VECTOR-VALUED EISENSTEIN SERIES OF SMALL WEIGHT

BRANDON WILLIAMS

ABSTRACT. We study the (mock) Eisenstein series of weight $k \le 2$ for the Weil representation on an even lattice. We describe the transformation laws in general and give examples where the coefficients contain interesting arithmetic information.

Key words and phrases: Modular forms; Mock modular forms; Eisenstein series; Weil representation

1. Introduction

In [4], Bruinier and Kuss give an expression for the Fourier coefficients of the Eisenstein series E_k of weight $k \ge 5/2$ for the Weil representation attached to a discriminant form. These coefficients involve special values of L-functions and zero counts of polynomials modulo prime powers, and they also make sense for $k \in \{1, 3/2, 2\}$ and sometimes $k = 1/2$. Unfortunately, the q-series obtained in this way often fail to be modular forms. In particular, in weight $k = 3/2$ and $k = 2$, the Eisenstein series may be a mock modular form that requires a real-analytic correction in order to transform as a modular form. Many examples of this phenomenon of the Eisenstein series are well-known (although perhaps less familiar in a vector-valued setting). We will list a few examples of this:

Example 1. The Eisenstein series of weight 2 for a unimodular lattice Λ is the quasimodular form

$$E_2(\tau) = 1 - 24\sum_{n=1}^\infty \sigma_1(n)q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - ...$$

where $\sigma_1(n) = \sum_{d|n} d$, which transforms under the modular group by

$$E_2\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2 E_2(\tau) + \frac{6}{\pi i}c(c\tau+d).$$

Example 2. The Eisenstein series of weight 3/2 for the quadratic form $q_2(x) = x^2$ is essentially Zagier's mock Eisenstein series:

$$E_{3/2}(\tau) = \Big(1 - 6q - 12q^2 - 16q^3 - ...\Big)\mathfrak{e}_0 + \Big(-4q^{3/4} - 12q^{7/4} - 12q^{11/4} - ...\Big)\mathfrak{e}_{1/2},$$

in which the coefficient of $q^{n/4}\mathfrak{e}_{n/2}$ is -12 times the Hurwitz class number $H(n)$. It transforms under the modular group by

$$E_{3/2}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{3/2}\rho^*\Big(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\Big)\Big[E_{3/2}(\tau) - \frac{3}{\pi}\sqrt{\frac{i}{2}}\int_{d/c}^{i\infty}(\tau+t)^{-3/2}\vartheta(t)\,\mathrm{d}t\Big],$$

where ϑ is the theta series

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/4}\mathfrak{e}_{n/2}.$$

Example 3. In the Eisenstein series of weight 3/2 for the quadratic form $q_3(x) = -12x^2$, the components of $\mathfrak{e}_{1/12}$, $\mathfrak{e}_{5/12}$, $\mathfrak{e}_{7/12}$ and $\mathfrak{e}_{11/12}$ are

$$\Big(-3q^{23/24} - 5q^{47/24} - 7q^{71/24} - 8q^{95/24} - 10q^{119/24} - 10q^{143/24} - ...\Big)\mathfrak{e}_\gamma, \;\; \gamma \in \{1/12, 5/12, 7/12, 11/12\}.$$

We verified by computer that the coefficient of $q^{n-1/24}$ above is (-1) times the degree of the n-th partition class polynomial considered by Bruinier and Ono [5] for $1 \le n \le 750$, which is not surprising in view of

2010 *Mathematics Subject Classification.* 11F27, 11F30, 11F37.

example 2 since this degree also counts equivalence classes of certain binary quadratic forms. This Eisenstein series is not a modular form.

Example 4. The Eisenstein series of weight 3/2 for the quadratic form $q_4(x,y,z) = x^2 + y^2 - z^2$ is a mock modular form that is related to the functions considered by Bringmann and Lovejoy [1] in their work on overpartitions. More specifically, the component of $\mathfrak{e}_{(0,0,0)}$ in $E_{3/2}$ is

$$1 - 2q - 4q^2 - 8q^3 - 10q^4 - ... = 1 - \sum_{n=1}^{\infty} |\overline{\alpha}(n)| q^n,$$

where $\overline{\alpha}(n)$ is the difference between the number of even-rank and odd-rank overpartitions of n. Similarly, the $M2$-rank differences considered in [1] appear to occur in the Eisenstein series of weight 3/2 for the quadratic form $q_5(x,y,z) = 2x^2 + 2y^2 - z^2$, whose $\mathfrak{e}_{(0,0,0)}$-component is

$$1 - 2q - 4q^2 - 2q^4 - 8q^5 - 8q^6 - 8q^7 - ...$$

Example 5. Unlike the previous examples, the Eisenstein series of weight 3/2 for the quadratic form $q_6(x,y,z) = -x^2 - y^2 - z^2$ is a true modular form; in fact, it is the theta series for the cubic lattice and the Fourier coefficients of its $\mathfrak{e}_{(0,0,0)}$-component count the representations of integers as sums of three squares. The results of this note imply that we should expect different behavior in example 4 and example 5 because there are many more isotropic vectors for q_4 and q_5 modulo large powers of 2 than there are for q_6.

Among negative-definite lattices of small dimension there are lots of examples where the Eisenstein series equals the theta series. (Note that we find theta series for negative-definite lattices instead of positive-definite because we consider the dual Weil representation ρ^*.) When the lattice is even-dimensional this immediately leads to formulas for representation numbers in terms of twisted divisor sums. These formulas are well-known from the theory of scalar modular forms of higher level but the vector-valued derivations of these formulas seem more natural. We give several examples of this throughout the note.

Acknowledgments: I thank Kathrin Bringmann for discussing the examples involving overpartition rank differences with me.

Contents

2. Notation

Λ denotes an even lattice with nondegenerate quadratic form q. The signature of Λ is (b^+, b^-) and its dimension is e. The dual lattice is Λ'. The natural basis of the group ring $\mathbb{C}[\Lambda'/\Lambda]$ is denoted $\mathfrak{e}_\gamma$, $\gamma \in \Lambda'/\Lambda$.

We denote by ρ the Weil representation of $Mp_2(\mathbb{Z})$ on the group ring $\mathbb{C}[\Lambda'/\Lambda]$ and ρ^* its (unitary) dual. Elements of Λ'/Λ are usually denoted γ or β and $\mathfrak{e}_\gamma, \mathfrak{e}_\beta$ denotes the corresponding element in the natural basis of $\mathbb{C}[\Lambda'/\Lambda]$. We denote the **denominator** of $\gamma \in \Lambda'/\Lambda$ by d_γ; i.e. $d_\gamma \in \mathbb{N}$ is minimal such that $d_\gamma \gamma \in \Lambda$.

$E_k(\tau)$ denotes the weight-k (mock) **Eisenstein series**, which is the result of Bruinier and Kuss's formula [4] for $E_{k,0}$ naively evaluated at $k \leq 2$. (However, we divide their formula by 2 such that the result has constant term $\mathfrak{e}_0$.) $E_k^*(\tau, s)$ is the weight-k real-analytic Eisenstein series described in section 4.

For a **discriminant** D (an integer congruent to 0 or 1 mod 4), we denote by χ_D the Dirichlet character mod $|D|$ given by the Kronecker symbol,

$$\chi_D(n) = \Big(\frac{D}{n}\Big).$$

The associated **L-function** is

$$L(s, \chi_D) = \sum_{n=1}^{\infty} \chi_D(n) n^{-s},$$

converging when $\mathrm{Re}[s] > 1$ and extended to all $s \in \mathbb{C}$ by analytic continuation. We denote by $\sigma_k(n, \chi_D)$ the **twisted divisor sum**

$$\sigma_k(n, \chi_D) = \sum_{d|n} \chi_D(n/d) d^k.$$

We use the abbreviation $\mathbf{e}(x) = e^{2\pi i x}$ and the notation $\mathbf{e}(\tau) = q$, where $\tau = x + iy$ is a complex variable confined to the upper half-plane $\mathbb{H}$.

3. Background

In this section we review some facts about the metaplectic group and vector-valued modular forms, as well as Dirichlet L-functions, which will be useful later.

Recall that the **metaplectic group** $Mp_2(\mathbb{Z})$ is the double cover of $SL_2(\mathbb{Z})$ consisting of pairs (M, ϕ), where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and ϕ is a branch of $\sqrt{c\tau + d}$ on the upper half-plane

$$\mathbb{H} = \{\tau = x + iy \in \mathbb{C} : \, y > 0\}.$$

We will usually omit ϕ. $Mp_2(\mathbb{Z})$ is generated by the elements

$$T = \Big(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \Big), \;\; S = \Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \Big),$$

with defining relations $S^8 = I$ and $S^2 = (ST)^3$.

