$	$q^{4/11}(90 + 11q + 638q^2 - 341q^3 + 682q^4 - 649q^5 + O(q^6))$
$\mathfrak{e}_{5/22}, \mathfrak{e}_{17/22}$	$q^{25/44}(76 + 11q + 33q^2 + 319q^3 - 693q^4 + 165q^5 + O(q^6))$
$\mathfrak{e}_{3/11}, \mathfrak{e}_{8/11}$	$q^{9/11}(120 - 330q - 187q^2 - 286q^3 + 462q^4 - 144q^5 + O(q^6))$
$\mathfrak{e}_{7/22}, \mathfrak{e}_{15/22}$	$q^{5/44}(-11 - 53q - 319q^2 - 627q^3 - 451q^4 + 1168q^5 + O(q^6))$
$\mathfrak{e}_{4/11}, \mathfrak{e}_{7/11}$	$q^{5/11}(-121 - 58q - 869q^2 - 242q^3 + 239q^4 - 572q^5 + O(q^6))$
$\mathfrak{e}_{9/22}, \mathfrak{e}_{13/22}$	$q^{37/44}(-275 - 335q - 715q^2 - 445q^3 + 55q^4 + 110q^5 + O(q^6))$
$\mathfrak{e}_{5/11}, \mathfrak{e}_{6/11}$	$q^{3/11}(-55 - 550q - 444q^2 - 312q^3 - 396q^4 - 748q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{3/4}(-330 - 660q + 330q^2 - 1078q^3 - 1144q^4 + 418q^5 + O(q^6))$

	$\frac{671}{4}(E_{5/2} - Q_{5/2,1/11,1/11})$
$\mathfrak{e}_0$	$0 + 374q - 3564q^2 - 1782q^3 - 660q^4 + 3190q^5 + O(q^6)$
$\mathfrak{e}_{1/22}, \mathfrak{e}_{21/22}$	$q^{1/44}(32 - 704q - 528q^2 - 3168q^3 - 1584q^4 + 352q^5 + O(q^6))$
$\mathfrak{e}_{1/11}, \mathfrak{e}_{10/11}$	$q^{1/11}(-81 - 286q - 1903q^2 - 418q^3 - 1034q^4 - 176q^5 + O(q^6))$
$\mathfrak{e}_{3/22}, \mathfrak{e}_{19/22}$	$q^{9/44}(68 - 968q - 484q^2 - 1144q^3 - 924q^4 + 110q^5 + O(q^6))$
$\mathfrak{e}_{2/11}, \mathfrak{e}_{9/11}$	$q^{4/11}(-192 - 605q - 198q^2 + 1309q^3 - 2618q^4 + 803q^5 + O(q^6))$
$\mathfrak{e}_{5/22}, \mathfrak{e}_{17/22}$	$q^{25/44}(-520 + 66q + 1540q^2 - 770q^3 - 132q^4 - 1694q^5 + O(q^6))$
$\mathfrak{e}_{3/11}, \mathfrak{e}_{8/11}$	$q^{9/11}(-256 + 704q - 451q^2 + 2310q^3 - 1254q^4 + 844q^5 + O(q^6))$
$\mathfrak{e}_{7/22}, \mathfrak{e}_{15/22}$	$q^{5/44}(-66 + 292q + 770q^2 + 264q^3 + 2662q^4 - 1776q^5 + O(q^6))$
$\mathfrak{e}_{4/11}, \mathfrak{e}_{7/11}$	$q^{5/11}(-55 + 750q + 2167q^2 - 110q^3 - 823q^4 + 594q^5 + O(q^6))$
$\mathfrak{e}_{9/22}, \mathfrak{e}_{13/22}$	$q^{37/44}(1034 - 180q + 1078q^2 + 1844q^3 + 330q^4 + 660q^5 + O(q^6))$
$\mathfrak{e}_{5/11}, \mathfrak{e}_{6/11}$	$q^{3/11}(341 + 726q + 1484q^2 - 408q^3 + 308q^4 + 3564q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$q^{3/4}(704 + 1408q - 704q^2 + 4268q^3 - 1496q^4 - 2860q^5 + O(q^6))$

Table A.22: Holomorphic products of weight less than 15

	Principal part	Weyl vector	Character
ψ_3	$6\mathfrak{e}_0 + 2q^{-1/44}(\mathfrak{e}_{1/22} + \mathfrak{e}_{21/22}) +$ $+ 2q^{-5/44}(\mathfrak{e}_{7/22} + \mathfrak{e}_{15/22}) + q^{-9/44}(\mathfrak{e}_{3/22} + \mathfrak{e}_{19/22})$	$(\frac{1}{2}, \frac{5}{44}, \frac{1}{2})$	$\chi_2\mu$

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$\psi_4^{(1)}$	$8\mathfrak{e}_0 - q^{-1/44}(\mathfrak{e}_{1/22} + \mathfrak{e}_{21/22}) +$ $+ 3q^{-1/11}(\mathfrak{e}_{1/11} + \mathfrak{e}_{10/11}) + q^{-5/44}(\mathfrak{e}_{7/22} + \mathfrak{e}_{15/22}) +$ $+ q^{-3/11}(\mathfrak{e}_{5/11} + \mathfrak{e}_{6/11})$	$(\frac{1}{2}, \frac{5}{44}, \frac{1}{2})$	χ_2
$\psi_4^{(2)}$	$8\mathfrak{e}_0 + 5q^{-1/44}(\mathfrak{e}_{1/22} + \mathfrak{e}_{21/22}) +$ $+ 2q^{-1/11}(\mathfrak{e}_{1/11} + \mathfrak{e}_{10/11}) + q^{-5/44}(\mathfrak{e}_{7/22} + \mathfrak{e}_{15/22}) +$ $+ q^{-9/44}(\mathfrak{e}_{3/22} + \mathfrak{e}_{19/22})$	$(1, \frac{3}{11}, 1)$	$-$
ψ_9	$18\mathfrak{e}_0 + q^{-1/11}(\mathfrak{e}_{1/11} + \mathfrak{e}_{10/11}) +$ $+ 2q^{-9/44}(\mathfrak{e}_{3/22} + \mathfrak{e}_{19/22}) + q^{-5/11}(\mathfrak{e}_{4/11} + \mathfrak{e}_{7/11})$	$(1, \frac{2}{11}, 1)$	μ
$\psi_{10}^{(1)}$	$20\mathfrak{e}_0 - 3q^{-1/44}(\mathfrak{e}_{1/22} + \mathfrak{e}_{21/22}) +$ $+ 4q^{-1/11}(\mathfrak{e}_{1/11} + \mathfrak{e}_{10/11}) + q^{-9/44}(\mathfrak{e}_{3/22} + \mathfrak{e}_{19/22}) +$ $+ q^{-3/4}\mathfrak{e}_{1/2}$	$(1, \frac{2}{11}, 1)$	μ
$\psi_{10}^{(2)}$	$20\mathfrak{e}_0 + 7q^{-1/44}(\mathfrak{e}_{1/22} + \mathfrak{e}_{21/22}) +$ $+ q^{-1/11}(\mathfrak{e}_{1/11} + \mathfrak{e}_{10/11}) + 5q^{-5/44}(\mathfrak{e}_{7/22} + \mathfrak{e}_{15/22}) +$ $+ q^{-1}\mathfrak{e}_0$	$(\frac{3}{2}, \frac{9}{44}, \frac{1}{2})$	$\chi_2\mu$
ψ_{14}	$28\mathfrak{e}_0 + 22q^{-1/44}(\mathfrak{e}_{1/22} + \mathfrak{e}_{21/22}) +$ $+ q^{-45/44}(\mathfrak{e}_{1/22} + \mathfrak{e}_{21/22})$	$(3, \frac{1}{2}, 1)$	$-$

A.12 Level $t = 12$

Table A.23: Obstruction space, level $t = 12$

	$E_{5/2}$
$\mathfrak{e}_0$	$1 - \frac{48}{5}q - \frac{216}{5}q^2 - \frac{408}{5}q^3 - \frac{696}{5}q^4 - \frac{576}{5}q^5 + O(q^6)$
$\mathfrak{e}_{1/24}, \mathfrak{e}_{7/24}, \mathfrak{e}_{17/24}, \mathfrak{e}_{23/24}$	$q^{1/48}(-\frac{1}{20} - \frac{337}{20}q - \frac{204}{5}q^2 - \frac{384}{5}q^3 - \frac{588}{5}q^4 - \frac{852}{5}q^5 + O(q^6))$
$\mathfrak{e}_{1/12}, \mathfrak{e}_{5/12}, \mathfrak{e}_{7/12}, \mathfrak{e}_{11/12}$	$q^{1/12}(-\frac{1}{2} - 12q - \frac{121}{2}q^2 - 60q^3 - \frac{337}{2}q^4 - 132q^5 + O(q^6))$
$\mathfrak{e}_{1/8}, \mathfrak{e}_{7/8}$	$q^{3/16}(-\frac{17}{10} - \frac{84}{5}q - \frac{216}{5}q^2 - 96q^3 - \frac{588}{5}q^4 - \frac{828}{5}q^5 + O(q^6))$
$\mathfrak{e}_{1/6}, \mathfrak{e}_{5/6}$	$q^{1/3}(-\frac{23}{5} - \frac{96}{5}q - \frac{288}{5}q^2 - \frac{336}{5}q^3 - 168q^4 - \frac{768}{5}q^5 + O(q^6))$
$\mathfrak{e}_{5/24}, \mathfrak{e}_{11/24}, \mathfrak{e}_{13/24}, \mathfrak{e}_{19/24}$	$q^{25/48}(-\frac{121}{20} - \frac{132}{5}q - \frac{1321}{20}q^2 - \frac{437}{4}q^3 - \frac{696}{5}q^4 - 192q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	$q^{3/4}(-17 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 + O(q^6))$

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$\mathfrak{e}_{1/3}, \mathfrak{e}_{2/3}$	$q^{1/3}(-\frac{12}{5} - \frac{179}{5}q - \frac{192}{5}q^2 - \frac{504}{5}q^3 - 96q^4 - \frac{1427}{5}q^5 + O(q^6))$
$\mathfrak{e}_{3/8}, \mathfrak{e}_{5/8}$	$q^{11/16}(-\frac{36}{5} - \frac{449}{10}q - 60q^2 - \frac{468}{5}q^3 - \frac{2057}{10}q^4 - \frac{888}{5}q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$0 - \frac{72}{5}q - \frac{144}{5}q^2 - \frac{782}{5}q^3 - \frac{384}{5}q^4 - \frac{864}{5}q^5 + O(q^6)$

	$20(E_{5/2} - Q_{5/2,1/48,1/24})$
$\mathfrak{e}_0$	$0 + 288q + 576q^2 + 288q^3 + 576q^4 + 576q^5 + O(q^6)$
$\mathfrak{e}_{1/24}, \mathfrak{e}_{23/24}$	$q^{1/48}(-11 + 143q + 624q^2 + 384q^3 + 1008q^4 - 528q^5 + O(q^6))$
$\mathfrak{e}_{1/12}, \mathfrak{e}_{11/12}$	$q^{1/12}(10 + 240q + 230q^2 + 720q^3 - 10q^4 + 240q^5 + O(q^6))$
$\mathfrak{e}_{1/8}, \mathfrak{e}_{7/8}$	$q^{3/16}(36 + 144q + 576q^2 + 1008q^4 - 432q^5 + O(q^6))$
$\mathfrak{e}_{1/6}, \mathfrak{e}_{5/6}$	$q^{1/3}(48 + 96q + 288q^2 + 576q^3 + 0q^4 - 192q^5 + O(q^6))$
$\mathfrak{e}_{5/24}, \mathfrak{e}_{19/24}$	$q^{25/48}(109 - 48q + 119q^2 - 265q^3 + 576q^4 - 960q^5 + O(q^6))$
$\mathfrak{e}_{1/4}, \mathfrak{e}_{3/4}$	0
$\mathfrak{e}_{1/3}, \mathfrak{e}_{2/3}$	$q^{1/3}(-48 - 96q - 288q^2 - 576q^3 + 0q^4 + 192q^5 + O(q^6))$
$\mathfrak{e}_{3/8}, \mathfrak{e}_{5/8}$	$q^{11/16}(-144 - 108q - 720q^2 - 432q^3 + 36q^4 + 288q^5 + O(q^6))$
$\mathfrak{e}_{5/12}, \mathfrak{e}_{7/12}$	$q^{1/12}(-10 - 240q - 230q^2 - 720q^3 + 10q^4 - 240q^5 + O(q^6))$
$\mathfrak{e}_{11/24}, \mathfrak{e}_{13/24}$	$q^{25/48}(-121 - 528q - 131q^2 - 35q^3 - 864q^4 - 480q^5 + O(q^6))$
$\mathfrak{e}_{1/2}$	$0 - 288q - 576q^2 - 288q^3 - 576q^4 - 576q^5 + O(q^6)$

