Divisors and the Riemann-Roch theorem

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Math 274 final paper

This is an expository paper submitted for Bernd Sturmfels' Math 274 course in Spring 2016. It is mostly based on the exposition in [5].

Recall the Riemann-Roch theorem for Riemann surfaces:

Theorem 1. *Let X be a Riemann surface of genus g with canonical divisor K, and let D be any divisor on X. Then*

$$\ell(D) - \ell(K - D) = \deg(D) - (g - 1).$$

Here, a divisor is a formal sum of points of X. K is the divisor of any meromorphic one-form (since X is a complex curve, all meromorphic one-forms are equal up to multiplying by a function); i.e. $K = \sum_{p \in X} n_p(p)$, where n_p is the order of the one-form in p. The genus g is the dimension of the space of holomorphic one-forms. $\ell(D)$ is the dimension of the space of meromorphic functions where the order of poles does not exceed the value prescribed by D. The degree $\deg(D)$ is the sum of all coefficients of D.

For example, on the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, the meromorphic one-form $\mathrm{d}z$ has a pole of order 2 in ∞, so we can take $K = 2(\infty)$. There are no holomorphic one-forms, so $g = 0$. If D denotes the divisor $(\infty) - (0)$, then $\ell(D)$ is the dimension of the space of holomorphic functions with at most a pole of order 1 in ∞ and a zero in 0; this is one-dimensional, spanned by the function $f(z) = z$..

Divisors on Tropical Curves

Definition 1. A **tropical curve** is a finite graph G together with a metric

$$d : G \times G \to [0, \infty],$$

where the distance between two points of G is allowed to be infinite if and only if one of the points is a leaf.

More precisely, it is a graph G with a topology and with fixed homeomorphisms between each edge and a closed interval $(0, a]$, $a \in [0, \infty]$, such that only leaves of G are allowed to correspond to ∞. The interval $[0, \infty]$ is understood as the one-point compactification of the nonnegative reals $\mathbb{R}_{\geq 0}$.

Any two graph structures on the same space with the same topology are understood to be the same tropical curve; in particular, subdividing edges does not change the tropical curve.

Definition 2. Let G be a tropical curve. A **divisor** on G is a formal $\mathbb{Z}$-linear combination of points. The group of divisors is denoted $\mathrm{Div}(G)$.

For any divisor $D = \sum_{P \in G} n_P(P)$, the sum $\sum_{P \in G} n_P$ is the **degree** $\deg(D)$.

Definition 3. Let G be a tropical curve. A **rational function** on G is a continuous map $f : G \to \mathbb{R} \cup \{\infty\}$ that is piecewise-linear with integer slopes on each edge. The **divisor** of f is

$$(f) = \sum_{P \in G} f_P(P),$$

where f_P is the sum of all slopes of f in directions coming out of P.
Any divisor on G of the form (f) is a **principal divisor**; the **class group**, or **Picard group**, of G is the quotient

$$\mathrm{Cl}(G) = \mathrm{Div}(G)/\mathrm{Prin}(G)$$

of divisors by principal divisors.

By subdividing G, we may assume that rational functions are linear between the vertices of G.

Proposition 4. *Every principal divisor has degree* 0*.*

Proof. This is because the sum on every edge of the graph is zero: if f is a rational function that is linear between points P and Q on an edge of the graph, then the slope exiting P cancels with the slope exiting Q. $\square$

Generalizing this argument slightly proves that every divisor class is represented by a divisor supported on only finite vertices, since the infinite vertices can only occur on leaves. From now on, we will only need to consider true metric graphs.

Definition 5. Let G be a tropical curve. The **canonical divisor** is the divisor

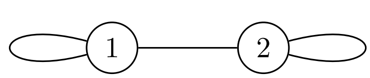
$$K = \sum_{P \in \mathrm{Vert}(G)} K_P(P),$$

where K_P is the number of rays incident to P in G minus 2.