Let Λ be a lattice (which we can always take as $\Lambda = \mathbb{Z}^e$ for some $e \in \mathbb{N}$) with an even quadratic form $q : \Lambda \to \mathbb{Z}$, and let

$$\Lambda' = \{v \in \mathbb{Q}^n : \, \langle v, w \rangle \in \mathbb{Z} \, \text{for all} \, w \in \Lambda\}$$

be the dual lattice. We denote by $\mathfrak{e}_\gamma$, $\gamma \in \Lambda'/\Lambda$ the natural basis of the group algebra $\mathbb{C}[\Lambda'/\Lambda]$. The **Weil representation** of $Mp_2(\mathbb{Z})$ attached to Λ is the map

$$\rho : Mp_2(\mathbb{Z}) \longrightarrow \mathrm{Aut}\, \mathbb{C}[\Lambda'/\Lambda]$$

defined by

$$\rho(T)\mathfrak{e}_\gamma = \mathbf{e}\Big(q(\gamma)\Big)\mathfrak{e}_\gamma, \;\; \rho(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|\Lambda'/\Lambda|}} \sum_{\beta \in \Lambda'/\Lambda} \mathbf{e}\Big(- \langle \gamma, \beta \rangle \Big) \mathfrak{e}_\beta.$$

In particular,

$$\rho(Z)\mathfrak{e}_\gamma = i^{b^- - b^+} \mathfrak{e}_{-\gamma}, \;\; \text{where} \;\; Z = (-I, i) = S^2 = (ST)^3.$$

Here we use $\mathbf{e}(x)$ to denote $e^{2\pi i x}$, and (b^+, b^-) is the signature of Λ.
We will usually consider the dual representation ρ^* of ρ (which also occurs as the Weil representation itself, for the lattice Λ and quadratic form $-q$).

A **modular form of weight** k for ρ^* is a holomorphic function $f : \mathbb{H} \to \mathbb{C}[\Lambda'/\Lambda]$ with the properties:
(i) f transforms under the action of $Mp_2(\mathbb{Z})$ by

$$f(M \cdot \tau) = (c\tau + d)^k \rho^*(M) f(\tau), \;\; M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z});$$

(ii) f is holomorphic in ∞. This means that in the Fourier expansion

$$f(\tau) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - q(\gamma)} c(n,\gamma) q^n \mathfrak{e}_\gamma,$$

all coefficients $c(n,\gamma)$ are zero for $n < 0$.

If N is the smallest natural number such that $N\langle \gamma, \beta \rangle$ and $Nq(\gamma) \in \mathbb{Z}$ for all $\beta, \gamma \in \Lambda'/\Lambda$, then ρ^* factors through $SL_2(\mathbb{Z}/N\mathbb{Z})$ if $e = \dim \Lambda$ is even, and through a double cover of $SL_2(\mathbb{Z}/N\mathbb{Z})$ if e is odd. This implies in particular that the component functions f_γ of f are scalar modular forms of level N.

We will also consider **harmonic weak Maass forms**, which have the same transformation behavior as modular forms but for which the holomorphy assumption is weakened to real-analyticity and the weight-k Laplace equation $\Delta f(\tau) = 2iky \frac{\partial}{\partial \overline{\tau}} f(\tau)$, where $\Delta = y^2 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the hyperbolic Laplacian on $\mathbb{H}$. Harmonic weak Maass forms are also required to satisfy a (weakened) growth condition at cusps. We refer to [3] and [9] for details.

The weights of modular forms are restricted due to

$$f(\tau) = f(Z \cdot \tau) = i^{2k} \rho^*(Z) f(\tau) = i^{2k + b^+ - b^-} \sum_{\gamma \in \Lambda'/\Lambda} f_{-\gamma}(\tau) \mathfrak{e}_\gamma.$$

In particular, if $2k + b^+ - b^-$ is not an even integer, then there are no nonzero modular forms. In the case $2k + b^+ - b^- \equiv 2\,(4)$ (which seems to be of less interest), the components satisfy $f_\gamma = -f_{-\gamma}$ and in particular the $\mathfrak{e}_0$-component of f must be zero. We will focus on the case $2k + b^+ - b^- \equiv 0\,(4)$ which is the only case in which the Eisenstein series E_k with constant term $\mathfrak{e}_0$ can be defined.

Remark 6. There is an involution $\sim$ of the metaplectic group given on the standard generators by

$$\tilde{S} = S^{-1}, \;\; \tilde{T} = T^{-1},$$

which is well-defined because

$$(\tilde{S}\tilde{T})^3 = S^{-1}(ST)^{-3}S = S^{-1}S^{-2}S = \tilde{S}^2$$

and $\tilde{S}^8 = I$. On matrices it is given by

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

and it acts on the branches of square roots by $\tilde{\phi}(\tau) = \overline{\phi(-\overline{\tau})}$, where $\phi(\tau)^2 = c\tau + d$. One can check on the generators S, T that this intertwines the Weil representation ρ and its dual ρ^* in the sense that

$$\rho(\tilde{M}) = \rho^*(M) = \overline{\rho(M)}, \;\; M \in Mp_2(\mathbb{Z}).$$

Remark 7. At many points in this note we will need to consider the L-function

$$L(s, \chi_D) = \sum_{n=1}^\infty \chi_D(n) n^{-s}$$

attached to the Dirichlet character mod $|D|$,

$$\chi_D(n) = \Big(\frac{D}{n} \Big),$$

where D is a discriminant (i.e. $D \equiv 0, 1 \mod 4$). In particular, we recall the following properties of Dirichlet L-functions.

(i) Let χ be a Dirichlet character. Then $L(s,\chi)$ converges absolutely in some half-plane $\mathrm{Re}[s] > s_0$ and is given by an Euler product

$$L(s,\chi) = \prod_{p \text{ prime}} (1 - \chi(p) p^{-s})^{-1}$$

there.
(ii) $L(s,\chi)$ has a meromorphic extension to all $\mathbb{C}$ and satisfies the functional equation

$$\Gamma(s)\cos\Big(\frac{\pi(s-\delta)}{2}\Big)L(s,\chi)=\frac{\tau(\chi)}{2i^\delta}(2\pi/f)^sL(1-s,\overline{\chi}),$$

where f is the conductor of χ, $\tau(\chi)=\sum_{a=1}^f\chi(a)e^{2\pi ia/f}$ is the Gauss sum of χ, and

$$\delta=\begin{cases}1: & \chi(-1)=-1;\\ 0: & \chi(-1)=1.\end{cases}$$

(iii) $L(s,\chi)$ is never zero at $s=1$, and is holomorphic there unless χ is a trivial character, in which case it has a simple pole.
(iv) The values $L(1-n,\chi)$, $n\in\mathbb{N}$ are rational numbers, given by

$$L(1-n,\chi)=-\frac{B_{n,\chi}}{n},$$

where $B_{n,\chi}\in\mathbb{Q}$ is a generalized Bernoulli number.
We refer to section 4 of [10] for these and other results on Dirichlet L-functions.

4. The real-analytic Eisenstein series

Fix an even lattice Λ and let ρ^* be the dual Weil representation on $\mathbb{C}[\Lambda'/\Lambda]$.

Definition 8. The **real-analytic Eisenstein series** of weight k is

$$E_k^*(\tau,s)=\frac{1}{2}\sum_{c,d}(c\tau+d)^{-k}|c\tau+d|^{-2s}\rho^*(M)^{-1}\mathfrak{e}_0.$$

Here, (c,d) runs through all pairs of coprime integers and M is any element $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in Mp_2(\mathbb{Z})$ with bottom row (c,d).

This series converges absolutely and locally uniformly in the half-plane $\mathrm{Re}[s]>1-k/2$ and defines a holomorphic function in s. For fixed s, it transforms under the metaplectic group by

$$E_k^*\Big(M\cdot\tau,s\Big)=(c\tau+d)^k|c\tau+d|^{2s}\rho^*(M)E_k^*(\tau,s)$$

for any $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in Mp_2(\mathbb{Z})$. (Multiplying $E_k^*(\tau,s)$ by y^s results in a function that transforms like a modular form of weight k, but for the purposes of this note there seems to be no advantage in doing this.)