	$2(Q_{5/2,1/48,1/24} - Q_{5/2,1/48,7/24})$
$\mathfrak{e}_{1/24}, \mathfrak{e}_{23/24}$	$q^{1/48}(1 - q - 96q^2 - 48q^3 - 144q^4 + 48q^5 + O(q^6))$
$\mathfrak{e}_{1/12}, \mathfrak{e}_{11/12}$	$q^{1/12}(-2 - 48q - 46q^2 - 144q^3 + 2q^4 - 48q^5 + O(q^6))$
$\mathfrak{e}_{5/24}, \mathfrak{e}_{19/24}$	$q^{25/48}(-23 - 48q - 25q^2 + 23q^3 - 144q^4 + 48q^5 + O(q^6))$
$\mathfrak{e}_{7/24}, \mathfrak{e}_{17/24}$	$q^{1/48}(-1 + q + 96q^2 + 48q^3 + 144q^4 - 48q^5 + O(q^6))$
$\mathfrak{e}_{5/12}, \mathfrak{e}_{7/12}$	$q^{1/12}(2 + 48q + 46q^2 + 144q^3 - 2q^4 + 48q^5 + O(q^6))$
$\mathfrak{e}_{11/24}, \mathfrak{e}_{13/24}$	$q^{25/48}(23 + 48q + 25q^2 - 23q^3 + 144q^4 - 48q^5 + O(q^6))$
other components	0

Table A.24: Holomorphic products of weight less than 10

	Principal part	Weyl v.	Char.
ψ_1	$2\mathfrak{e}_0 + q^{-1/12}(\mathfrak{e}_{1/12} + \mathfrak{e}_{5/12} + \mathfrak{e}_{7/12} + \mathfrak{e}_{11/12})$	$(\frac{1}{6}, \frac{1}{24}, \frac{1}{6})$	$\chi_{12}^2\mu$
$\psi_{3/2}$	$3\mathfrak{e}_0 - q^{-1/48}(\mathfrak{e}_{1/24} + \mathfrak{e}_{23/24}) + q^{-1/48}(\mathfrak{e}_{7/24} + \mathfrak{e}_{17/24}) +$ $+ q^{-1/12}(\mathfrak{e}_{1/12} + \mathfrak{e}_{11/12}) + 2q^{-1/12}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12})$	$(\frac{1}{8}, \frac{1}{48}, \frac{1}{8})$	$\sqrt{\chi_{12}^3\mu}$
ψ_2	$4\mathfrak{e}_0 + 3q^{-\frac{1}{48}}(\mathfrak{e}_{1/24} + \mathfrak{e}_{7/24} + \mathfrak{e}_{17/24} + \mathfrak{e}_{23/24}) + q^{-\frac{3}{16}}(\mathfrak{e}_{1/8} + \mathfrak{e}_{7/8})$	$(\frac{1}{2}, \frac{1}{8}, \frac{1}{2})$	χ_{12}^6
$\psi_4^{(1)}$	$8\mathfrak{e}_0 - q^{-1/48}(\mathfrak{e}_{1/24} + \mathfrak{e}_{7/24} + \mathfrak{e}_{17/24} + \mathfrak{e}_{23/24}) +$ $+ q^{-3/16}(\mathfrak{e}_{1/8} + \mathfrak{e}_{7/8}) + q^{-1/3}(\mathfrak{e}_{1/3} + \mathfrak{e}_{2/3})$	$(\frac{1}{3}, \frac{1}{24}, \frac{1}{3})$	χ_{12}^4
$\psi_4^{(2)}$	$8\mathfrak{e}_0 + 4q^{-1/48}(\mathfrak{e}_{1/24} + \mathfrak{e}_{7/24} + \mathfrak{e}_{17/24} + \mathfrak{e}_{23/24}) -$ $- q^{-1/12}(\mathfrak{e}_{1/12} + \mathfrak{e}_{5/12} + \mathfrak{e}_{7/12} + \mathfrak{e}_{11/12}) + q^{-1/3}(\mathfrak{e}_{1/6} + \mathfrak{e}_{5/6})$	$(\frac{2}{3}, \frac{1}{8}, \frac{2}{3})$	χ_{12}^8
ψ_6	$12\mathfrak{e}_0 + 12q^{-1/48}(\mathfrak{e}_{1/24} + \mathfrak{e}_{7/24} + \mathfrak{e}_{17/24} + \mathfrak{e}_{23/24}) + q^{-1}\mathfrak{e}_0$	$(\frac{3}{2}, \frac{1}{4}, \frac{1}{2})$	$\chi_2\mu$
$\psi_9^{(1)}$	$18\mathfrak{e}_0 - 6q^{-1/48}(\mathfrak{e}_{1/24} + \mathfrak{e}_{23/24}) + 5q^{-1/48}(\mathfrak{e}_{7/24} + \mathfrak{e}_{17/24}) +$ $+ 6q^{-1/12}(\mathfrak{e}_{1/12} + \mathfrak{e}_{11/12}) + q^{-25/48}(\mathfrak{e}_{11/24} + \mathfrak{e}_{13/24})$	$(\frac{3}{4}, \frac{1}{8}, \frac{3}{4})$	$\chi_{12}^9\mu$
$\psi_9^{(2)}$	$18\mathfrak{e}_0 + 5q^{-1/48}(\mathfrak{e}_{1/24} + \mathfrak{e}_{23/24}) - 6q^{-1/48}(\mathfrak{e}_{7/24} + \mathfrak{e}_{17/24}) +$ $+ 6q^{-1/12}(\mathfrak{e}_{5/12} + \mathfrak{e}_{7/12}) + q^{-25/48}(\mathfrak{e}_{11/24} + \mathfrak{e}_{13/24})$	$(\frac{5}{4}, \frac{5}{24}, \frac{5}{4})$	$\chi_{12}^3\mu$

Appendix B

Hermitian modular forms

Any complex matrix $z \in \mathbb{C}^{2\times 2}$ can be decomposed in the form $z = x + iy$ where x, y are hermitian, i.e. $x = \overline{x^T}$, $y = \overline{y^T}$. (Despite the notation, x and y are generally not real.) The **Hermitian upper half-space of degree two** is the set $\mathbf{H}$ of those matrices z for which y as above is positive-definite. It is acted upon by the split-unitary group

$$U_{2,2}(\mathbb{C}) = \Big\{ M \in GL_4(\mathbb{C}) : \; M^T J \overline{M} = J \Big\}, \text{ where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the usual way; that is,

$$M \cdot z = (az + b)(cz + d)^{-1} \text{ for a block matrix } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{2,2}(\mathbb{C}).$$

Let K be an imaginary-quadratic field with ring of integers $\mathcal{O}_K$. A **Hermitian modular form of degree two** of weight k and character χ for K is a holomorphic function $f : \mathbf{H} \to \mathbb{C}$ for which

$$f(M \cdot z) = \det(cz + d)^k \chi(M) f(z)$$

holds for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{2,2}(\mathbb{C})$ with $a, b, c, d \in \mathcal{O}_K$.

Hermitian modular forms of degree two can be interpreted as orthogonal modular forms for the lattice $\mathcal{O}_K \oplus II_{2,2}$, which has signature $(2, 4)$ when $\mathcal{O}_K$ is equipped with the negative norm-form $Q = -N_{K/\mathbb{Q}}$. We can write the Lorentzian Gram matrix of $\mathcal{O}_K \oplus II_{1,1}$ in block form

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{S}_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \ \mathbf{S}_0 = \begin{pmatrix} -2 & -2\mathrm{re}(\alpha) \\ -2\mathrm{re}(\alpha) & -2|\alpha|^2 \end{pmatrix},$$

where α is any (fixed) element such that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. In particular there is also a version of the Borcherds lift. A detailed reference for this is the dissertation [24], in particular chapter 5. The Hermitian upper half-space and the orthogonal upper half-space (in the convention of section 2.6) are identified by the map

$$\Phi : \mathbf{H} \longrightarrow \mathbb{H}_{\mathbf{S}}, \ \ \Phi(x+iy) = \Phi(x) + i\Phi(y),$$

where if $x = \begin{pmatrix} x_1 & x_2 \\ \overline{x_2} & x_4 \end{pmatrix}$ is Hermitian then we define

$$\Phi\begin{pmatrix} x_1 & x_2 \\ \overline{x_2} & x_4 \end{pmatrix} = \Big(x_1, \frac{\mathrm{im}(\alpha x_2)}{\mathrm{im}(\alpha)}, -\frac{\mathrm{im}(x_2)}{\mathrm{im}(\alpha)}, x_4 \Big)^T;$$

one can check that this map satisfies $\det(z) = \frac{1}{2}\Phi(z)^T \mathbf{S} \Phi(z)$ as well as

$$\mathrm{tr}(bz) = \Phi(b^{adj})^T \mathbf{S} \Phi(z)$$

for all $z \in \mathbf{H}$ and all Hermitian matrices $b = \begin{pmatrix} b_1 & b_2 \\ \overline{b_2} & b_4 \end{pmatrix}$ with adjugate $b^{adj} = \begin{pmatrix} b_4 & -b_2 \\ -\overline{b_2} & b_1 \end{pmatrix}$. In this way the Hermitian form of the Borcherds product with input function $F(\tau) = \sum_{\gamma,n} c(n,\gamma) q^n \mathfrak{e}_\gamma$ is

$$\Psi_F(z) = \mathbf{e}\Big(\mathrm{tr}(\Phi^{-1}(\rho)^{adj} z)\Big) \prod_{T>0} \Big(1 - \mathbf{e}(\mathrm{tr}(Tz))\Big)^{c(\det(T), \Phi(T))},$$

where $\rho = \rho_W$ is the Weyl vector of a particular Weyl chamber W for $\mathbf{S}$ and $T > 0$ means T is positive with respect to W. (See [24], Satz 5.4 and its proof for more details.)

The possible characters of the product Ψ_F are easy to determine using section 5.3 of [24]. If the discriminant of K is odd then Ψ_F transforms without character; if the discriminant of K is even, then there is a unique nontrivial character χ of $SU_{2,2}(\mathcal{O}_K)$ which is determined by

$$\chi\Big(z \mapsto z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Big) = -1,$$

and it appears in the character of Ψ_F if and only if

$$\sum_{\gamma \in A} \sum_{n=0}^{\infty} \sigma_1(n) c(-n - Q(\gamma), \gamma)$$

(which is always integral) is odd, where we set $\sigma_1(0) = -1/24$ as suggested by the Eisenstein series E_2. Note that χ restricts to the Siegel character of $SP_4(\mathbb{Z})$. In the cases $K = \mathbb{Q}(\sqrt{-3}), K = \mathbb{Q}(i)$ there is an additional character det of $U_{2,2}(\mathcal{O}_K)/SU_{2,2}(\mathcal{O}_K)$. In the first case this never occurs in the character of Ψ_F because all products for $\mathbb{Q}(\sqrt{-3})$ have weight divisible by 9 (cf. [26]); in the second case, all products for $\mathbb{Q}(i)$ have even weight (also cf. [26]) so that det occurs in the character of Ψ_F if and only if χ does. Also, the transformation $z \mapsto z^T$ belongs to the orthogonal group but not the Hermitian modular group; so Borcherds products also transform with a quadratic character under $z \mapsto z^T$, i.e. they are either **symmetric** or **skew-symmetric** depending on the sign in $\Psi_F(z^T) = \pm \Psi_F(z)$. As in [24] one can compute

$$\frac{\Psi_F(z^T)}{\Psi_F(z)} = \mathbf{e}\Big(\frac{1}{2} \sum_{\gamma \in \sqrt{d_K}^{-1} \cdot \mathbb{N}^2} c(-Q(\gamma), \gamma) \Big).$$

Explanation: In the tables below, we work out a basis of the obstruction space $\mathbb{C}E_3 \oplus S_3(\rho^*)$ for some lattices $\mathcal{O}_K$ of small discriminant, as well as the first few solutions to the obstruction problem. ψ_k denotes a Borcherds product of weight k. For each product, we include the principle part of the nearly-holomorphic modular form that produces it under Borcherds' lift as well as its Weyl vector. The components $\mathfrak{e}_\gamma$ of the principal part correspond to the cosets $\gamma \in \mathbf{S}_0^{-1}\mathbb{Z}^2/\mathbb{Z}^2$, where $\mathbf{S}_0 = \begin{pmatrix} -2 & -1 \\ -1 & \frac{d_K - 1}{2} \end{pmatrix}$ if the discriminant d_K is odd and $\mathbf{S}_0 = \begin{pmatrix} -2 & 0 \\ 0 & \frac{d_K}{2} \end{pmatrix}$ if d_K is even. We write "symm" or "skew" according to whether ψ_k is symmetric or skew-symmetric under $z \mapsto z^T$. If the discriminant is even then we include its character under $SU_{2,2}(\mathcal{O}_K)$; it is either trivial (-) or the character χ that restricts to the Siegel character of $SP_4(\mathbb{Z})$, depending on whether the first and last components of the Weyl vector are integral or half-integral. In the last few tables we use $\mathfrak{e}_{\pm\gamma}$ to denote either $\mathfrak{e}_\gamma + \mathfrak{e}_{-\gamma}$ or $\{\mathfrak{e}_\gamma, \mathfrak{e}_{-\gamma}\}$ in order to save space. **Note:** many of the Weyl vectors have been corrected.