Definition 6. Let G be a tropical curve.
(i) A divisor $D = \sum_{P \in G} n_P(P)$ is **effective** if $n_P \geq 0$ for all P.
(ii) Let D be a divisor. $|D|$ is the set of all effective divisors in the class of D.
(iii) Let D be a divisor. Its **rank** is

$$r(D) = \min\{\deg(E) : \ E \text{ effective with } |D - E| = \emptyset\} - 1.$$

Finding the correct definition of rank is probably the most difficult part of tropicalizing the Riemann-Roch theorem. The direct definition of $\ell(D)$ from the case of Riemann surfaces does not seem to have a natural analogy. Another approach is noticing that, for Riemann surfaces, $|D|$ is in bijection with the projective space $\Big(\Gamma(X, \mathcal{O}(D)) \backslash \{0\}\Big)/\mathbb{C}^\times$ (see [3], Proposition II.7.7, for example), so $\ell(D)$ can be recovered as $\dim |D| + 1$. This also leads to difficulties: in fact, for tropical graphs, $|D|$ is naturally a polyhedral complex but not of a pure dimension, and the naive dimension (dimension of the largest cell) does not make the Riemann-Roch theorem true. For example, in the graph



with canonical divisor $(1)+(2)$, the space $|K|$ consists of a two-dimensional cell (K is linearly equivalent to any divisor $(p)+(q)$ with points p, q on the line from 1 to 2) and two one-dimensional cells (K is only linearly equivalent to a divisor $(p)+(q)$ with p, q on one of the loops if p and q are at an equal distance to their nearest vertex). Since $|0| = \{0\}$ consists of a single point, the natural definition is $r(0) = 0$; then

$$r(K) - r(0) = \deg(K) - (g-1) = 2 - 1 = 1$$

implies that 1 is the correct 'dimension' for $|K|$.

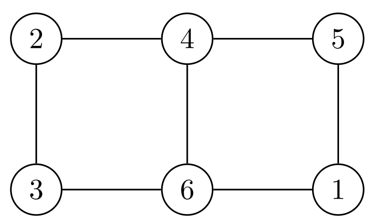
The definition of $r(D)$ presented above is constructed in a way that forces the Riemann-Roch theorem to be true. It also behaves nicely with respect to specialization from curves over the Puiseux series field $\mathbb{C}\{\{t\}\}$ to tropical curves; see [1].

Vertex orders and the proof of Riemann-Roch

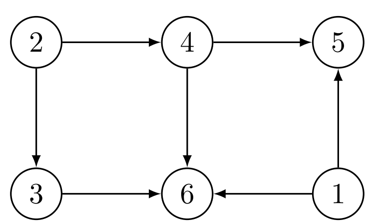
Definition 7. Let G be a tropical curve, and let $<$ be an order of the vertices of G. Orient all edges of G from the higher to lower vertices. The **order divisor** of $<$ is

$$D_< = \sum_{P \in G} n_P(P), \quad n_P = \#\{\text{edges oriented out of } P\} - 1.$$

For example, consider the following graph with six vertices (edge lengths have been omitted, since these are irrelevant for order divisors):



where the vertices are ordered $1 > 2 > 3 > 4 > 5 > 6$. The corresponding directed graph is



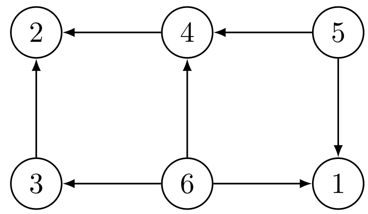
and the order divisor is

$$D_< = (1) + (2) + (4) - (5) - (6).$$

We can also consider the reverse order $\overline{<}$, in which the vertices are ordered

$$6 \overline{>} 5 \overline{>} 4 \overline{>} 3 \overline{>} 2 \overline{>} 1.$$

The directed graph is



with order divisor

$$D_{\overline{<}} = (5) + 2(6) - (2) - (1).$$

It is clear that the sum of $D_<$ and $D_{\overline{<}}$ will always be the canonical divisor: the number of rays incident to any vertex P is always the number of rays entering plus the number of rays exiting with respect to any orientation of edges. In this case, we can verify that the canonical divisor is

$$K = (4) + (6) = D_< + D_{\overline{<}}.$$

Proposition 8. *Let $<$ be a vertex order on a tropical curve G. Then $|D_<| = \emptyset$ and* $\deg D_< = g - 1$.