Writing the Fourier series of $E_k^*(\tau,s)$ in the form

$$E_k^*(\tau,s)=\mathfrak{e}_0+\sum_{\gamma\in\Lambda'/\Lambda}\sum_{n\in\mathbb{Z}-q(\gamma)}c(n,\gamma,s,y)q^n\mathfrak{e}_\gamma,$$

a computation analogous to section 1.2.3 of [2] (where it is carried out in more detail) using the exact formula for the coefficients $\rho(M)_{0,\gamma}$ of the Weil representation shows that

$$\begin{aligned}c(n,\gamma,s,y)&=\frac{1}{2}\sum_{c\neq0}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\rho(M)_{0,\gamma}\int_{-\infty+iy}^{\infty+iy}(c\tau+d)^{-k}|c\tau+d|^{-2s}\mathbf{e}(-n\tau)\,\mathrm{d}x\\&=\sum_{c=1}^\infty\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\rho(M)_{0,\gamma}c^{-k-2s}\mathbf{e}\Big(\frac{nd}{c}\Big)\int_{-\infty+iy}^{\infty+iy}\tau^{-k}|\tau|^{-2s}\mathbf{e}(-n\tau)\,\mathrm{d}x\\&=\frac{\sqrt{i}^{b^--b^+}}{\sqrt{|\Lambda'/\Lambda|}}\tilde{L}(n,\gamma,k+e/2+2s)I(k,y,n,s),\end{aligned}\tag{1}$$

where M is any element of $Mp_2(\mathbb{Z})$ whose bottom row is (c,d). Here, $\tilde{L}(n,\gamma,s)$ is the L-series

$$\begin{aligned}\tilde{L}(n,\gamma,s) &= \sum_{c=1}^{\infty} c^{-s+e/2} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \rho(M)_{0,\gamma} \mathbf{e}\Big(\frac{nd}{c}\Big) \\ &= \sum_{c=1}^{\infty} c^{-s} \sum_{\substack{v \in \Lambda/c\Lambda \\ d \in (\mathbb{Z}/c\mathbb{Z})^\times}} \mathbf{e}\Big(\frac{aq(v) - \langle \gamma, v\rangle + dq(\gamma) - nd}{c}\Big) \\ &= \sum_{c=1}^{\infty} c^{-s} \sum_{a|c} \Big[\mu(c/a)a(c/a)^e \cdot \#\Big\{v \in \Lambda/a\Lambda :\, q(v-\gamma)+n \equiv 0 \,(\mathrm{mod}\; a)\Big\}\Big] \\ &= \zeta(s-e)^{-1} L(n,\gamma,s-1),\end{aligned}$$

where $L(n,\gamma,s)$ is

$$L(n,\gamma,s) = \sum_{a=1}^{\infty} a^{-s}\mathbf{N}(a) = \prod_{p\,\mathrm{prime}} \Big(\sum_{\nu=0}^{\infty} p^{-\nu s}\mathbf{N}(p^\nu)\Big) = \prod_{p\,\mathrm{prime}} L_p(n,\gamma,s),$$

and $\mathbf{N}(p^\nu)$ is the number of zeros $v \in \Lambda/p^\nu\Lambda$ of the quadratic polynomial $q(v-\gamma)+n$; and $I(k,y,n,s)$ is the integral

$$\begin{aligned}I(k,y,n,s) &= \int_{-\infty+iy}^{\infty+iy} \tau^{-k}|\tau|^{-2s}\mathbf{e}(-n\tau)\,\mathrm{d}x \\ &= y^{1-k-2s}e^{2\pi ny}\int_{-\infty}^{\infty}(t+i)^{-k}(t^2+1)^{-s}\mathbf{e}(-nyt)\,\mathrm{d}t, \quad \tau = y(t+i).\end{aligned}$$

Remark 9. Both the L-series term $\tilde{L}(n,\gamma,s)$ and the integral term $I(k,y,n,s)$ of (1) have meromorphic continuations to all $s \in \mathbb{C}$. First we remark that the integral $I(k,y,n,s)$ was considered by Gross and Zagier [7], section IV.3., where it was shown that for $n \neq 0$, $I(k,y,n,s)$ is a finite linear combination of K-Bessel functions (we will not need the exact expression) and its value at $s=0$ is given by

$$I(k,y,n,0) = \begin{cases} 0: & n<0; \\ (-2\pi i)^k n^{k-1}\frac{1}{\Gamma(k)}: & n>0; \end{cases} \tag{2}$$

if $n \neq 0$; and when $n=0$,

$$I(k,y,0,s) = 2\pi(-i)^k(2y)^{1-k-2s}\frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)}. \tag{3}$$

In particular, the zero value of the latter expression is

$$I(k,y,0,0) = \begin{cases} 0: & k \neq 1; \\ -i\pi: & k=1. \end{cases}$$

The Euler factors $L_p(n,\gamma,s) = \sum_{\nu=0}^{\infty} p^{-\nu s}\mathbf{N}(p^\nu)$ are known to be rational functions in p^{-s} that can be calculated using the methods of [6] (see also section 6 of [11]). For generic primes (primes $p \neq 2$ that do not divide $|\Lambda'/\Lambda|$, or the numerator or denominator of n if $n \neq 0$) the result is that

$$L_p(n,\gamma,s) = \begin{cases} \frac{1}{1-p^{e-1-s}}\Big[1-\Big(\frac{D}{p}\Big)p^{e/2-s}\Big]: & n \neq 0; \\ \\ \frac{1-\Big(\frac{D'}{p}\Big)p^{e/2-1-s}}{(1-p^{e-1-s})\Big[1-\Big(\frac{D'}{p}\Big)p^{e/2-s}\Big]}: & n=0; \end{cases}$$

if e is even and

$$L_p(n,\gamma,s) = \begin{cases} \frac{1}{1-p^{e-1-s}}\Big[1+\Big(\frac{\mathcal{D}'}{p}\Big)p^{(e-1)/2-s}\Big]: & n \neq 0; \\ \\ \frac{1-p^{e-1-2s}}{(1-p^{e-1-s})(1-p^{e-2s})}: & n=0; \end{cases}$$

if e is odd. Here, D' and $\mathcal{D}'$ are defined by

$$D' = (-1)^k |\Lambda'/\Lambda| \text{ and } \mathcal{D}' = 2nd_\gamma^2 (-1)^{k-1/2} |\Lambda'/\Lambda|.$$

In particular, if we define $D = D' \cdot \prod_{\text{bad}\, p} p^2$ and $\mathcal{D} = \mathcal{D}' \cdot \prod_{\text{bad}\, p} p^2$, where the bad primes are 2 and any prime dividing $|\Lambda'/\Lambda|$ or n, then we get the meromorphic continuations

$$\tilde{L}(n,\gamma,s) = \begin{cases} \frac{1}{L(s-e/2,\chi_D)} \prod_{\text{bad}\, p} (1-p^{e-s}) L_p(n,\gamma,s-1) : & n \neq 0; \\ \frac{L(s-1-e/2,\chi_D)}{L(s-e/2,\chi_D)} \prod_{\text{bad}\, p} (1-p^{e-s}) L_p(s-1) : & n = 0; \end{cases}$$

if e is even and

$$\tilde{L}(n,\gamma,s) = \begin{cases} \frac{L(s-(e+1)/2,\chi_{\mathcal{D}})}{\zeta(2s-1-e)} \prod_{\text{bad}\, p} \frac{1-p^{e-s}}{1-p^{e+1-2s}} L_p(n,\gamma,s-1) : & n \neq 0; \\ \frac{\zeta(2s-2-e)}{\zeta(2s-1-e)} \prod_{\text{bad}\, p} \frac{(1-p^{e-s})(1-p^{e+2-2s})}{1-p^{e+1-2s}} L_p(s-1) : & n = 0; \end{cases}$$

if e is odd.

Remark 10. We denote by E_k the series

$$E_k(\tau) = \mathfrak{e}_0 + \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n>0} c(n,\gamma,0,y) q^n \mathfrak{e}_\gamma.$$

The formula (2) gives $I(k,y,n,0) = (-2\pi i)^k n^{k-1} \frac{1}{\Gamma(k)}$ independently of y, and so $E_k(\tau)$ is holomorphic. When $k > 2$, this is just the zero-value $E_k(\tau) = E_k^*(\tau,0)$ and therefore E_k is a modular form. In small weights this tends to fail because the terms

$$\lim_{s \to 0} \tilde{L}(n,\gamma,k+e/2+2s) I(k,y,n,s)$$

may have a pole of $\tilde{L}$ cancelling the zero of I for $n \leq 0$, resulting in nonzero (and often nonholomorphic) contributions to $E_k^*(\tau,0)$.