B.1 Discriminant -3

Table B.1: Obstruction space, discriminant -3

	E_3
$\mathfrak{e}_{(0,0)}$	$1 - 90q - 216q^2 - 738q^3 - 1170q^4 - 1728q^5 + O(q^6)$
$\mathfrak{e}_{(1/3,1/3)}, \mathfrak{e}_{(2/3,2/3)}$	$q^{1/3}(-9 - 117q - 450q^2 - 648q^3 - 1530q^4 - 1845q^5 + O(q^6))$

Table B.2: Holomorphic products of weight less than 200

	Principal part	Weyl vector	
ψ_9	$18\mathfrak{e}_{(0,0)} + q^{-1/3}(\mathfrak{e}_{(1/3,1/3)} + \mathfrak{e}_{(2/3,2/3)})$	$(1, 1/3, 1/3, 1)$	skew
ψ_{45}	$90\mathfrak{e}_{(0,0)} + q^{-1}\mathfrak{e}_{(0,0)}$	$(4, 1, 0, 3)$	symm
$\psi_{108}^{(1)}$	$216\mathfrak{e}_{(0,0)} - q^{-1/3}(\mathfrak{e}_{(1/3,1/3)} + \mathfrak{e}_{(2/3,2/3)}) +$ $+ q^{-4/3}(\mathfrak{e}_{(1/3,1/3)} + \mathfrak{e}_{(2/3,2/3)})$	$(9, 1/3, 1/3, 3)$	symm
$\psi_{108}^{(2)}$	$216\mathfrak{e}_{(0,0)} + q^{-2}\mathfrak{e}_{(0,0)}$	$(9, 0, 0, 0)$	symm

B.2 Discriminant -4

Table B.3: Obstruction space, discriminant -4

	E_3
$\mathfrak{e}_{(0,0)}$	$1 - 68q - 260q^2 - 480q^3 - 1028q^4 - 1768q^5 + O(q^6)$
$\mathfrak{e}_{(1/2,0)}, \mathfrak{e}_{(0,1/2)}$	$q^{1/4}(-4 - 104q - 292q^2 - 680q^3 - 1160q^4 - 1536q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{1/2}(-20 - 96q - 520q^2 - 576q^3 - 1460q^4 - 1440q^5 + O(q^6))$

	$\frac{1}{6}(E_3 - Q_{3,1/4,(0,1/2)})$
$\mathfrak{e}_{(0,0)}$	0
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(1 - 6q + 9q^2 + 10q^3 - 30q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{(0,1/2)}$	$q^{1/4}(-1 + 6q - 9q^2 - 10q^3 + 30q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	0

Table B.4: Holomorphic products of weight less than 150

	Principal part	Weyl vector	Char.	
ψ_4	$8\mathfrak{e}_{(0,0)} + q^{-1/4}\big(\mathfrak{e}_{(1/2,0)} + \mathfrak{e}_{(0,1/2)}\big)$	$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$	χ	skew
ψ_{10}	$20\mathfrak{e}_{(0,0)} + q^{-1/2}\mathfrak{e}_{(1/2,1/2)}$	$(1, \frac{1}{2}, 0, 1)$	-	symm
ψ_{30}	$60\mathfrak{e}_{(0,0)} - q^{-1/4}\big(\mathfrak{e}_{(1/2,0)} + \mathfrak{e}_{(0,1/2)}\big) + q^{-1}\mathfrak{e}_{(0,0)}$	$(\frac{5}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{2})$	χ	symm
ψ_{48}	$96\mathfrak{e}_{(0,0)} + q^{-3/2}\mathfrak{e}_{(1/2,1/2)}$	$(4, 0, 0, 0)$	$-$	symm
$\psi_{64}^{(1)}$	$128\mathfrak{e}_{(0,0)} + 6q^{-1/4}\mathfrak{e}_{(1/2,0)} + q^{-5/4}\mathfrak{e}_{(1/2,0)}$	$(6, 1, \frac{3}{2}, 4)$	-	symm
$\psi_{64}^{(2)}$	$128\mathfrak{e}_{(0,0)} + 6q^{-1/4}\mathfrak{e}_{(0,1/2)} + q^{-5/4}\mathfrak{e}_{(0,1/2)}$	$(6, 2, 0, 4)$	-	symm
ψ_{120}	$240\mathfrak{e}_{(0,0)} - q^{-1/2}\mathfrak{e}_{(1/2,1/2)} + q^{-2}\mathfrak{e}_{(0,0)}$	$(10, \frac{1}{2}, 0, 3)$	-	symm

B.3 Discriminant -7

Table B.5: Obstruction space, discriminant -7

	E_3
$\mathfrak{e}_{(0,0)}$	$1 - \frac{175}{4}q - \frac{875}{4}q^2 - 336q^3 - \frac{3675}{4}q^4 - 1008q^5 + O(q^6)$
$\mathfrak{e}_{(1/7,5/7)}, \mathfrak{e}_{(6/7,2/7)}$	$q^{1/7}(-\frac{7}{8} - \frac{595}{8}q - 168q^2 - \frac{2135}{4}q^3 - \frac{2947}{4}q^4 - \frac{10731}{8}q^5 + O(q^6))$
$\mathfrak{e}_{(4/7,6/7)}, \mathfrak{e}_{(3/7,1/7)}$	$q^{2/7}(-\frac{35}{8} - \frac{511}{8}q - \frac{2387}{8}q^2 - \frac{1855}{4}q^3 - 840q^4 - \frac{4795}{4}q^5 + O(q^6))$
$\mathfrak{e}_{(2/7,3/7)}, \mathfrak{e}_{(5/7,4/7)}$	$q^{4/7}(-\frac{147}{8} - \frac{427}{4}q - \frac{2555}{8}q^2 - \frac{4207}{8}q^3 - \frac{9555}{8}q^4 - 1176q^5 + O(q^6))$
	$\frac{8}{15}\big(E_3 - Q_{3,1/7,(1/7,5/7)}\big)$
$\mathfrak{e}_{(0,0)}$	$0 + 14q - 42q^2 + 0q^3 + 70q^4 + 0q^5 + O(q^6)$
$\mathfrak{e}_{(1/7,5/7)}, \mathfrak{e}_{(6/7,2/7)}$	$q^{1/7}(-1 + 3q + 0q^2 - 18q^3 + 54q^4 - 45q^5 + O(q^6))$
$\mathfrak{e}_{(4/7,6/7)}, \mathfrak{e}_{(3/7,1/7)}$	$q^{2/7}(3 - 9q + 11q^2 - 18q^3 + 0q^4 + 38q^5 + O(q^6))$
$\mathfrak{e}_{(2/7,3/7)}, \mathfrak{e}_{(5/7,4/7)}$	$q^{4/7}(-5 + 6q + 27q^2 - 25q^3 - 45q^4 + 0q^5 + O(q^6))$

Table B.6: Holomorphic products of weight less than 175

	Principal part	Weyl vector	
ψ_7	$14\mathfrak{e}_{(0,0)} + 3q^{-1/7}\big(\mathfrak{e}_{(1/7,5/7)} + \mathfrak{e}_{(6/7,2/7)}\big) + q^{-2/7}\big(\mathfrak{e}_{(4/7,6/7)} + \mathfrak{e}_{(3/7,1/7)}\big)$	$(1, \frac{2}{7}, \frac{3}{7}, 1)$	skew

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$\psi_{28}^{(1)}$	$56\mathfrak{e}_{(0,0)} + q^{-1/7}(\mathfrak{e}_{(1/7,5/7)} + \mathfrak{e}_{(6/7,2/7)}) + 2q^{-2/7}(\mathfrak{e}_{(4/7,6/7)}+$ $+ \mathfrak{e}_{(3/7,1/7)}) + q^{-4/7}(\mathfrak{e}_{(2/7,3/7)} + \mathfrak{e}_{(5/7,4/7)})$	$(3, \frac{9}{7}, \frac{3}{7}, 3)$	symm
$\psi_{28}^{(2)}$	$56\mathfrak{e}_{(0,0)} + 7q^{-1/7}(\mathfrak{e}_{(1/7,5/7)} + \mathfrak{e}_{(6/7,2/7)}) + q^{-1}\mathfrak{e}_{(0,0)}$	$(3, 0, 1, 2)$	skew
ψ_{70}	$140\mathfrak{e}_{(0,0)} - q^{-2/7}(\mathfrak{e}_{(4/7,6/7)} + \mathfrak{e}_{(3/7,1/7)})+$ $+ q^{-8/7}(\mathfrak{e}_{(1/7,5/7)} + \mathfrak{e}_{(6/7,2/7)})$	$(6, 1, 0, 4)$	symm
ψ_{77}	$154\mathfrak{e}_{(0,0)} + 3q^{-2/7}(\mathfrak{e}_{(4/7,6/7)} + \mathfrak{e}_{(3/7,1/7)})+$ $+ q^{-9/7}(\mathfrak{e}_{(4/7,6/7)} + \mathfrak{e}_{(3/7,1/7)})$	$(7, \frac{9}{7}, \frac{3}{7}, 3)$	skew
ψ_{112}	$224\mathfrak{e}_{(0,0)} + 6q^{-1/7}(\mathfrak{e}_{(1/7,5/7)} + \mathfrak{e}_{(6/7,2/7)})+$ $+ q^{-11/7}(\mathfrak{e}_{(2/7,3/7)} + \mathfrak{e}_{(5/7,4/7)})$	$(10, \frac{4}{7}, \frac{6}{7}, 4)$	symm
ψ_{140}	$280\mathfrak{e}_{(0,0)} + 7q^{-2/7}(\mathfrak{e}_{(3/7,1/7)} + \mathfrak{e}_{(4/7,6/7)}) + q^{-2}\mathfrak{e}_{(0,0)}$	$(13, 4, 0, 8)$	symm
$\psi_{168}^{(1)}$	$336\mathfrak{e}_{(0,0)} + q^{-15/7}(\mathfrak{e}_{(1/7,5/7)} + \mathfrak{e}_{(6/7,2/7)})$	$(14, 0, 0, 0)$	symm
$\psi_{168}^{(2)}$	$336\mathfrak{e}_{(0,0)} + q^{-3}\mathfrak{e}_{(0,0)}$	$(14, 0, 0, 0)$	symm

B.4 Discriminant -8

Table B.7: Obstruction space, discriminant -8

	E_3
$\mathfrak{e}_{(0,0)}$	$1 - \frac{130}{3}q - \frac{514}{3}q^2 - \frac{1300}{3}q^3 - \frac{2050}{3}q^4 - 1008q^5 + O(q^6)$
$\mathfrak{e}_{(0,1/4)}, \mathfrak{e}_{(0,3/4)}$	$q^{1/8}(-\frac{2}{3} - \frac{182}{3}q - \frac{580}{3}q^2 - \frac{1202}{3}q^3 - \frac{2440}{3}q^4 - \frac{3364}{3}q^5 + O(q^6))$
$\mathfrak{e}_{(0,1/2)}$	$q^{1/2}(-\frac{34}{3} - \frac{340}{3}q - 240q^2 - 480q^3 - \frac{3094}{3}q^4 - \frac{4148}{3}q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(-\frac{10}{3} - 48q - \frac{910}{3}q^2 - 336q^3 - \frac{2900}{3}q^4 - 960q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/4)}, \mathfrak{e}_{(1/2,3/4)}$	$q^{3/8}(-\frac{20}{3} - \frac{244}{3}q - \frac{724}{3}q^2 - \frac{1640}{3}q^3 - 768q^4 - \frac{3700}{3}q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{3/4}(-\frac{100}{3} - 96q - \frac{1220}{3}q^2 - 480q^3 - \frac{3620}{3}q^4 - 1056q^5 + O(q^6))$