Remark 11. Suppose the dimension e is even and $k \geq e/2$. By appending hyperbolic planes to the lattice Λ, we can assume that $k = e/2$ and apply theorem 4.8 of [4]. It follows that the coefficient $c(n,0)$ of $q^n \mathfrak{e}_0$ in E_k is

$$c(n,0) = \frac{(2\pi)^k}{L(k,\chi_D)\sqrt{|\Lambda'/\Lambda|}\Gamma(k)} \cdot \sigma_{k-1}(n,\chi_D) \cdot \prod_{p|D'} \Big[(1-p^{e/2-k}) L_p(n,0,k+e/2-1)\Big],$$

where $\sigma_{k-1}(n,\chi_D)$ is the twisted divisor sum

$$\sigma_{k-1}(n,\chi_D) = \sum_{d|n} \chi_D(n/d) d^{k-1}$$

and $D' = 4|\Lambda'/\Lambda|$. (The assumption $k \geq e/2$ is probably not necessary but it holds in all examples we will consider anyway.) For a fixed lattice Λ, the expression $\prod_{p|D'} \Big[(1-p^{e/2-k}) L_p(n,0,k+e/2-1)\Big]$ can always be worked out exactly with some effort (we give some examples of this in the following sections), but there seems to be no nice closed form for it in general. Theorem 4.8 of [4] also gives an interpretation of the coefficients when e is odd but this is more complicated.

5. Weight 1/2

In weight $k = 1/2$, the L-series factor is

$$\tilde{L}\Big(n,\gamma,\frac{e+1}{2}+2s\Big) = \begin{cases} \frac{L_{\mathcal{D}}(2s)}{\zeta(4s)} \prod_{\text{bad}\, p} \frac{1-p^{(e-1)/2-2s}}{1-p^{-4s}} L_p\Big(n,\gamma,\frac{e-1}{2}+2s\Big) : & n \neq 0; \\ \frac{\zeta(4s-1)}{\zeta(4s)} \prod_{\text{bad}\, p} \frac{(1-p^{(e-1)/2-2s})(1-p^{1-4s})}{1-p^{-4s}} L_p\Big(n,\gamma,\frac{e-1}{2}+2s\Big) : & n = 0. \end{cases}$$

The factors $1 - p^{-4s}$ appearing in the denominator lead to a singularity at $s = 0$ and there does not appear to be an obvious way around this in general (multiplying the Eisenstein series $E_k^*(\tau,s)$ by the factors

$(1-p^{-4s})$, for example, does not seem to lead to interesting results at $s=0$). However, in dimension $e=1$, where $q(x)=-mx^2$ for some $m\in\mathbb{N}$, this is canceled by the numerator and the L-series factor in this case is

$$\tilde{L}(n,\gamma,1+2s)=\begin{cases}\frac{L(2s,\chi_{\mathcal{D}})}{\zeta(4s)}\prod_{\mathrm{bad}\,p}\frac{L_p(n,\gamma,2s)}{1+p^{-2s}}: & n\neq 0;\\ \frac{\zeta(4s-1)}{\zeta(4s)}\prod_{\mathrm{bad}\,p}\frac{(1-p^{1-4s})L_p(n,\gamma,2s)}{1+p^{-2s}}: & n=0.\end{cases}$$

Here, $\mathcal{D}$ is the discriminant

$$\mathcal{D}=2d_\gamma^2 n|\Lambda'/\Lambda|\prod_{\mathrm{bad}\,p}p^2=4mnd_\gamma^2\prod_{\mathrm{bad}\,p}p^2.$$

Write $\mathcal{D}=\mathcal{D}_0 f^2$ where $\mathcal{D}_0\equiv 0,1\,(4)$ is the fundamental discriminant. Then

$$L(2s,\chi_{\mathcal{D}})=L(2s,\chi_{\mathcal{D}_0})\prod_{p|f}(1-p^{-2s})$$

has a zero at $s=0$ of order

$$\#\{\text{prime factors of } f\}+\begin{cases}1: & n>0,\,\mathcal{D}_0\neq\square;\\ 0: & \text{otherwise.}\end{cases}$$

The local L-functions

$$L_p(n,\gamma,s)=\sum_{\nu=0}^{\infty}p^{-\nu s}\#\Big\{x\in\mathbb{Z}/p^\nu\mathbb{Z}:\,-m(x-\gamma)^2+n\equiv 0\Big\}$$

can be calculated by elementary means (for example, with Hensel's lemma). When $n=0$, it turns out that L_p is holomorphic in $s=0$ for all p (and therefore $\tilde{L}(1)=0$.) Suppose $n\neq 0$. When $p|n$ but $p\nmid f$, one can show that

$$L_p(n,\gamma,s)=1+p^{-s},$$

and when $p|m$ but $p\nmid f$, one can show that $L_p(n,\gamma,s)=1$. Since $L_p(n,\gamma,s)$ can have at worst a simple pole in $s=0$ for primes $p|f$, it follows that $\tilde{L}(1+2s)$ is holomorphic in $s=0$ and $\tilde{L}(1)=0$ unless $\mathcal{D}_0$, or equivalently mn, is a square.

Suppose that mn is a square and $p|mn$. Write $\gamma=\frac{a}{2m}$. For large enough ν, the number of solutions to

$$-m(x-a/2m)^2+n=-mx^2+ax+\tilde{n}\equiv 0\,(p^\nu),$$

where $\tilde{n}=n-\frac{a^2}{4m}$, is

$$\begin{cases}p^{v_p(\sqrt{mn})}: & v_p(a)\leq v_p(\tilde{n}),\ p\neq 2;\\ 4\cdot p^{v_p(\sqrt{mn})}: & v_p(a)\leq v_p(\tilde{n}),\ p=2;\\ 0: & v_p(a)>v_p(\tilde{n});\end{cases}\tag{4}$$

if $v_p(m)>v_p(n)$ and

$$\begin{cases}2\cdot p^{v_p(\sqrt{mn})}: & p\neq 2;\\ 4\cdot p^{v_p(\sqrt{mn})}: & p=2;\end{cases}\tag{5}$$

if $v_p(m)\leq v_p(n)$. Here $v_p(n)$ is the valuation of n at p, i.e. $n=\pm\prod_p p^{v_p(n)}$. Note that the result in (4) and (5) is also the limit $\lim_{s\to 0}(1-p^{-2s})L_p(2s)$.

It follows that

$$\begin{aligned}\lim_{s\to 0}\frac{L(2s,\chi_{\mathcal{D}})}{\zeta(4s)}\prod_{\mathrm{bad}\,p}\frac{L_p(n,\gamma,2s)}{1+p^{-2s}}&=\lim_{s\to 0}\Big[\frac{\zeta(2s)}{\zeta(4s)}\Big(\prod_{p|f}\frac{1-p^{-2s}}{1+p^{-2s}}L_p(n,\gamma,2s)\Big)\Big]\\ &=2\sqrt{mn}\cdot\Big(\frac{1}{2}\Big)^{\varepsilon},\end{aligned}$$

where

$$\varepsilon = \#\{\text{primes}\, p \neq 2\, \text{dividing}\, d_\gamma\} + \begin{cases} 1: & 4|d_\gamma; \\ 0: & \text{otherwise.} \end{cases}$$

We conclude that the coefficient $c(n,\gamma)$ of $q^n \mathfrak{e}_\gamma$ in $E_{1/2}$ is

$$\begin{aligned} c(n,\gamma) &= \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{2m}}(-2\pi i)^{1/2} n^{-1/2} \frac{1}{\Gamma(1/2)} \cdot \lim_{s\to 0} \frac{L(2s,\chi_\mathcal{D})}{\zeta(4s)} \prod_{\text{bad}\, p} \frac{L_p(n,\gamma,2s)}{1+p^{-2s}} \\ &= 2 \cdot \Big(\frac{1}{2}\Big)^\varepsilon. \end{aligned}$$

In particular, $E_{1/2}$ is the theta function ϑ for the quadratic form $q(x) = mx^2$ if and only if m is a prime power. In all other cases, $E_{1/2} - \vartheta$ is a nontrivial cusp form.