	$\frac{3}{5}(E_3 - Q_{3,1/8,(0,1/4)})$
$\mathfrak{e}_{(0,0)}$	$0 + 16q - 32q^2 - 32q^3 + 64q^4 + 0q^5 + O(q^6)$
$\mathfrak{e}_{(0,1/4)}, \mathfrak{e}_{(0,3/4)}$	$q^{1/8}(-1 + 5q - 2q^2 - 25q^3 + 28q^4 + 46q^5 + O(q^6))$
$\mathfrak{e}_{(0,1/2)}$	$q^{1/2}(-8 + 16q + 0q^2 + 0q^3 + 40q^4 - 112q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(4 + 0q - 20q^2 + 0q^3 + 8q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/4)}, \mathfrak{e}_{(1/2,3/4)}$	$q^{3/8}(2 - 14q + 34q^2 - 28q^3 + 0q^4 - 14q^5 + O(q^6))$

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$\mathfrak{e}_{(1/2,1/2)}$	$q^{3/4}(-8+0q+56q^2+0q^3-136q^4+0q^5+O(q^6))$

Table B.8: Holomorphic products of weight less than 80

	Principal part	Weyl vector	Char.	
ψ_3	$6\mathfrak{e}_{(0,0)}+2q^{-1/8}(\mathfrak{e}_{(0,1/4)}+\mathfrak{e}_{(0,3/4)})+q^{-1/4}\mathfrak{e}_{(1/2,0)}$	$(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{2})$	χ	symm
ψ_8	$16\mathfrak{e}_{(0,0)}+2q^{-1/8}(\mathfrak{e}_{(0,1/4)}+\mathfrak{e}_{(0,3/4)})+$ $+q^{-3/8}(\mathfrak{e}_{(1/2,1/4)}+\mathfrak{e}_{(1/2,3/4)})$	$(1,\frac{1}{2},\frac{1}{4},1)$	$-$	symm
ψ_9	$18\mathfrak{e}_{(0,0)}+2q^{-1/4}\mathfrak{e}_{(1/2,0)}+q^{-1/2}\mathfrak{e}_{(0,1/2)}$	$(1,\frac{1}{2},\frac{1}{4},1)$	$-$	skew
ψ_{20}	$40\mathfrak{e}_{(0,0)}+2q^{-1/4}\mathfrak{e}_{(1/2,0)}+q^{-3/4}\mathfrak{e}_{(1/2,1/2)}$	$(2,1,0,2)$	$-$	symm
$\psi_{24}^{(1)}$	$48\mathfrak{e}_{(0,0)}+q^{-5/4}\mathfrak{e}_{(1/2,0)}$	$(2,0,0,0)$	$-$	symm
$\psi_{24}^{(2)}$	$48\mathfrak{e}_{(0,0)}+6q^{-1/8}(\mathfrak{e}_{(0,1/4)}+\mathfrak{e}_{(0,3/4)})-$ $-q^{-1/4}\mathfrak{e}_{(1/2,0)}+q^{-1}\mathfrak{e}_{(0,0)}$	$(\frac{5}{2},\frac{1}{4},\frac{3}{4},\frac{3}{2})$	χ	symm
ψ_{48}	$96\mathfrak{e}_{(0,0)}+q^{-7/4}\mathfrak{e}_{(1/2,1/2)}$	$(4,0,0,0)$	$-$	symm
ψ_{54}	$108\mathfrak{e}_{(0,0)}+6q^{-1/8}(\mathfrak{e}_{(0,1/4)}+\mathfrak{e}_{(0,3/4)})-$ $-q^{-3/8}(\mathfrak{e}_{(1/2,1/4)}+\mathfrak{e}_{(1/2,3/4)})+q^{-3/2}\mathfrak{e}_{(0,1/2)}$	$(5,\frac{1}{2},\frac{3}{4},3)$	$-$	symm
ψ_{64}	$128\mathfrak{e}_{(0,0)}+5q^{-1/8}(\mathfrak{e}_{(0,1/4)}+\mathfrak{e}_{(0,3/4)})+$ $+q^{-9/8}(\mathfrak{e}_{(0,1/4)}+\mathfrak{e}_{(0,3/4)})$	$(6,1,1,4)$	$-$	symm

B.5 Discriminant -11

Table B.9: Obstruction space, discriminant -11

	E_3
$\mathfrak{e}_{(0,0)}$	$1-\frac{122}{3}q-120q^2-\frac{1220}{3}q^3-\frac{1586}{3}q^4-\frac{3172}{3}q^5+O(q^6)$
$\mathfrak{e}_{(1/11,9/11)},\mathfrak{e}_{(10/11,2/11)}$	$q^{1/11}(-\frac{1}{3}-\frac{130}{3}q-\frac{530}{3}q^2-288q^3-\frac{2366}{3}q^4-816q^5+O(q^6))$
$\mathfrak{e}_{(5/11,1/11)},\mathfrak{e}_{(6/11,10/11)}$	$q^{3/11}(-\frac{10}{3}-48q-217q^2-\frac{1183}{3}q^3-\frac{2210}{3}q^4-840q^5+O(q^6))$
$\mathfrak{e}_{(2/11,7/11)},\mathfrak{e}_{(9/11,4/11)}$	$q^{4/11}(-\frac{13}{3}-\frac{260}{3}q-168q^2-\frac{1370}{3}q^3-\frac{2050}{3}q^4-\frac{3482}{3}q^5+O(q^6))$
$\mathfrak{e}_{(4/11,3/11)},\mathfrak{e}_{(7/11,8/11)}$	$q^{5/11}(-\frac{26}{3}-\frac{205}{3}q-\frac{820}{3}q^2-360q^3-\frac{2353}{3}q^4-\frac{3380}{3}q^5+O(q^6))$
$\mathfrak{e}_{(3/11,5/11)},\mathfrak{e}_{(8/11,6/11)}$	$q^{9/11}(-\frac{91}{3}-\frac{338}{3}q-\frac{962}{3}q^2-480q^3-\frac{2810}{3}q^4-\frac{3277}{3}q^5+O(q^6))$

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	$\frac{3}{4}(E_3 - Q_{3,1/11,(1/11,9/11)})$
$\mathfrak{e}_{(0,0)}$	$0 + 22q + 0q^2 - 110q^3 + 88q^4 - 22q^5 + O(q^6)$
$\mathfrak{e}_{(1/11,9/11)}, \mathfrak{e}_{(10/11,2/11)}$	$q^{1/11}(-1 + 20q - 35q^2 + 0q^3 + 16q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{(5/11,1/11)}, \mathfrak{e}_{(6/11,10/11)}$	$q^{3/11}(5 + 0q + 24q^2 - 64q^3 - 50q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{(2/11,7/11)}, \mathfrak{e}_{(9/11,4/11)}$	$q^{4/11}(-4 - 5q + 0q^2 + 25q^3 + 80q^4 - 107q^5 + O(q^6))$
$\mathfrak{e}_{(4/11,3/11)}, \mathfrak{e}_{(7/11,8/11)}$	$q^{5/11}(1 - 16q + 35q^2 + 0q^3 - 49q^4 - 20q^5 + O(q^6))$
$\mathfrak{e}_{(3/11,5/11)}, \mathfrak{e}_{(8/11,6/11)}$	$q^{9/11}(-16 + 4q + 37q^2 + 0q^3 + 70q^4 - 64q^5 + O(q^6))$

Abbreviate $\mathfrak{e}_\gamma + \mathfrak{e}_{-\gamma}$ by $\mathfrak{e}_{\pm\gamma}$.

Table B.10: Holomorphic products of weight less than 80

	Principal part	Weyl vector	
ψ_5	$10\mathfrak{e}_{(0,0)} + 5q^{-1/11}\mathfrak{e}_{\pm(1/11,9/11)} + q^{-3/11}\mathfrak{e}_{\pm(5/11,1/11)}$	$(1, \frac{3}{11}, \frac{5}{11}, 1)$	skew
ψ_8	$16\mathfrak{e}_{(0,0)} + q^{-1/11}\mathfrak{e}_{\pm(1/11,9/11)} +$ $+ q^{-3/11}\mathfrak{e}_{\pm(5/11,1/11)} + q^{-4/11}\mathfrak{e}_{\pm(2/11,7/11)}$	$(1, \frac{4}{11}, \frac{3}{11}, 1)$	symm
ψ_9	$18\mathfrak{e}_{(0,0)} + q^{-1/11}\mathfrak{e}_{\pm(1/11,9/11)} + q^{-5/11}\mathfrak{e}_{\pm(4/11,3/11)}$	$(1, \frac{5}{11}, \frac{1}{11}, 1)$	skew
ψ_{24}	$48\mathfrak{e}_{(0,0)} + 11q^{-1/11}\mathfrak{e}_{\pm(1/11,9/11)} + q^{-1}\mathfrak{e}_{(0,0)}$	$(3, 0, 1, 2)$	skew
ψ_{40}	$80\mathfrak{e}_{(0,0)} - q^{-1/11}\mathfrak{e}_{\pm(1/11,9/11)} +$ $+ 3q^{-3/11}\mathfrak{e}_{\pm(5/11,1/11)} + q^{-9/11}\mathfrak{e}_{\pm(3/11,5/11)}$	$(4, \frac{21}{11}, \frac{2}{11}, 4)$	symm
ψ_{45}	$90\mathfrak{e}_{(0,0)} + 15q^{-1/11}\mathfrak{e}_{\pm(1/11,9/11)} -$ $- q^{-3/11}\mathfrak{e}_{\pm(5/11,1/11)} + q^{-12/11}\mathfrak{e}_{\pm(1/11,9/11)}$	$(5, -\frac{2}{11}, \frac{15}{11}, 3)$	skew
ψ_{48}	$96\mathfrak{e}_{(0,0)} + q^{-14/11}(\mathfrak{e}_{(5/11,1/11)} + \mathfrak{e}_{(6/11,10/11)})$	$(4, 0, 0, 0)$	symm
ψ_{60}	$120\mathfrak{e}_{(0,0)} + q^{-2}\mathfrak{e}_{(0,0)}$	$(5, 0, 0, 0)$	symm
ψ_{75}	$150\mathfrak{e}_{(0,0)} + 3q^{-1/11}\mathfrak{e}_{\pm(1/11,9/11)} + 3q^{-3/11}\mathfrak{e}_{\pm(5/11,1/11)} -$ $- q^{-4/11}\mathfrak{e}_{\pm(2/11,7/11)} + q^{-16/11}\mathfrak{e}_{\pm(4/11,3/11)}$	$(7, \frac{14}{11}, \frac{5}{11}, 3)$	skew

B.6 Discriminant -15

Table B.11: Obstruction space, discriminant -15

	E_3
$\mathfrak{e}_{(0,0)}$	$1-\frac{65}{2}q-120q^2-240q^3-\frac{1365}{2}q^4-624q^5+O(q^6)$
$\mathfrak{e}_{(1/15,13/15)}, \mathfrak{e}_{(4/15,7/15)}, \mathfrak{e}_{(11/15,8/15)}, \mathfrak{e}_{(14/15,2/15)}$	$q^{1/15}(-\frac{1}{8}-\frac{341}{8}q-\frac{481}{4}q^2-\frac{1325}{4}q^3-\frac{1861}{4}q^4-\frac{3801}{4}q^5+O(q^6))$
$\mathfrak{e}_{(2/15,11/15)}, \mathfrak{e}_{(7/15,1/15)}, \mathfrak{e}_{(8/15,14/15)}, \mathfrak{e}_{(13/15,4/15)}$	$q^{4/15}(-\frac{21}{8}-\frac{181}{4}q-\frac{725}{4}q^2-\frac{2353}{8}q^3-\frac{5461}{8}q^4-\frac{3121}{4}q^5+O(q^6))$
$\mathfrak{e}_{(2/5,1/5)}, \mathfrak{e}_{(3/5,4/5)}$	$q^{2/5}(-\frac{25}{4}-48q-\frac{861}{4}q^2-\frac{725}{2}q^3-600q^4-\frac{3281}{4}q^5+O(q^6))$
$\mathfrak{e}_{(1/5,3/5)}, \mathfrak{e}_{(4/5,2/5)}$	$q^{3/5}(-\frac{41}{4}-\frac{425}{4}q-168q^2-\frac{1825}{4}q^3-\frac{1325}{2}q^4-1008q^5+O(q^6))$
$\mathfrak{e}_{(1/3,1/3)}, \mathfrak{e}_{(2/3,2/3)}$	$q^{2/3}(-\frac{65}{4}-\frac{313}{4}q-\frac{1105}{4}q^2-360q^3-720q^4-\frac{1885}{2}q^5+O(q^6))$

	$8(E_3-Q_{3,1/15,(1/15,13/15)})$
$\mathfrak{e}_{(0,0)}$	$0+300q+0q^2+0q^3-900q^4+0q^5+O(q^6)$
$\mathfrak{e}_{(1/15,13/15)}, \mathfrak{e}_{(14/15,2/15)}$	$q^{1/15}(-9+219q-2q^2-650q^3+1062q^4-1298q^5+O(q^6))$
$\mathfrak{e}_{(4/15,7/15)}, \mathfrak{e}_{(11/15,8/15)}$	$q^{1/15}(-1-269q-18q^2+310q^3+118q^4+638q^5+O(q^6))$
$\mathfrak{e}_{(2/15,11/15)}, \mathfrak{e}_{(13/15,4/15)}$	$q^{4/15}(-29+198q-410q^2-49q^3+939q^4-98q^5+O(q^6))$
$\mathfrak{e}_{(7/15,1/15)}, \mathfrak{e}_{(8/15,14/15)}$	$q^{4/15}(59+22q+550q^2-441q^3-1069q^4-882q^5+O(q^6))$
$\mathfrak{e}_{(2/5,1/5)}, \mathfrak{e}_{(3/5,4/5)}$	$q^{2/5}(30+0q+270q^2-420q^3+0q^4-810q^5+O(q^6))$
$\mathfrak{e}_{(1/5,3/5)}, \mathfrak{e}_{(4/5,2/5)}$	$q^{3/5}(-90-210q+0q^2+270q^3+1020q^4+0q^5+O(q^6))$
$\mathfrak{e}_{(1/3,1/3)}, \mathfrak{e}_{(2/3,2/3)}$	$q^{2/3}(-50-250q+350q^2+0q^3+0q^4+700q^5+O(q^6))$

	$Q_{3,1/15,(1/15,13/15)}-Q_{3,1/15,(4/15,7/15)}$
$\mathfrak{e}_{(1/15,13/15)}, \mathfrak{e}_{(14/15,2/15)}$	$q^{1/15}(1-61q-2q^2+120q^3-118q^4+242q^5+O(q^6))$
$\mathfrak{e}_{(4/15,7/15)}, \mathfrak{e}_{(11/15,8/15)}$	$q^{1/15}(-1+61q+2q^2-120q^3+118q^4-242q^5+O(q^6))$
$\mathfrak{e}_{(2/15,11/15)}, \mathfrak{e}_{(13/15,4/15)}$	$q^{4/15}(11-22q+120q^2-49q^3-251q^4-98q^5+O(q^6))$
$\mathfrak{e}_{(7/15,1/15)}, \mathfrak{e}_{(8/15,14/15)}$	$q^{4/15}(-11+22q-120q^2+49q^3+251q^4+98q^5+O(q^6))$
other components	0

Abbreviate $\mathfrak{e}_\gamma + \mathfrak{e}_{-\gamma}$ by $\mathfrak{e}_{\pm\gamma}$.

Table B.12: Holomorphic products of weight less than 40

	Principal part	Weyl vector	
ψ_6	$12\mathfrak{e}_{(0,0)} + 3q^{-1/15}(\mathfrak{e}_{\pm(1/15,13/15)} + \mathfrak{e}_{\pm(4/15,7/15)}) + q^{-4/15}(\mathfrak{e}_{\pm(2/15,11/15)} + \mathfrak{e}_{\pm(7/15,1/15)})$	$(1, \frac{1}{3}, \frac{1}{3}, 1)$	symm
ψ_7	$14\mathfrak{e}_{(0,0)} + 3q^{-1/15}(\mathfrak{e}_{\pm(1/15,13/15)} + \mathfrak{e}_{\pm(4/15,7/15)}) + q^{-2/5}\mathfrak{e}_{\pm(2/5,1/5)}$	$(1, \frac{2}{5}, \frac{1}{5}, 1)$	skew
ψ_9	$18\mathfrak{e}_{(0,0)} - q^{-1/15}\mathfrak{e}_{\pm(1/15,13/15)} + 10q^{-1/15}\mathfrak{e}_{\pm(4/15,7/15)} + 2q^{-4/15}\mathfrak{e}_{\pm(2/15,11/15)} + q^{-4/15}\mathfrak{e}_{\pm(7/15,1/15)}$	$(1, \frac{2}{5}, \frac{1}{5}, 1)$	skew
$\psi_{20}^{(1)}$	$40\mathfrak{e}_{(0,0)} - 3q^{-1/15}(\mathfrak{e}_{\pm(1/15,13/15)} + \mathfrak{e}_{\pm(4/15,7/15)}) + 2q^{-4/15}(\mathfrak{e}_{\pm(2/15,11/15)} + \mathfrak{e}_{\pm(7/15,1/15)}) + q^{-3/5}\mathfrak{e}_{\pm(1/5,3/5)}$	$(2, \frac{13}{15}, \frac{4}{15}, 2)$	symm
$\psi_{20}^{(2)}$	$40\mathfrak{e}_{(0,0)} + 15q^{-1/15}(\mathfrak{e}_{\pm(1/15,13/15)} + \mathfrak{e}_{\pm(4/15,7/15)}) + q^{-1}\mathfrak{e}_{(0,0)}$	$(3, 0, 1, 2)$	skew
ψ_{27}	$54\mathfrak{e}_{(0,0)} + q^{-1/15}(\mathfrak{e}_{\pm(1/15,13/15)} + \mathfrak{e}_{\pm(4/15,7/15)}) + 2q^{-4/15}(\mathfrak{e}_{\pm(2/15,11/15)} + \mathfrak{e}_{\pm(7/15,1/15)}) + q^{-2/3}\mathfrak{e}_{\pm(1/3,1/3)}$	$(3, \frac{4}{3}, \frac{1}{3}, 3)$	skew

B.7 Discriminant −19

Table B.13: Obstruction space, discriminant −19

	E_3
$\mathfrak{e}_{(0,0)}$	$1 - \frac{362}{11}q - \frac{1080}{11}q^2 - \frac{2880}{11}q^3 - \frac{4706}{11}q^4 - \frac{9412}{11}q^5 + O(q^6)$
$\mathfrak{e}_{\pm(1/19,17/19)}$	$q^{1/19}(-\frac{1}{11} - \frac{338}{11}q - \frac{1344}{11}q^2 - \frac{2520}{11}q^3 - \frac{6100}{11}q^4 - \frac{6552}{11}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(2/19,15/19)}$	$q^{4/19}(-\frac{13}{11} - \frac{530}{11}q - \frac{1200}{11}q^2 - \frac{3722}{11}q^3 - \frac{5330}{11}q^4 - \frac{8906}{11}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(9/19,1/19)}$	$q^{5/19}(-\frac{26}{11} - \frac{408}{11}q - \frac{1850}{11}q^2 - \frac{2880}{11}q^3 - \frac{5905}{11}q^4 - \frac{8463}{11}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(5/19,9/19)}$	$q^{6/19}(-\frac{24}{11} - \frac{651}{11}q - \frac{1586}{11}q^2 - \frac{3650}{11}q^3 - \frac{5040}{11}q^4 - \frac{10202}{11}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(8/19,3/19)}$	$q^{7/19}(-\frac{50}{11} - \frac{504}{11}q - \frac{1898}{11}q^2 - \frac{3277}{11}q^3 - \frac{6890}{11}q^4 - \frac{6960}{11}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(3/19,13/19)}$	$q^{9/19}(-\frac{73}{11} - \frac{650}{11}q - \frac{2210}{11}q^2 - \frac{2928}{11}q^3 - \frac{7540}{11}q^4 - \frac{8568}{11}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(7/19,5/19)}$	$q^{11/19}(-\frac{122}{11} - \frac{624}{11}q - \frac{2451}{11}q^2 - \frac{3770}{11}q^3 - \frac{6720}{11}q^4 - \frac{8424}{11}q^5 + O(q^6))$

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$\mathfrak{e}_{\pm(4/19,11/19)}$	$q^{16/19}(-\frac{205}{11}-\frac{1300}{11}q-\frac{1968}{11}q^2-\frac{5330}{11}q^3-\frac{6890}{11}q^4-\frac{10944}{11}q^5+O(q^6))$
$\mathfrak{e}_{\pm(6/19,7/19)}$	$q^{17/19}(-\frac{290}{11}-\frac{949}{11}q-\frac{3172}{11}q^2-\frac{4104}{11}q^3-\frac{7680}{11}q^4-\frac{10250}{11}q^5+O(q^6))$

	$\frac{11}{12}(E_3-Q_{3,1/19,(1/19,17/19)})$
$\mathfrak{e}_{(0,0)}$	$0+34q+20q^2-20q^3+112q^4-282q^5+O(q^6)$
$\mathfrak{e}_{\pm(1/19,17/19)}$	$q^{1/19}(-1+36q-2q^2+10q^3+5q^4-18q^5+O(q^6))$
$\mathfrak{e}_{\pm(2/19,15/19)}$	$q^{4/19}(-2+20q+10q^2-81q^3+104q^4-29q^5+O(q^6))$
$\mathfrak{e}_{\pm(9/19,1/19)}$	$q^{5/19}(7+10q+75q^2-20q^3-53q^4-202q^5+O(q^6))$
$\mathfrak{e}_{\pm(5/19,9/19)}$	$q^{6/19}(-2-46q-24q^2+35q^3+20q^4+94q^5+O(q^6))$
$\mathfrak{e}_{\pm(8/19,3/19)}$	$q^{7/19}(5+2q+71q^2-54q^3-70q^4-30q^5+O(q^6))$
$\mathfrak{e}_{\pm(3/19,13/19)}$	$q^{9/19}(-7+10q-65q^2+20q^3+105q^4-10q^5+O(q^6))$
$\mathfrak{e}_{\pm(7/19,5/19)}$	$q^{11/19}(-1-8q+24q^2-30q^3-10q^4-42q^5+O(q^6))$
$\mathfrak{e}_{\pm(4/19,11/19)}$	$q^{16/19}(-18-35q-10q^2+5q^3+150q^4+12q^5+O(q^6))$
$\mathfrak{e}_{\pm(6/19,7/19)}$	$q^{17/19}(-15-36q+29q^2-12q^3+20q^4+90q^5+O(q^6))$

	$11(E_3-Q_{3,4/19,(2/19,15/19)})$
$\mathfrak{e}_{(0,0)}$	$0+78q-200q^2+200q^3+552q^4-942q^5+O(q^6)$
$\mathfrak{e}_{\pm(1/19,17/19)}$	$q^{1/19}(-1+36q+20q^2-100q^3+115q^4+180q^5+O(q^6))$
$\mathfrak{e}_{\pm(2/19,15/19)}$	$q^{4/19}(-24+130q-100q^2-323q^3+544q^4-7q^5+O(q^6))$
$\mathfrak{e}_{\pm(9/19,1/19)}$	$q^{5/19}(29-100q+185q^2+200q^3-361q^4-444q^5+O(q^6))$
$\mathfrak{e}_{\pm(5/19,9/19)}$	$q^{6/19}(20-156q+108q^2+145q^3-200q^4+182q^5+O(q^6))$
$\mathfrak{e}_{\pm(8/19,3/19)}$	$q^{7/19}(5-20q+181q^2-164q^3-290q^4+300q^5+O(q^6))$
$\mathfrak{e}_{\pm(3/19,13/19)}$	$q^{9/19}(-29+120q-175q^2-200q^3+435q^4+100q^5+O(q^6))$
$\mathfrak{e}_{\pm(7/19,5/19)}$	$q^{11/19}(-23+80q+24q^2-360q^3+100q^4+420q^5+O(q^6))$
$\mathfrak{e}_{\pm(4/19,11/19)}$	$q^{16/19}(4-145q+100q^2+225q^3-180q^4-120q^5+O(q^6))$
$\mathfrak{e}_{\pm(6/19,7/19)}$	$q^{17/19}(-15-36q+7q^2+120q^3-200q^4-20q^5+O(q^6))$

Abbreviate $\mathfrak{e}_\gamma + \mathfrak{e}_{-\gamma}$ by $\mathfrak{e}_{\pm\gamma}$.