6. Weight 1

In weight 1, the L-series term is always holomorphic at $s = 0$. However, the zero-value

$$I(1,y,0,0) = -i\pi$$

being nonzero means that E_k still needs a correction term. Setting $s = 0$ in the real-analytic Eisenstein series gives

$$E_1^*(\tau,0) = E_1(\tau) - \pi \frac{(-1)^{(2+b^- - b^+)/4}}{\sqrt{|\Lambda'/\Lambda|}} \frac{L(0,\chi_D)}{L(1,\chi_D)} \sum_{\substack{\gamma\in\Lambda'/\Lambda \\ q(\gamma)\in\mathbb{Z}}} \Big[\prod_{\text{bad}\, p} \lim_{s\to 0}(1-p^{e/2-1-2s}) L_p(0,\gamma,e/2+2s)\Big]\mathfrak{e}_\gamma,$$

where D is the discriminant $D = -4|\Lambda'/\Lambda|$ and the bad primes are the primes dividing D. In particular, E_1 may differ from the true modular form $E_1^*(\tau,0)$ by a constant. (Of course, $E_1^*(\tau,0)$ may be identically zero.)

For two-dimensional negative-definite lattices, the corrected Eisenstein series $E_1^*(\tau,0)$ is often a multiple of the theta series. This leads to identities relating representation numbers of quadratic forms and divisor counts. Of course, such identities are well-known from the theory of modular forms of higher level. The vector-valued proofs tend to be shorter since $M_k(\rho^*)$ is generally much smaller than the space of modular forms of higher-level in which the individual components lie, so there is less algebra (although computing the local factors takes some work).

Example 12. Consider the quadratic form $q(x,y) = -x^2 - xy - y^2$, with $|\Lambda'/\Lambda| = 3$. The L-function values are

$$L(0,\chi_{-12}) = \frac{2}{3}, \;\; L(1,\chi_{-12}) = \frac{\pi\sqrt{3}}{6}$$

and the local L-series are

$$L_2(0,0,s) = \frac{1+2^{-s}}{1-2^{2-2s}}, \;\; L_3(0,0,s) = \frac{1}{1-3^{1-s}}$$

with

$$\lim_{s\to 0}(1-2^{-2s})L_2(0,0,1+2s) = \frac{3}{4}, \;\; \lim_{s\to 0}(1-3^{-2s})L_3(0,0,1+2s) = 1,$$

and therefore $E_1^*(\tau,0) = E_1(\tau) + \mathfrak{e}_0$. Since $M_1(\rho^*)$ is one-dimensional, comparing constant terms shows that

$$E_1(\tau) + \mathfrak{e}_0 = 2\vartheta.$$

By remark 11, the coefficient $c(n,0)$ of $q^n \mathfrak{e}_0$ in E_1 is

$$c(n,0) = \frac{2\pi}{L(1,\chi_{-12})\cdot\sqrt{3}} \cdot \sigma_0(n,\chi_{-12}) \cdot \underbrace{\begin{cases} 3/2: & v_2(n)\,\text{even}; \\ 0: & v_2(n)\,\text{odd}; \end{cases}}_{\text{local factor at 2}} \cdot \underbrace{\begin{cases} 2: & n \neq (3a+2)3^b \text{ for any } a,b \in \mathbb{N}_0; \\ 0: & n = (3a+2)3^b \text{ for some } a,b\in\mathbb{N}_0; \end{cases}}_{\text{local factor at 3}}$$

$$= 12\Big[\sum_{d|n} \Big(\frac{-12}{d}\Big)\Big] \cdot \begin{cases} 1: & n \neq (3a+2)3^b; \\ 0: & n = (3a+2)3^b. \end{cases}$$

This implies the identity

$$\#\{(a,b)\in\mathbb{Z}^2 :\, a^2+ab+b^2 = n\} = 6\varepsilon \cdot \Big(\#\{\text{divisors}\, d = 6\ell+1 \,\text{of}\, n\} - \#\{\text{divisors}\, d = 6\ell-1\,\text{of}\, n\}\Big),$$

valid for $n \geq 1$, where $\varepsilon = 1$ unless n has the form $(3a+2)3^b$ for $a,b\in\mathbb{N}_0$, in which case $\varepsilon = 0$.

Example 13. Consider the quadratic form $q(x,y) = -x^2-y^2$, with $|\Lambda'/\Lambda| = 4$ and $\chi_{-16} = \chi_{-4}$. The L-function values are

$$L(0,\chi_{-4}) = \frac{1}{2},\;\; L(1,\chi_{-4}) = \frac{\pi}{4},$$

and the only bad prime is 2 with $L_2(0,0,s) = \frac{1}{1-2^{1-s}}$ and therefore

$$\lim_{s\to 0}(1-2^{e/2-1-2s})L_2(0,0,e/2+2s) = 1.$$

Therefore,

$$E_1^*(\tau,0) = E_1(\tau) + \mathfrak{e}_0.$$

Since $M_1(\rho^*)$ is one-dimensional, comparing constant terms gives $E_1(\tau)+\mathfrak{e}_0 = 2\vartheta(\tau)$.

By remark 11, the coefficient $c(n,0)$ of $q^n\mathfrak{e}_0$ in E_1 is

$$c(n,0) = \frac{2\pi}{L(1,\chi_{-4})\cdot 2}\cdot\sigma_0(n,\chi_{-4})\cdot \underbrace{\begin{cases} 2: & \big(\frac{-4}{n}\big) \neq -1; \\ 0: & \big(\frac{-4}{n}\big) = -1; \end{cases}}_{\text{local factor at 2}} = 8\sum_{d|n}\Big(\frac{-4}{d}\Big),$$

and therefore

$$\#\Big\{(a,b)\in\mathbb{Z}^2 :\, a^2+b^2 = n\Big\} = 4\sum_{d|n}\Big(\frac{-4}{d}\Big) = 4\cdot\Big(\#\{\text{divisors}\, d = 4\ell+1\,\text{of}\, n\} - \#\{\text{divisors}\, d = 4\ell+3\,\text{of}\, n\}\Big).$$

Remark 14. Similarly to the case of scalar-valued Eisenstein series, relations between vector-valued Eisenstein series lead to identities of twisted divisor sums that are difficult to prove by other means. For the quadratic form $q(x,y) = -x^2-y^2$, the coefficient $c_k(n,0)$ of $q^n\mathfrak{e}_{0,0}$, $n>0$ in the Eisenstein series of weight k is

$$c_k(n,0) = \frac{(2\pi)^k(-1)^{(k-1)/2}}{2\cdot(k-1)!L(k,\chi_{-4})}\tilde{\sigma}_{k-1}(n),$$

where

$$\tilde{\sigma}_{k-1}(n) = \Big[\sum_{d|n}\chi_{-4}(n/d)d^{k-1}\Big]\cdot\Big[1+2^{-(a+2)(k-1)}\Big(\frac{-4}{m}\Big)\Big]$$

if $n = 2^a m$ with odd m. By comparing the first coefficients we see that E_3 is (-12) times the Serre derivative of $\vartheta = \frac{E_1+1}{2}$, which implies an identity of the form

$$32\tilde{\sigma}_2(n) = (12n-1)\tilde{\sigma}_0(n) + 12\sigma_1(n) + 24\sum_{a=1}^{n-1}\sigma_1(a)\tilde{\sigma}_0(n-a),\;\; n\geq 1.$$

7. Weight 3/2

In weight 3/2, the L-series term is

$$\tilde{L}(n,\gamma,3/2+e/2+2s)=\begin{cases}\frac{L(1+2s,\chi_{\mathcal{D}})}{\zeta(4s+2)}\prod_{\text{bad }p}\frac{1-p^{(e-3)/2-2s}}{1-p^{-2-4s}}L_p(n,\gamma,1/2+e/2+2s): & n\neq 0;\\ \\ \frac{\zeta(4s+1)}{\zeta(4s+2)}\prod_{\text{bad }p}\frac{(1-p^{(e-3)/2-2s})(1-p^{-1-4s})}{1-p^{-2-4s}}L_p(n,\gamma,1/2+e/2+2s): & n=0;\end{cases}$$

and it is holomorphic in $s=0$ unless $n=0$ or

$$\mathcal{D}=-2nd_\gamma^2|\Lambda'/\Lambda|\prod_{\text{bad }p}p^2$$

is a square. In these cases, $\tilde{L}(n,\gamma,3/2+e/2+2s)$ has a simple pole with residue

$$\mathrm{Res}\Big(\tilde{L}(n,\gamma,3/2+e/2+2s),s=0\Big)=\frac{3}{\pi^2}\prod_{\text{bad }p}\lim_{s\to 0}\frac{(1-p^{e/2-3/2-2s})(1-p^{-1})}{1-p^{-2}}L_p(n,\gamma,1/2+e/2+2s)$$

if $n\neq 0$, and

$$\mathrm{Res}\Big(\tilde{L}(n,\gamma,3/2+e/2+2s),s=0\Big)=\frac{3}{2\pi^2}\prod_{\text{bad }p}\lim_{s\to 0}\frac{(1-p^{e/2-3/2-2s})(1-p^{-1})}{1-p^{-2}}L_p(n,\gamma,1/2+e/2+2s)$$

if $n=0$.