Table B.14: Holomorphic products of weight less than 35

	Principal part	Weyl vector	
ψ_4	$8\mathfrak{e}_{(0,0)} + 5q^{-1/19}\mathfrak{e}_{\pm(1/19,17/19)} + q^{-4/19}\mathfrak{e}_{\pm(2/19,15/19)} + q^{-5/19}\mathfrak{e}_{\pm(9/19,1/19)}$	$(1, \frac{6}{19}, \frac{7}{19}, 1)$	symm
ψ_5	$10\mathfrak{e}_{(0,0)} + 5q^{-1/19}\mathfrak{e}_{\pm(1/19,17/19)} + q^{-7/19}\mathfrak{e}_{\pm(8/19,3/19)}$	$(1, \frac{7}{19}, \frac{5}{19}, 1)$	skew
ψ_7	$14\mathfrak{e}_{(0,0)} + q^{-1/19}\mathfrak{e}_{\pm(1/19,17/19)} + 2q^{-4/19}\mathfrak{e}_{\pm(2/19,15/19)} + q^{-5/19}\mathfrak{e}_{\pm(9/19,1/19)} + q^{-6/19}\mathfrak{e}_{\pm(5/19,9/19)}$	$(1, \frac{7}{19}, \frac{5}{19}, 1)$	skew
ψ_9	$18\mathfrak{e}_{(0,0)} + q^{-5/19}\mathfrak{e}_{\pm(9/19,1/19)} + q^{-9/19}\mathfrak{e}_{\pm(3/19,13/19)}$	$(1, \frac{8}{19}, \frac{3}{19}, 1)$	skew
ψ_{14}	$28\mathfrak{e}_{(0,0)} + 6q^{-1/19}\mathfrak{e}_{\pm(1/19,17/19)} + q^{-5/19}\mathfrak{e}_{\pm(9/19,1/19)} + q^{-11/19}\mathfrak{e}_{\pm(7/19,5/19)}$	$(2, \frac{16}{19}, \frac{6}{19}, 2)$	symm
ψ_{19}	$38\mathfrak{e}_{(0,0)} + 15q^{-1/19}\mathfrak{e}_{\pm(1/19,17/19)} + q^{-4/19}\mathfrak{e}_{\pm(2/19,15/19)} + q^{-1}\mathfrak{e}_{(0,0)}$	$(3, \frac{1}{19}, \frac{17}{19}, 2)$	symm
ψ_{30}	$60\mathfrak{e}_{(0,0)} - 5q^{-1/19}\mathfrak{e}_{\pm(1/19,17/19)} + 4q^{-4/19}\mathfrak{e}_{\pm(2/19,15/19)} + 3q^{-5/19}\mathfrak{e}_{\pm(9/19,1/19)} + q^{-16/19}\mathfrak{e}_{\pm(4/19,11/19)}$	$(3, \frac{25}{19}, \frac{7}{19}, 3)$	symm
ψ_{33}	$66\mathfrak{e}_{(0,0)} + 27q^{-1/19}\mathfrak{e}_{\pm(1/19,17/19)} - q^{-5/19}\mathfrak{e}_{\pm(9/19,1/19)} + q^{-6/19}\mathfrak{e}_{\pm(5/19,9/19)} + q^{-20/19}\mathfrak{e}_{\pm(1/19,17/19)}$	$(5, -\frac{4}{19}, \frac{27}{19}, 3)$	skew

B.8 Discriminant -20

Table B.15: Obstruction space, discriminant -20

	E_3
$\mathfrak{e}_{(0,0)}$	$1 - \frac{442}{15}q - \frac{504}{5}q^2 - 240q^3 - \frac{6682}{15}q^4 - \frac{10642}{15}q^5 + O(q^6)$
$\mathfrak{e}_{\pm(0,1/10)}, \mathfrak{e}_{\pm(1/2,2/5)}$	$q^{1/20}(-\frac{1}{15} - \frac{100}{3}q - \frac{1682}{15}q^2 - \frac{3722}{15}q^3 - \frac{7381}{15}q^4 - \frac{10202}{15}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,1/5)}$	$q^{1/5}(-\frac{17}{15} - \frac{130}{3}q - 120q^2 - \frac{4097}{15}q^3 - \frac{1700}{3}q^4 - \frac{3528}{5}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,3/10)}, \mathfrak{e}_{\pm(1/2,1/5)}$	$q^{9/20}(-\frac{91}{15} - \frac{842}{15}q - \frac{817}{5}q^2 - \frac{1060}{3}q^3 - \frac{7922}{15}q^4 - \frac{11882}{15}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,2/5)}$	$q^{4/5}(-\frac{257}{15} - \frac{1547}{15}q - \frac{650}{3}q^2 - 360q^3 - \frac{2050}{3}q^4 - \frac{14314}{15}q^5 + O(q^6))$
$\mathfrak{e}_{(0,1/2)}, \mathfrak{e}_{(1/2,0)}$	$q^{1/4}(-\frac{26}{15} - \frac{626}{15}q - \frac{2366}{15}q^2 - \frac{1344}{5}q^3 - \frac{2304}{5}q^4 - \frac{2600}{3}q^5 + O(q^6))$

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$\mathfrak{e}_{\pm(1/2,1/10)}$	$q^{3/10}(-\frac{10}{3} - \frac{168}{5}q - \frac{530}{3}q^2 - 240q^3 - \frac{1850}{3}q^4 - \frac{2808}{5}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,3/10)}$	$q^{7/10}(-\frac{50}{3} - \frac{288}{5}q - \frac{820}{3}q^2 - \frac{1368}{5}q^3 - \frac{2210}{3}q^4 - 720q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{1/2}(-\frac{24}{5} - \frac{260}{3}q - \frac{624}{5}q^2 - \frac{1300}{3}q^3 - \frac{2184}{5}q^4 - 960q^5 + O(q^6))$

	$15(E_3 - Q_{3,1/20,(0,1/10)})$
$\mathfrak{e}_{(0,0)}$	$0 + 608q + 288q^2 + 0q^3 + 1568q^4 - 2752q^5 + O(q^6)$
$\mathfrak{e}_{\pm(0,1/10)}$	$q^{1/20}(-16 + 550q + 118q^2 - 122q^3 + 854q^4 - 1232q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,2/5)}$	$q^{1/20}(-1 - 350q - 812q^2 + 748q^3 - 31q^4 + 598q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,1/5)}$	$q^{1/5}(-32 + 400q - 512q^3 + 800q^4 - 864q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,3/10)}$	$q^{9/20}(-106 + 208q - 666q^2 - 650q^3 + 2128q^4 - 362q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/5)}$	$q^{9/20}(59 - 122q + 1299q^2 - 1550q^3 - 2q^4 - 932q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,2/5)}$	$q^{4/5}(-272 - 512q - 400q^2 + 0q^3 + 1600q^4 + 1856q^5 + O(q^6))$
$\mathfrak{e}_{(0,1/2)}$	$q^{1/4}(-26 - 656q - 266q^2 - 432q^3 + 288q^4 + 3500q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(124 + 94q + 1384q^2 - 432q^3 + 288q^4 - 5500q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/10)}$	$q^{3/10}(100 + 216q + 1100q^2 - 1900q^4 - 504q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,3/10)}$	$q^{7/10}(-100 - 144q - 200q^2 + 216q^3 - 100q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{1/2}(-72 - 1000q - 432q^2 + 1000q^3 + 648q^4 + 0q^5 + O(q^6))$

	$Q_{3,1/20,(0,1/10)} - Q_{3,1/20,(1/2,2/5)}$
$\mathfrak{e}_{\pm(0,1/10)}$	$q^{1/20}(1 - 60q - 62q^2 + 58q^3 - 59q^4 + 122q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,2/5)}$	$q^{1/20}(-1 + 60q + 62q^2 - 58q^3 + 59q^4 - 122q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,3/10)}$	$q^{9/20}(11 - 22q + 131q^2 - 60q^3 - 142q^4 - 38q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/5)}$	$q^{9/20}(-11 + 22q - 131q^2 + 60q^3 + 142q^4 + 38q^5 + O(q^6))$
$\mathfrak{e}_{(0,1/2)}$	$q^{1/4}(10 + 50q + 110q^2 + 0q^3 + 0q^4 - 600q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(-10 - 50q - 110q^2 + 0q^3 + 0q^4 + 600q^5 + O(q^6))$
other components	0

	$15(E_3 - Q_{3,1/5,(0,1/5)})$
$\mathfrak{e}_{(0,0)}$	$0 + 128q - 192q^2 + 0q^3 + 1088q^4 - 832q^5 + O(q^6)$
$\mathfrak{e}_{\pm(0,1/10)}, \mathfrak{e}_{\pm(1/2,3/5)}$	$q^{1/20}(-1 + 40q - 182q^2 + 178q^3 + 359q^4 - 602q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,1/5)}$	$q^{1/5}(-32 + 160q - 512q^3 + 320q^4 + 576q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,3/10)}, \mathfrak{e}_{\pm(1/2,1/5)}$	$q^{9/20}(-31 + 118q + 9q^2 - 440q^3 + 238q^4 + 178q^5 + O(q^6))$

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$\mathfrak{e}_{\pm(0,2/5)}$	$q^{4/5}(-32-32q-160q^2+0q^3+640q^4-64q^5+O(q^6))$
$\mathfrak{e}_{(0,1/2)},\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(34-146q+94q^2+288q^3-192q^4-400q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/10)}$	$q^{3/10}(40-144q+440q^2+0q^3-760q^4+336q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,3/10)}$	$q^{7/10}(-40+96q-80q^2-144q^3-40q^4+0q^5+O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{1/2}(48-400q+288q^2+400q^3-432q^4+0q^5+O(q^6))$

Table B.16: Holomorphic products of weight less than 18

	Principal part	Weyl vector	Char.	
ψ_3	$6\mathfrak{e}_{(0,0)}+q^{-1/20}(\mathfrak{e}_{\pm(0,1/10)}+\mathfrak{e}_{\pm(1/2,2/5)})+$ $+q^{-1/5}\mathfrak{e}_{\pm(0,1/5)}+q^{-1/4}(\mathfrak{e}_{(0,1/2)}+\mathfrak{e}_{(1/2,0)})$	$(\frac{1}{2},\frac{1}{4},\frac{3}{20},\frac{1}{2})$	χ	symm
ψ_5	$10\mathfrak{e}_{(0,0)}+4q^{-1/20}(\mathfrak{e}_{\pm(0,1/10)}+\mathfrak{e}_{\pm(1/2,2/5)})+$ $+q^{-1/5}\mathfrak{e}_{\pm(0,1/5)}+q^{-3/10}\mathfrak{e}_{\pm(1/2,1/10)}$	$(1,\frac{1}{2},\frac{3}{10},1)$	$-$	skew
$\psi_8^{(1)}$	$16\mathfrak{e}_{(0,0)}+2q^{-1/20}\mathfrak{e}_{\pm(0,1/10)}+q^{-1/20}\mathfrak{e}_{\pm(1/2,2/5)}+$ $+2q^{-1/4}\mathfrak{e}_{(0,1/2)}+q^{-9/20}\mathfrak{e}_{\pm(1/2,1/5)}$	$(1,\frac{1}{2},\frac{1}{10},1)$	$-$	symm
$\psi_8^{(2)}$	$16\mathfrak{e}_{(0,0)}+q^{-1/20}\mathfrak{e}_{\pm(0,1/10)}+2q^{-1/20}\mathfrak{e}_{\pm(1/2,2/5)}+$ $+2q^{-1/4}\mathfrak{e}_{(1/2,0)}+q^{-9/20}\mathfrak{e}_{\pm(0,3/10)}$	$(1,\frac{1}{2},\frac{1}{5},1)$	$-$	symm
$\psi_8^{(3)}$	$16\mathfrak{e}_{(0,0)}+q^{-3/10}\mathfrak{e}_{\pm(1/2,1/10)}+$ $+2q^{-1/5}\mathfrak{e}_{\pm(0,1/5)}+q^{-1/2}\mathfrak{e}_{(1/2,1/2)}$	$(1,\frac{1}{2},\frac{1}{5},1)$	$-$	symm
$\psi_8^{(4)}$	$16\mathfrak{e}_{(0,0)}+10q^{-1/20}\mathfrak{e}_{\pm(0,1/10)}+q^{-3/10}\mathfrak{e}_{\pm(1/2,1/10)}+$ $+2q^{-1/5}\mathfrak{e}_{\pm(0,1/5)}+2q^{-1/4}\mathfrak{e}_{(1/2,0)}$	$(2,1,\frac{7}{10},2)$	$-$	symm
ψ_{15}	$30\mathfrak{e}_{(0,0)}+15q^{-1/20}(\mathfrak{e}_{\pm(0,1/10)}+\mathfrak{e}_{\pm(1/2,2/5)})-$ $-q^{-1/4}(\mathfrak{e}_{(0,1/2)}+\mathfrak{e}_{(1/2,0)})+q^{-1}\mathfrak{e}_{(0,0)}$	$(\frac{5}{2},\frac{1}{4},\frac{3}{4},\frac{3}{2})$	χ	skew