This pole cancels with the zero of $I(k,y,n,s)$ at $s=0$, whose derivative there is

$$\frac{d}{ds}\Big|_{s=0}I(k,y,n,s)=-16\pi^2(1+i)y^{-1/2}\beta(4\pi|n|y),\quad\text{where }\beta(x)=\frac{1}{16\pi}\int_1^\infty u^{-3/2}e^{-xu}\,du,$$

as calculated in [8], section 2.2. This expression is also valid for $n=0$, where it reduces to

$$\frac{d}{ds}\Big|_{s=0}I(k,y,0,s)=2\pi(-i)^{3/2}\frac{d}{ds}\Big|_{s=0}(2y)^{-1/2-2s}\frac{\Gamma(2s+1/2)}{\Gamma(s)\Gamma(s+3/2)}=-\frac{2\pi}{\sqrt{y}}(1+i).$$

Therefore, $E^*_{3/2}(\tau,0)$ is a harmonic weak Maass form that is not generally holomorphic:

$$E^*_{3/2}(\tau,0)=E_{3/2}(\tau)+\frac{3(-1)^{(3+b^+-b^-)/4}\sqrt{2}}{\pi\sqrt{y|\Lambda'/\Lambda|}}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ q(\gamma)\in\mathbb{Z}}}\prod_{p|\#(\Lambda'/\Lambda)}\frac{1-p^{(e-3)/2}}{1+p^{-1}}L_p(0,\gamma,1/2+e/2)\mathfrak{e}_\gamma+$$

$$+\frac{48(-1)^{(3+b^+-b^-)/4}\sqrt{2}}{\sqrt{y|\Lambda'/\Lambda|}}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ n\in\mathbb{Z}-q(\gamma)\\ -2n|\Lambda'/\Lambda|=\square}}\Big[\beta(4\pi|n|y)\prod_{\text{bad }p}\frac{1-p^{(e-3)/2}}{1+p^{-1}}L_p(n,\gamma,1/2+e/2)\Big]q^n\mathfrak{e}_\gamma,$$

where $-2n|\Lambda'/\Lambda|=\square$ means that $-2n|\Lambda'/\Lambda|$ should be a rational square. (In particular, the real-analytic correction involves only exponents $n\leq 0$.)

Example 15. Zagier's Eisenstein series [8] occurs as the Eisenstein series for the quadratic form $q(x)=x^2$. The underlying harmonic weak Maass form is

$$E^*_{3/2}(\tau,0)=E_{3/2}(\tau)-\frac{3}{\pi\sqrt{y}}\mathfrak{e}_0-\frac{48}{\sqrt{y}}\sum_{\gamma\in\Lambda'/\Lambda}\sum_{\substack{n\in\mathbb{Z}-q(\gamma)\\ -n=\square}}\beta(4\pi|n|y)\underbrace{\prod_{\text{bad }p}\frac{1-p^{-1}}{1+p^{-1}}L_p(n,\gamma,1)}_{=1}q^n\mathfrak{e}_\gamma$$

$$=E_{3/2}(\tau)-\frac{24}{\sqrt{y}}\sum_{n=-\infty}^{\infty}\beta(4\pi(n/2)^2y)q^{-(n/2)^2}\mathfrak{e}_{n/2}.$$

The coefficient of $q^{n/4}$ in

$$E_{3/2}(\tau)=\Big(1-6q-12q^2-16q^3-...\Big)\mathfrak{e}_0+\Big(-4q^{3/4}-12q^{7/4}-12q^{11/4}-...\Big)\mathfrak{e}_{1/2}$$

is -12 times the Hurwitz class number $H(n)$. (We obtain Zagier's Eisenstein series in its usual form by summing the components, replacing τ by 4τ and y by $4y$, and dividing by -12.)

Example 16. For certain lattices, the correction term above may vanish entirely. This is the case for the quadratic form $q(x,y,z) = -x^2 - y^2 - z^2$. Here the local L-series at $p = 2$, i.e. $L_2(n,\gamma,2+2s)$, is holomorphic at $s = 0$ whenever n is such that $\mathcal{D}$ is a square, and therefore $\tilde{L}(n,\gamma,3/2+e/2)$ is annihilated by the factor $(1-2^{-2s})$ at $s = 0$. In fact, the Eisenstein series $E_{3/2}$ is exactly the theta series for q. (This is easier to see by computing $E_{3/2}$ directly.) See also remark 18 below.

Remark 17. We can use essentially the same argument as Hirzebruch and Zagier [8] to derive the transformation law of the general $E_{3/2}$. Write $E^*_{3/2}(\tau,0)$ in the form

$$E^*_{3/2}(\tau,0) = E_{3/2} + \frac{1}{\sqrt{y}} \sum_{\gamma \in \Lambda'/\Lambda} \sum_{\substack{n \in \mathbb{Z}-q(\gamma) \\ n \le 0}} a(n,\gamma)\beta(-4\pi n y) q^n \mathfrak{e}_\gamma$$

with coefficients $a(n,\gamma)$. Applying the ξ-operator $\xi = y^{3/2}\overline{\frac{\partial}{\partial \overline{\tau}}}$ of [3] to $E^*_{3/2}(\tau,0)$ and using

$$\frac{d}{dy}\Big[\frac{1}{\sqrt{y}}\beta(y)\Big] = \frac{1}{16\pi}\frac{d}{dy}\Big[\int_y^\infty v^{-3/2}e^{-v}\,\mathrm{d}v\Big] = -\frac{1}{16\pi}y^{-3/2}e^{-y}$$

shows that the "shadow"

$$\vartheta(\tau) = \sum_{\gamma,n} a(n,\gamma) q^{-n}\mathfrak{e}_\gamma$$

is a modular form of weight $1/2$ for the representation ρ (not its dual!), and

$$\begin{aligned}
E^*_{3/2}(\tau,0) - E_{3/2}(\tau) &= y^{-1/2} \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z}-q(\gamma)} a(n,\gamma)\beta(-4\pi n y) q^n \mathfrak{e}_\gamma \\
&= \frac{1}{16\pi} y^{-1/2} \int_1^\infty \sum_{\gamma,n} u^{-3/2} e^{-4\pi n u y} q^n \mathfrak{e}_\gamma \,\mathrm{d}u \\
&= \frac{1}{16\pi} y^{-1/2} \int_1^\infty u^{-3/2}\vartheta(2iuy - \tau)\,\mathrm{d}u \\
&= \frac{\sqrt{2i}}{16\pi} \int_{-x+iy}^{i\infty} (v+\tau)^{-3/2}\vartheta(v)\,\mathrm{d}v, \quad v = 2iuy - \tau.
\end{aligned}$$

For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$, defining $\tilde{M} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ as in remark 6 and substituting $v = \tilde{M}\cdot t$ gives

$$\begin{aligned}
E^*_{3/2}(M\cdot\tau,0) - E_{3/2}(M\cdot\tau) &= \frac{\sqrt{2i}}{16\pi}\int_{-\overline{M\cdot\tau}}^{i\infty} \Big(\frac{a\tau+b}{c\tau+d} + v\Big)^{-3/2}\vartheta(v)\,\mathrm{d}v \\
&= \frac{\sqrt{2i}}{16\pi}\int_{-\overline{\tau}}^{d/c} \Big(\frac{a\tau+b}{c\tau+d} + \frac{at-b}{-ct+d}\Big)^{-3/2}\vartheta(\tilde{M}\cdot t)\,\frac{\mathrm{d}t}{(ct-d)^2} \\
&= \frac{\sqrt{2i}}{16\pi}(c\tau+d)^{3/2}\int_{-\overline{\tau}}^{d/c} (\tau+t)^{-3/2}\rho(\tilde{M})\vartheta(t)\,\mathrm{d}t \\
&= (c\tau+d)^{3/2}\rho^*(M)\Big[\frac{\sqrt{2i}}{16\pi}\int_{-\overline{\tau}}^{d/c}(\tau+t)^{-3/2}\vartheta(t)\,\mathrm{d}t\Big].
\end{aligned}$$