B.9 Discriminant -23

Table B.17: Obstruction space, discriminant -23

	E_3
$\mathfrak{e}_{(0,0)}$	$1-\frac{265}{12}q-\frac{1325}{12}q^2-\frac{1325}{6}q^3-\frac{1855}{4}q^4-528q^5+O(q^6)$

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$\mathfrak{e}_{\pm(1/23,21/23)}$	$q^{1/23}(-\frac{1}{24} - \frac{425}{12}q - \frac{1105}{12}q^2 - 240q^3 - \frac{2405}{6}q^4 - \frac{2947}{4}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(5/23,13/23)}$	$q^{2/23}(-\frac{5}{24} - \frac{601}{24}q - \frac{1705}{12}q^2 - \frac{2521}{12}q^3 - \frac{5525}{12}q^4 - \frac{7735}{12}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(7/23,9/23)}$	$q^{3/23}(-\frac{5}{12} - \frac{425}{12}q - \frac{2353}{24}q^2 - \frac{7735}{24}q^3 - 360q^4 - \frac{8705}{12}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(2/23,19/23)}$	$q^{4/23}(-\frac{7}{8} - \frac{205}{6}q - \frac{3005}{24}q^2 - \frac{2665}{12}q^3 - \frac{2275}{4}q^4 - 576q^5 + O(q^6))$
$\mathfrak{e}_{\pm(11/23,1/23)}$	$q^{6/23}(-\frac{25}{12} - \frac{421}{12}q - \frac{595}{4}q^2 - \frac{3005}{12}q^3 - \frac{11765}{24}q^4 - \frac{14521}{24}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(10/23,3/23)}$	$q^{8/23}(-\frac{85}{24} - \frac{481}{12}q - \frac{1025}{6}q^2 - 240q^3 - \frac{4207}{8}q^4 - \frac{4205}{6}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(3/23,17/23)}$	$q^{9/23}(-\frac{91}{24} - \frac{455}{8}q - 120q^2 - \frac{2125}{6}q^3 - \frac{5101}{12}q^4 - \frac{3367}{4}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(9/23,5/23)}$	$q^{12/23}(-\frac{35}{4} - 48q - \frac{2105}{12}q^2 - \frac{7381}{24}q^3 - \frac{7225}{12}q^4 - \frac{8065}{12}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(6/23,11/23)}$	$q^{13/23}(-\frac{85}{12} - \frac{637}{8}q - \frac{1741}{12}q^2 - \frac{4205}{12}q^3 - 480q^4 - \frac{21845}{24}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(4/23,15/23)}$	$q^{16/23}(-\frac{341}{24} - \frac{425}{6}q - \frac{2405}{12}q^2 - 288q^3 - \frac{1435}{2}q^4 - \frac{8581}{12}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(8/23,7/23)}$	$q^{18/23}(-\frac{455}{24} - \frac{841}{12}q - \frac{5461}{24}q^2 - \frac{2105}{6}q^3 - 600q^4 - 720q^5 + O(q^6))$

	$\frac{24}{5}(E_3 - Q_{3,1/23,(1/23,21/23)})$
$\mathfrak{e}_{(0,0)}$	$0 + 230q + 46q^2 + 92q^3 + 414q^4 + 0q^5 + O(q^6)$
$\mathfrak{e}_{\pm(1/23,21/23)}$	$q^{1/23}(-5 + 166q + 134q^2 + 0q^3 + 716q^4 - 954q^5 + O(q^6))$
$\mathfrak{e}_{\pm(5/23,13/23)}$	$q^{2/23}(-1 - 125q - 298q^2 - 202q^3 + 142q^4 + 698q^5 + O(q^6))$
$\mathfrak{e}_{\pm(7/23,9/23)}$	$q^{3/23}(-2 - 122q - 245q^2 - 11q^3 + 0q^4 - 26q^5 + O(q^6))$
$\mathfrak{e}_{\pm(2/23,19/23)}$	$q^{4/23}(-9 + 172q - 25q^2 + 86q^3 - 42q^4 + 0q^5 + O(q^6))$
$\mathfrak{e}_{\pm(11/23,1/23)}$	$q^{6/23}(38 + 62q + 486q^2 - 50q^3 - 49q^4 - 605q^5 + O(q^6))$
$\mathfrak{e}_{\pm(10/23,3/23)}$	$q^{8/23}(31 + 38q + 380q^2 + 0q^3 - 225q^4 - 724q^5 + O(q^6))$
$\mathfrak{e}_{\pm(3/23,17/23)}$	$q^{9/23}(-23 + 63q + 0q^2 - 500q^3 + 830q^4 - 306q^5 + O(q^6))$
$\mathfrak{e}_{\pm(9/23,5/23)}$	$q^{12/23}(6 + 0q + 358q^2 - 329q^3 - 250q^4 - 346q^5 + O(q^6))$
$\mathfrak{e}_{\pm(6/23,11/23)}$	$q^{13/23}(-34 - 339q - 130q^2 + 94q^3 + 0q^4 + 575q^5 + O(q^6))$
$\mathfrak{e}_{\pm(4/23,15/23)}$	$q^{16/23}(-73 - 4q - 338q^2 + 0q^3 + 396q^4 + 254q^5 + O(q^6))$
$\mathfrak{e}_{\pm(8/23,7/23)}$	$q^{18/23}(-43 - 106q + 103q^2 - 196q^3 + 0q^4 + 0q^5 + O(q^6))$

	$24(E_3 - Q_{3,2/23,(5/23,13/23)})$
$\mathfrak{e}_{(0,0)}$	$0 + 46q + 1334q^2 - 1748q^3 - 1242q^4 + 0q^5 + O(q^6)$
$\mathfrak{e}_{\pm(1/23,21/23)}$	$q^{1/23}(-1 - 274q + 142q^2 + 0q^3 + 796q^4 + 2430q^5 + O(q^6))$
$\mathfrak{e}_{\pm(5/23,13/23)}$	$q^{2/23}(-29 - 25q + 622q^2 - 386q^3 - 1210q^4 - 398q^5 + O(q^6))$
$\mathfrak{e}_{\pm(7/23,9/23)}$	$q^{3/23}(38 + 350q - 49q^2 - 1423q^3 + 0q^4 - 754q^5 + O(q^6))$

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$\mathfrak{e}_{\pm(2/23,19/23)}$	$q^{4/23}(27+380q-725q^2+1054q^3-2226q^4+0q^5+O(q^6))$
$\mathfrak{e}_{\pm(11/23,1/23)}$	$q^{6/23}(-2+358q-738q^2+950q^3-1421q^4-121q^5+O(q^6))$
$\mathfrak{e}_{\pm(10/23,3/23)}$	$q^{8/23}(-109-338q+1084q^2+0q^3+675q^4-1796q^5+O(q^6))$
$\mathfrak{e}_{\pm(3/23,17/23)}$	$q^{9/23}(-43-189q+0q^2+1292q^3+166q^4-2970q^5+O(q^6))$
$\mathfrak{e}_{\pm(9/23,5/23)}$	$q^{12/23}(318+0q-706q^2-1525q^3+2254q^4+46q^5+O(q^6))$
$\mathfrak{e}_{\pm(6/23,11/23)}$	$q^{13/23}(-122-183q-26q^2+854q^3+0q^4+547q^5+O(q^6))$
$\mathfrak{e}_{\pm(4/23,15/23)}$	$q^{16/23}(187-500q-154q^2+0q^3+1836q^4-986q^5+O(q^6))$
$\mathfrak{e}_{\pm(8/23,7/23)}$	$q^{18/23}(-431+94q+827q^2+844q^3+O(q^6))$

	$24(E_3-Q_{3,3/23,(7/23,9/23)})$
$\mathfrak{e}_{(0,0)}$	$0+46q-874q^2+1564q^3-138q^4+0q^5+O(q^6)$
$\mathfrak{e}_{\pm(1/23,21/23)}$	$q^{1/23}(-1+134q+262q^2+0q^3-812q^4-1938q^5+O(q^6))$
$\mathfrak{e}_{\pm(5/23,13/23)}$	$q^{2/23}(19-25q-362q^2-266q^3+1070q^4+610q^5+O(q^6))$
$\mathfrak{e}_{\pm(7/23,9/23)}$	$q^{3/23}(-34-250q-49q^2+905q^3+0q^4+494q^5+O(q^6))$
$\mathfrak{e}_{\pm(2/23,19/23)}$	$q^{4/23}(3-268q+475q^2-554q^3+1470q^4+0q^5+O(q^6))$
$\mathfrak{e}_{\pm(11/23,1/23)}$	$q^{6/23}(-26-98q+654q^2-850q^3+931q^4-121q^5+O(q^6))$
$\mathfrak{e}_{\pm(10/23,3/23)}$	$q^{8/23}(83+358q-956q^2+0q^3+75q^4+1204q^5+O(q^6))$
$\mathfrak{e}_{\pm(3/23,17/23)}$	$q^{9/23}(77-21q+0q^2-748q^3+166q^4+1878q^5+O(q^6))$
$\mathfrak{e}_{\pm(9/23,5/23)}$	$q^{12/23}(-210+0q+422q^2+1211q^3-986q^4-986q^5+O(q^6))$
$\mathfrak{e}_{\pm(6/23,11/23)}$	$q^{13/23}(-2+225q-26q^2-898q^3+0q^4-173q^5+O(q^6))$
$\mathfrak{e}_{\pm(4/23,15/23)}$	$q^{16/23}(-149+196q+398q^2+0q^3-2004q^4+1438q^5+O(q^6))$
$\mathfrak{e}_{\pm(8/23,7/23)}$	$q^{18/23}(361-362q-517q^2-188q^3+0q^4+0q^5+O(q^6))$

Table B.18: Holomorphic products of weight less than 18

	Principal part	Weyl vector	
ψ_5	$10\mathfrak{e}_{(0,0)}+3q^{-1/23}\mathfrak{e}_{\pm(1/23,21/23)}+$ $+3q^{-2/23}\mathfrak{e}_{\pm(5/23,13/23)}+q^{-3/23}\mathfrak{e}_{\pm(7/23,9/23)}+$ $+2q^{-4/23}\mathfrak{e}_{\pm(2/23,19/23)}+q^{-6/23}\mathfrak{e}_{\pm(11/23,1/23)}$	$(1,\frac{8}{23},\frac{7}{23},1)$	skew

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ψ_6	$12\mathfrak{e}_{(0,0)} + 3q^{-1/23}\mathfrak{e}_{\pm(1/23,21/23)}+$ $+ q^{-2/23}\mathfrak{e}_{\pm(5/23,13/23)} + 3q^{-3/23}\mathfrak{e}_{\pm(7/23,9/23)}+$ $+ q^{-4/23}\mathfrak{e}_{\pm(2/23,19/23)} + q^{-8/23}\mathfrak{e}_{\pm(10/23,3/23)}$	$(1, \frac{9}{23}, \frac{5}{23}, 1)$	symm
ψ_7	$14\mathfrak{e}_{(0,0)} + 2q^{-1/23}\mathfrak{e}_{\pm(1/23,21/23)}+$ $+ q^{-2/23}\mathfrak{e}_{\pm(5/23,13/23)} + 2q^{-3/23}\mathfrak{e}_{\pm(7/23,9/23)}+$ $+ q^{-6/23}\mathfrak{e}_{\pm(11/23,1/23)} + q^{-9/23}\mathfrak{e}_{\pm(3/23,17/23)}$	$(1, \frac{9}{23}, \frac{5}{23}, 1)$	skew
ψ_8	$16\mathfrak{e}_{(0,0)} - q^{-1/23}\mathfrak{e}_{\pm(1/23,21/23)}+$ $+ 8q^{-2/23}\mathfrak{e}_{\pm(5/23,13/23)} + 4q^{-3/23}\mathfrak{e}_{\pm(7/23,9/23)}+$ $+ 3q^{-4/23}\mathfrak{e}_{\pm(2/23,19/23)} + q^{-6/23}\mathfrak{e}_{\pm(11/23,1/23)}$	$(1, \frac{9}{23}, \frac{5}{23}, 1)$	symm
ψ_{12}	$24\mathfrak{e}_{(0,0)} + 23q^{-1/23}\mathfrak{e}_{\pm(1/23,21/23)} + q^{-1}\mathfrak{e}_{(0,0)}$	$(3, 0, 1, 2)$	skew
ψ_{14}	$28\mathfrak{e}_{(0,0)} + 6q^{-1/23}\mathfrak{e}_{\pm(1/23,21/23)}+$ $+ 7q^{-2/23}\mathfrak{e}_{\pm(5/23,13/23)} + q^{-8/23}\mathfrak{e}_{\pm(10/23,3/23)}+$ $+ q^{-12/23}\mathfrak{e}_{\pm(9/23,5/23)}$	$(2, \frac{20}{23}, \frac{6}{23}, 2)$	symm