Since $E^*_{3/2}(M\cdot\tau,0) = (c\tau+d)^{3/2}\rho^*(M)E^*_{3/2}(\tau,0)$, we conclude that

$$E_{3/2}(M\cdot\tau) = (c\tau+d)^{3/2}\rho^*(M)\Big[E_{3/2}(\tau) + \frac{\sqrt{2i}}{16\pi}\int_{d/c}^{i\infty}(\tau+t)^{-3/2}\vartheta(t)\,\mathrm{d}t\Big]. \tag{6}$$

Remark 18. The transformation law (6) can be used to give an easier sufficient condition for when $E_{3/2}$ is actually a modular form. For example, one can show that $M_{1/2}(\rho) = 0$ for the quadratic form $q(x,y,z) = -x^2 - y^2 - z^2$, which implies that the series ϑ defined above must be identically 0 and therefore

$$E_{3/2}(M \cdot \tau) = (c\tau + d)^{3/2} \rho^*(M) E_{3/2}(\tau),$$

so $E_{3/2}$ is a true modular form. It must be the theta series because $M_{3/2}(\rho^*)$ is one-dimensional. Even when $M_{1/2}(\rho) \neq 0$, we can identify ϑ in $M_{1/2}(\rho)$ by computing finitely many coefficients.

Example 19. Let q be the quadratic form $q(x,y,z) = x^2 + y^2 - z^2$. Using the program PSS [11], we can verify that $M_{1/2}(\rho)$ (which is $M_{1/2}(\rho^*)$ for $-q$) is two-dimensional, spanned by

$$\vartheta_1(\tau) = \Big(1 + 2q + 2q^4 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/2,0,1/2)}) + \Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(0,1/2,0)} + \mathfrak{e}_{(1/2,1/2,1/2)}),$$

$$\vartheta_2(\tau) = \Big(1 + 2q + 2q^4 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(0,1/2,1/2)}) + \Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(1/2,0,0)} + \mathfrak{e}_{(1/2,1/2,1/2)}).$$

(The methods of [11] do not apply in weight 1/2 but we can multiply by the discriminant Δ and do all computations in weight 25/2.) The local L-series at the bad prime $p = 2$ for the constant term $n = 0$ are

$$(1 - 2^{-2s}) L_p(0, 0, 2 + 2s) = \frac{1}{1 - 2^{-1-2s}} \text{ and } (1 - 2^{-2s}) L_p(0, \gamma, 2 + 2s) = 1$$

for $\gamma \in \{(1/2, 0, 1/2), (0, 1/2, 1/2)\}$, which implies that

$$E^*_{3/2}(\tau, 0) = E_{3/2}(\tau) - \frac{3}{2\pi\sqrt{y}} \Big(\frac{4}{3}\mathfrak{e}_{(0,0,0)} + \frac{2}{3}\mathfrak{e}_{(1/2,0,1/2)} + \frac{2}{3}\mathfrak{e}_{(0,1/2,1/2)}\Big) + \Big(\text{negative powers of } q\Big)$$

and therefore the shadow in equation (6) must be

$$\vartheta(\tau) = -8\Big(\vartheta_1(\tau) + \vartheta_2(\tau)\Big).$$

In particular, the $\mathfrak{e}_0$-component $E_{3/2}(\tau)_0$ of $E_{3/2}(\tau)$ is a mock modular form of level 4 that transforms under $\Gamma(4)$ by

$$E_{3/2}(M \cdot \tau)_0 = (c\tau + d)^{3/2} \Big[E_{3/2}(\tau)_0 - \frac{\sqrt{2i}}{\pi} \int_{d/c}^{i\infty} (\tau + t)^{-3/2} \Theta(t) \, \mathrm{d}t \Big], \;\; \Theta(t) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2 t).$$

It was shown by Bringmann and Lovejoy [1] that the series

$$\overline{\mathcal{M}}(\tau + 1/2) = 1 - \sum_{n=1}^{\infty} |\overline{\alpha}(n)| q^n = 1 - 2q - 4q^2 - 8q^3 - 10q^4 - ...$$

of example 4, where $|\overline{\alpha}(n)|$ counts overpartition rank differences of n, has the same transformation behavior under the group $\Gamma_0(16)$, which implies that the difference between $\overline{\mathcal{M}}(\tau + 1/2)$ and the $\mathfrak{e}_0$-component of $E_{3/2}$ is a true modular form of level 16. We can verify that these are the same by comparing all Fourier coefficients up to the Sturm bound.

8. Weight 2

In weight $k = 2$, the L-series term is

$$\tilde{L}(n, \gamma, 2 + e/2 + 2s) = \begin{cases} \frac{1}{L(1+2s, \chi_D)} \prod_{\text{bad } p} (1 - p^{e/2 - 2 - 2s}) L_p(n, \gamma, 1 + e/2 + 2s) : & n \neq 0; \\ \frac{L(1+2s,\chi_D)}{L(2+2s,\chi_D)} \prod_{\text{bad } p} (1 - p^{e/2-2-2s}) L_p(n, \gamma, 1 + e/2 + 2s) : & n = 0. \end{cases}$$

Since $L(1, \chi)$ is never zero for any Dirichlet character, the only way a pole can occur at $s = 0$ is if $n = 0$ and $D = |\Lambda'/\Lambda|$ is square. (In particular, when $|\Lambda'/\Lambda|$ is not square, E_2 is a modular form.)

Assume that $|\Lambda'/\Lambda|$ is square. Then

$$L(1 + 2s, \chi_D) = \zeta(1 + 2s) \prod_{\text{bad } p} (1 - p^{-1-2s}),$$

and therefore $\tilde{L}(0,\gamma,2+e/2+2s)$, has a simple pole at $s=0$ with residue

$$\begin{aligned}\mathrm{Res}\Big(\tilde{L}(0,\gamma,2+e/2+2s),s=0\Big) &= \frac{1}{2L(2,\chi_D)}\prod_{\text{bad } p}\Big[(1-p^{-1})\lim_{s\to 0}(1-p^{e/2-2-2s})L_p(0,\gamma,1+e/2+2s)\Big] \\ &= \frac{3}{\pi^2}\lim_{s\to 0}\prod_{\text{bad } p}\frac{1-p^{e/2-2-2s}}{1+p^{-1}}L_p(0,\gamma,1+e/2+2s)\end{aligned}$$

for any $\gamma\in\Lambda'/\Lambda$ with $q(\gamma)\in\mathbb{Z}$. This pole is canceled by the zero of $I(2,y,0,s)$ at $s=0$ which has derivative

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}I(2,y,0,s) &= -2\pi(2y)^{-1}\frac{d}{ds}\Big|_{s=0}(2y)^{-2s}\frac{\Gamma(2s+1)}{\Gamma(s)\Gamma(s+2)} \\ &= -\frac{\pi}{y},\end{aligned}$$

so

$$E_2^*(\tau,0)=E_2(\tau)-\frac{3}{\pi y\sqrt{|\Lambda'/\Lambda|}}\lim_{s\to 0}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ q(\gamma)\in\mathbb{Z}}}\prod_{\text{bad } p}\frac{1-p^{e/2-2-2s}}{1+p^{-1}}L_p(0,\gamma,1+e/2+2s)\mathfrak{e}_\gamma. \tag{7}$$

Example 20. Let Λ be a unimodular lattice. The only bad prime is $p=2$. Using the hyperbolic plane $q(x,y)=xy$ to define Λ, the local L-function is

$$L_2(0,0,s)=\frac{1-2^{-s}}{(1-2^{1-s})^2}$$

with $L_2(0,0,2)=3$, so we obtain the well-known result

$$E_2^*(\tau,0)=E_2(\tau)-\frac{3}{\pi y}\cdot\frac{1-1/2}{1+1/2}L_2(0,0,2)=E_2(\tau)-\frac{3}{\pi y}.$$