B.10 Discriminant −24

Table B.19: Obstruction space, discriminant −24

	E_3
$\mathfrak{e}_{(0,0)}$	$1 - \frac{650}{23}q - \frac{2040}{23}q^2 - \frac{5040}{23}q^3 - \frac{10250}{23}q^4 - \frac{13104}{23}q^5 + O(q^6)$
$\mathfrak{e}_{\pm(0,1/12)}, \mathfrak{e}_{\pm(0,5/12)}$	$q^{1/24}(-\frac{1}{23} - \frac{651}{23}q - \frac{2451}{23}q^2 - \frac{5330}{23}q^3 - \frac{9410}{23}q^4 - \frac{14763}{23}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,1/6)}$	$q^{1/6}(-\frac{17}{23} - \frac{850}{23}q - \frac{2520}{23}q^2 - \frac{5400}{23}q^3 - \frac{11067}{23}q^4 - \frac{16354}{23}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,1/4)}$	$q^{3/8}(-\frac{82}{23} - \frac{1220}{23}q - \frac{2880}{23}q^2 - \frac{6562}{23}q^3 - \frac{13000}{23}q^4 - \frac{14784}{23}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(0,1/3)}$	$q^{2/3}(-\frac{257}{23} - \frac{1690}{23}q - \frac{4097}{23}q^2 - \frac{7930}{23}q^3 - \frac{12850}{23}q^4 - \frac{18144}{23}q^5 + O(q^6))$
$\mathfrak{e}_{(0,1/2)}$	$q^{1/2}(-\frac{120}{23} - \frac{1394}{23}q - \frac{4420}{23}q^2 - \frac{6000}{23}q^3 - \frac{10920}{23}q^4 - \frac{20740}{23}q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(-\frac{50}{23} - \frac{624}{23}q - \frac{3650}{23}q^2 - \frac{5040}{23}q^3 - \frac{11520}{23}q^4 - \frac{12000}{23}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/12)}, \mathfrak{e}_{\pm(1/2,5/12)}$	$q^{7/24}(-\frac{50}{23} - \frac{962}{23}q - \frac{3172}{23}q^2 - \frac{6242}{23}q^3 - \frac{10610}{23}q^4 - \frac{16130}{23}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/6)}$	$q^{5/12}(-\frac{130}{23} - \frac{864}{23}q - \frac{4210}{23}q^2 - \frac{5040}{23}q^3 - \frac{14050}{23}q^4 - \frac{13104}{23}q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/4)}$	$q^{5/8}(-\frac{260}{23} - \frac{1344}{23}q - \frac{4100}{23}q^2 - \frac{8420}{23}q^3 - \frac{10944}{23}q^4 - \frac{18980}{23}q^5 + O(q^6))$

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$\mathfrak{e}_{\pm(1/2,1/3)}$	$q^{11/12}(-\frac{610}{23}-\frac{1584}{23}q-\frac{6500}{23}q^2-288q^3-\frac{17410}{23}q^4-\frac{15120}{23}q^5+O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{3/4}(-\frac{240}{23}-\frac{2500}{23}q-\frac{2928}{23}q^2-\frac{10660}{23}q^3-\frac{10800}{23}q^4-\frac{21120}{23}q^5+O(q^6))$

	$23(E_3-Q_{3,1/24,(0,1/12)})$
$\mathfrak{e}_{(0,0)}$	$0+960q+720q^2+480q^3+2400q^4-960q^5+O(q^6)$
$\mathfrak{e}_{\pm(0,1/12)}$	$q^{1/24}(-24+959q+309q^2+190q^3+3240q^4-2642q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,1/6)}$	$q^{1/6}(-40+760q+240q^2+120q^3+1560q^4-2600q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,1/4)}$	$q^{3/8}(-105+390q-120q^2-1065q^3+1260q^4+120q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,1/3)}$	$q^{2/3}(-280-80q-1360q^2-800q^3+2560q^4-480q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,5/12)}$	$q^{1/24}(-1-674q-864q^2-960q^3-1130q^4+3407q^5+O(q^6))$
$\mathfrak{e}_{(0,1/2)}$	$q^{1/2}(-120-1440q-1200q^2-480q^3+120q^4+4560q^5+O(q^6))$
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(180+480q+2100q^2+480q^3-480q^4-960q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/12)}$	$q^{7/24}(180+142q+2578q^2-722q^3+430q^4-4860q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/6)}$	$q^{5/12}(100+240q+1540q^2+480q^3-2780q^4-960q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/4)}$	$q^{5/8}(-30-240q+1650q^2-2670q^3+1200q^4-2190q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/3)}$	$q^{11/12}(-380-480q-520q^2+0q^3-620q^4+1440q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,5/12)}$	$q^{7/24}(-50-732q-1838q^2+612q^3+660q^4+430q^5+O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{3/4}(-240-2040q-720q^2+840q^3+240q^4+960q^5+O(q^6))$

	$Q_{3,1/24,(0,1/12)}-Q_{3,1/24,(0,5/12)}$
$\mathfrak{e}_{\pm(0,1/12)}$	$q^{1/24}(1-71q-51q^2-50q^3-190q^4+263q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,5/12)}$	$q^{1/24}(-1+71q+51q^2+50q^3+190q^4-263q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/12)}$	$q^{7/24}(-10-38q-192q^2+58q^3+10q^4+230q^5+O(q^6))$
$\mathfrak{e}_{\pm(1/2,5/12)}$	$q^{7/24}(10+38q+192q^2-58q^3-10q^4-230q^5+O(q^6))$
other components	0

	$23(E_3-Q_{3,1/6,(0,1/6)})$
$\mathfrak{e}_{(0,0)}$	$0+224q-384q^2-256q^3+1664q^4+512q^5+O(q^6)$
$\mathfrak{e}_{\pm(0,1/12)},\mathfrak{e}_{\pm(0,5/12)}$	$q^{1/24}(-1+85q-243q^2-178q^3+894q^4-43q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,1/6)}$	$q^{1/6}(-40+208q-128q^2-64q^3+456q^4-944q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,1/4)}$	$q^{3/8}(-82+252q+64q^2-674q^3+248q^4-64q^5+O(q^6))$
$\mathfrak{e}_{\pm(0,1/3)}$	$q^{2/3}(-96+288q-256q^2-800q^3+1088q^4+256q^5+O(q^6))$

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$\mathfrak{e}_{(0,1/2)}$	$q^{1/2}(64 - 336q - 96q^2 + 256q^3 - 64q^4 + 1248q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,0)}$	$q^{1/4}(88 - 256q + 536q^2 - 256q^3 + 256q^4 + 512q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/12)}, \mathfrak{e}_{\pm(1/2,5/12)}$	$q^{7/24}(42 - 134q - 44q^2 + 474q^3 - 214q^4 - 582q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/6)}$	$q^{5/12}(8 - 128q + 712q^2 - 256q^3 - 1400q^4 + 512q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/4)}$	$q^{5/8}(-76 + 128q + 500q^2 - 876q^3 - 640q^4 + 340q^5 + O(q^6))$
$\mathfrak{e}_{\pm(1/2,1/3)}$	$q^{11/12}(-104 + 256q - 336q^2 + 0q^3 + 24q^4 - 768q^5 + O(q^6))$
$\mathfrak{e}_{(1/2,1/2)}$	$q^{3/4}(128 - 752q + 384q^2 + 656q^3 - 128q^4 - 512q^5 + O(q^6))$

Table B.20: Holomorphic products of weight less than 17

	Principal part	Weyl vector	Char.	
ψ_2	$4\mathfrak{e}_{(0,0)} + 2q^{-1/24}(\mathfrak{e}_{\pm(0,1/12)} + \mathfrak{e}_{\pm(0,5/12)}) + {}$ $+ q^{-1/6}\mathfrak{e}_{\pm(0,1/6)} + q^{-1/4}\mathfrak{e}_{(1/2,0)}$	$(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{2})$	χ	skew
$\psi_5^{(1)}$	$10\mathfrak{e}_{(0,0)} - 2q^{-1/24}\mathfrak{e}_{\pm(0,1/12)} + 8q^{-1/24}\mathfrak{e}_{\pm(0,5/12)} + {}$ $+ 2q^{-1/6}\mathfrak{e}_{\pm(0,1/6)} + q^{-1/4}\mathfrak{e}_{(1/2,0)} + q^{-7/24}\mathfrak{e}_{\pm(1/2,5/12)}$	$(\frac{1}{2}, \frac{1}{4}, \frac{1}{12}, \frac{1}{2})$	χ	symm
$\psi_5^{(2)}$	$10\mathfrak{e}_{(0,0)} + 8q^{-1/24}\mathfrak{e}_{\pm(0,1/12)} - 2q^{-1/24}\mathfrak{e}_{\pm(0,5/12)} + {}$ $+ 2q^{-1/6}\mathfrak{e}_{\pm(0,1/6)} + q^{-1/4}\mathfrak{e}_{(1/2,0)} + {}$ $+ q^{-7/24}\mathfrak{e}_{\pm(1/2,5/12)}$	$(\frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \frac{3}{2})$	χ	symm
$\psi_5^{(3)}$	$10\mathfrak{e}_{(0,0)} - 2q^{-1/24}(\mathfrak{e}_{\pm(0,1/12)} + \mathfrak{e}_{\pm(0,5/12)}) + {}$ $+ 2q^{-1/6}\mathfrak{e}_{\pm(0,1/6)} + q^{-1/4}\mathfrak{e}_{(1/2,0)} + {}$ $+ q^{-1/2}\mathfrak{e}_{(0,1/2)}$	$(\frac{1}{2}, \frac{1}{4}, \frac{1}{12}, \frac{1}{2})$	χ	symm
$\psi_6^{(1)}$	$12\mathfrak{e}_{(0,0)} + 4q^{-1/24}(\mathfrak{e}_{\pm(0,1/12)} + \mathfrak{e}_{(0,5/12)}) + q^{-5/12}\mathfrak{e}_{\pm(1/2,1/6)}$	$(1, \frac{1}{2}, \frac{1}{6}, 1)$	$-$	symm
$\psi_6^{(2)}$	$12\mathfrak{e}_{(0,0)} + 3q^{-1/24}(\mathfrak{e}_{\pm(0,1/12)} + \mathfrak{e}_{\pm(0,5/12)}) + {}$ $+ 2q^{-1/4}\mathfrak{e}_{(1/2,0)} + q^{-3/8}\mathfrak{e}_{\pm(0,1/4)}$	$(1, \frac{1}{2}, \frac{1}{4}, 1)$	$-$	symm
ψ_{10}	$20\mathfrak{e}_{(0,0)} - 4q^{-1/24}(\mathfrak{e}_{\pm(0,1/12)} + \mathfrak{e}_{\pm(0,5/12)}) + {}$ $+ 4q^{-1/6}\mathfrak{e}_{\pm(0,1/6)} + 2q^{-1/4}\mathfrak{e}_{(1/2,0)} + {}$ $+ q^{-3/4}\mathfrak{e}_{(1/2,1/2)}$	$(1, \frac{1}{2}, \frac{1}{6}, 1)$	$-$	symm
ψ_{14}	$28\mathfrak{e}_{(0,0)} + 6q^{-1/24}(\mathfrak{e}_{\pm(0,1/12)} + \mathfrak{e}_{\pm(0,5/12)}) + {}$ $+ 2q^{-1/4}\mathfrak{e}_{(1/2,0)} + q^{-5/8}\mathfrak{e}_{\pm(1/2,1/4)}$	$(2, 1, \frac{1}{4}, 2)$	$-$	symm