Remark 21. We can summarize the above by saying that

$$E_2^*(\tau,0)=E_2(\tau)-\frac{1}{y}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ q(\gamma)\in\mathbb{Z}}}A(\gamma)\mathfrak{e}_\gamma$$

is a Maass form for some constants $A(\gamma)$. For $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in Mp_2(\mathbb{Z})$, since

$$E_2^*(M\cdot\tau,0)=(c\tau+d)^2\rho^*(M)E_2^*(\tau,0),$$

we find the transformation law

$$\begin{aligned}E_2(M\cdot\tau) &= E_2^*(M\cdot\tau,0)+\frac{|c\tau+d|^2}{y}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ q(\gamma)\in\mathbb{Z}}}A(\gamma)\mathfrak{e}_\gamma \\ &= (c\tau+d)^2\Big[\rho^*(M)E_2(\tau)-2ic(c\tau+d)\sum_{q(\gamma)\in\mathbb{Z}}A(\gamma)\rho^*(M)\mathfrak{e}_\gamma\Big].\end{aligned}$$

Example 22. The weight-2 Eisenstein series for the quadratic form $q(x,y)=x^2+3xy+y^2$ is a true modular form because the discriminant 5 of q is not a square. In particular, the $\mathfrak{e}_0$-component

$$1-30q-20q^2-40q^3-90q^4-130q^5-60q^6-120q^7-100q^8-210q^9-...$$

is a modular form of weight 2 for the congruence subgroup $\Gamma_1(5)$. Using remark 11, we see that the coefficient $c(n)$ of q^n for n coprime to 10 is

$$c(n)=\begin{cases}-30\sum_{d|n}\left(\frac{5}{n/d}\right)d: & n\equiv\pm 1 \bmod 10;\\ -20\sum_{d|n}\left(\frac{5}{n/d}\right)d: & n\equiv\pm 3 \bmod 10;\end{cases}$$

with a more complicated expression for other n involving the local factors at 2 and 5.

Example 23. The weight-2 Eisenstein series for the quadratic form $q(x,y) = 2xy$ is

$$\begin{aligned} E_2(\tau) &= \Big(1 - 8q - 40q^2 - 32q^3 - 104q^4 - ...\Big)\mathfrak{e}_{(0,0)} \\ &\quad + \Big(-16q - 32q^2 - 64q^3 - 64q^4 - 96q^5 - ...\Big)(\mathfrak{e}_{(0,1/2)} + \mathfrak{e}_{(1/2,0)}) \\ &\quad + \Big(-8q^{1/2} - 32q^{3/2} - 48q^{5/2} - 64q^{7/2} - 104q^{9/2} - ...\Big)\mathfrak{e}_{(1/2,1/2)} \\ &= \Big(1 - 8\sum_{n=1}^\infty \Big[\sum_{d|2n}(-1)^d d\Big]q^n\Big)\mathfrak{e}_{(0,0)} \\ &\quad + \Big(-8\sum_{n=1}^\infty \Big[\sum_{d|n}(1-(-1)^{n/d})d\Big]q^n\Big)(\mathfrak{e}_{(0,1/2)} + \mathfrak{e}_{(1/2,0)}) \\ &\quad + \Big(-8\sum_{n=0}^\infty \sigma_1(2n+1)q^{n+1/2}\Big)\mathfrak{e}_{(1/2,1/2)}. \end{aligned}$$

It is not a modular form. (In fact, there are no modular forms of weight 2 for this lattice.) On the other hand, the real-analytic correction (7) only involves the components $\mathfrak{e}_\gamma$ for which $q(\gamma) \in \mathbb{Z}$, i.e. $\mathfrak{e}_{(0,0)}, \mathfrak{e}_{(0,1/2)}, \mathfrak{e}_{(1/2,0)}$, so the components

$$1 - 8\sum_{n=1}^\infty \Big[\sum_{d|2n}(-1)^d d\Big]q^n, \quad \sum_{n=1}^\infty \Big[\sum_{d|n}(1-(-1)^{n/d})d\Big]q^n$$

are only quasimodular forms of level 4, while $\sum_{n=0}^\infty \sigma_1(2n+1)q^{2n+1}$ is a true modular form.

Example 24. Although the discriminant group of the quadratic form $q(x_1,x_2,x_3,x_4) = -x_1^2 - x_2^2 - x_3^2 - x_4^2$ has square order 16, the correction term still vanishes in this case. This is because the local L-functions for $p = 2$,

$$L_2(0,\gamma,3+s) = \begin{cases} \frac{2+2^{-s}}{2-2^{-s}} : & \gamma = 0; \\ 1 : & \gamma = (1/2,1/2,1/2,1/2); \end{cases}$$

are both holomorphic at $s = 0$ and therefore annihilated by the term $(1 - p^{4/2-2-2s})$ at $s = 0$. (Another way to see this is that $\sum_{\substack{\gamma\in\Lambda'/\Lambda \\ q(\gamma)\in\mathbb{Z}}} A(\gamma)\mathfrak{e}_\gamma$ is invariant under ρ due to the transformation law of E_2, but there are no nonzero invariants of ρ in this case.) In fact, the Eisenstein series E_2 for this lattice is exactly the theta series as one can see by calculating the first few coefficients. Comparing coefficients of the $\mathfrak{e}_0$-component leads immediately to Jacobi's formula:

$$\begin{aligned} \#\{(a,b,c,d) \in \mathbb{Z}^4 : a^2+b^2+c^2+d^2 = n\} &= \frac{(2\pi)^2}{L(2,\chi_{64})\cdot 4}\cdot \sigma_1(n,\chi_{64})\cdot L_2(n,0,3) \\ &= 8\cdot\Big[\sum_{d|n}\Big(\frac{4}{n/d}\Big)d\Big]\cdot\begin{cases} 1 : & n\,\text{odd}; \\ 3\cdot 2^{-v_2(n)} : & n\,\text{even}; \end{cases} \\ &= \begin{cases} 8\sum_{d|n} d : & n\,\text{odd}; \\ 24\sum_{\substack{d|n \\ d\,\text{odd}}} d : & n\,\text{even}; \end{cases} \end{aligned}$$

for all $n \in \mathbb{N}$.

References

[1] Kathrin Bringmann and Jeremy Lovejoy. Overpartitions and class numbers of binary quadratic forms. *Proc. Natl. Acad. Sci. USA*, 106(14):5513–5516, 2009. ISSN 1091-6490. doi: 10.1073/pnas.0900783106. URL `http://dx.doi.org/10.1073/pnas.0900783106`.

[2] Jan Bruinier. *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, volume 1780 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. ISBN 3-540-43320-1. doi: 10.1007/b83278. URL http://dx.doi.org/10.1007/b83278.

[3] Jan Bruinier and Jens Funke. On two geometric theta lifts. *Duke Math. J.*, 125(1):45–90, 2004. ISSN 0012-7094. doi: 10.1215/S0012-7094-04-12513-8. URL http://dx.doi.org/10.1215/S0012-7094-04-12513-8.

[4] Jan Bruinier and Michael Kuss. Eisenstein series attached to lattices and modular forms on orthogonal groups. *Manuscripta Math.*, 106(4):443–459, 2001. ISSN 0025-2611. doi: 10.1007/s229-001-8027-1. URL http://dx.doi.org/10.1007/s229-001-8027-1.

[5] Jan Bruinier and Ken Ono. Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms. *Adv. Math.*, 246:198–219, 2013. ISSN 0001-8708. doi: 10.1016/j.aim.2013.05.028. URL https://dx.doi.org/10.1016/j.aim.2013.05.028.

[6] Raemeon Cowan, Daniel Katz, and Lauren White. A new generating function for calculating the Igusa local zeta function. *Adv. Math.*, 304:355–420, 2017. ISSN 0001-8708. doi: 10.1016/j.aim.2016.09.003. URL http://dx.doi.org/10.1016/j.aim.2016.09.003.

[7] Benedict Gross and Don Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986. ISSN 0020-9910. doi: 10.1007/BF01388809. URL http://dx.doi.org/10.1007/BF01388809.

[8] Friedrich Hirzebruch and Don Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36:57–113, 1976. ISSN 0020-9910. doi: 10.1007/BF01390005. URL http://dx.doi.org/10.1007/BF01390005.

[9] Ken Ono. Unearthing the visions of a master: harmonic Maass forms and number theory. In *Current developments in mathematics, 2008*, pages 347–454. Int. Press, Somerville, MA, 2009.

[10] Lawrence Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997. ISBN 0-387-94762-0. doi: 10.1007/978-1-4612-1934-7. URL http://dx.doi.org/10.1007/978-1-4612-1934-7.

[11] Brandon Williams. Poincaré square series for the Weil representation. Preprint. URL https://arxiv.org/abs/1704.06758.

Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: btw@math.berkeley.edu