Proof. The degree of $D_<$ is

$$\sum_{P \in \mathrm{Vert}(G)} n_P = \#E - \#V = g - 1.$$

$|D_<| = 0$, because: for any rational function f, consider the vertices that maximize f, and choose a vertex P among these that is minimal with respect to the order. Then the coefficient of P in $D_< + (f)$ is negative, because: every outward-oriented edge incident to P leads to a vertex on which f has a strictly lesser value, so f has a slope less than or equal to -1 on that edge. $\square$

Definition 9. Let G be a tropical curve and let P be a point on G.
(i) Let A, B be divisors on G. A is **nearer to** P than B if there is an open neighborhood $P \in U \subseteq G$ such that for any $Q \in U$, the coefficient of Q in A is greater than or equal to the coefficient of Q in B, and at least one point has strictly larger coefficient.
(ii) A divisor A on G is **tight** at P if it is effective away from P, and no divisor of the form $A + (f)$ that is effective away from P is nearer to P.

Remember that we have assumed G is a metric graph (with finite edge lengths) - otherwise, this definition does not make sense.

Proposition 10. *A divisor A satisfies $|A| = \emptyset$ if and only if there is a vertex order $<$ and a rational function f such that $D_< - A - (f)$ is effective.*

Proof. Any divisor of the form $A = D_< + (f) - E$ for some effective divisor E satisfies $|A| = \emptyset$, because: if g were a rational function such that $A + (g)$ is effective, then

$$D_< + (f + g) = A + (g) + E$$

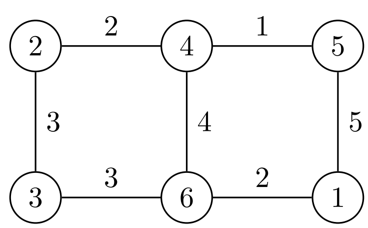
is effective; contradiction to the previous proposition.

On the other hand, assume that $|A| = \emptyset$, and assume G is subdivided such that every

point appearing in A is a vertex of G. The finite vertices of G are ordered as follows:
(i) Choose a rational function f such that $D+(f)$ is tight at some vertex P. P will be the smallest vertex.
(ii) Assume that a set of vertices S has already been ordered and define $T = \mathrm{Vert}(G)\backslash S$. Choose the shortest length ℓ between a vertex from S and a vertex not from S, and define a rational function that is constant 0 on S, constant ℓ on T, and has slope -1 on the part of every edge leaving T toward S. Since $A+(f)+(g)$ is nearer to P than $A+(f)$, it must not be effective, so there is a vertex Q with negative coefficient; let this be the next vertex in the order.

Then $D_< - A - (f)$ is effective, because: by construction, each vertex other than P has at least as many edges oriented towards it than the coefficient of $A+(f)$. Also, since $A+(f)$ is not effective, its coefficient at P is at most -1; this is exactly the coefficient of $D_<$ at P, so the coefficient of $D_< - A - (f)$ is nonnegative. $\square$

For example, consider the graph from before, with edge lengths:



and the divisor

$$A = (2) - 2(1).$$

Certainly, $|A| = \emptyset$ because A has negative degree. It is also tight at (1): if f is a rational function such that $A+(f)$ is effective away from 1, then the sum of slopes of f exiting (2) can be at most 1. However, if the sum of slopes were 1, this would contradict continuity of f on the square $(2)-(4)-(6)-(3)$ (since f is not allowed to decrease from any other vertex); so f must be constant going into 2, and therefore constant everywhere.

This means that we can let (1) be the lowest element in our order $<$. In the second step, we consider the divisor $A+(g) = (2)-(6)-(5)+(x)$, where x is a new vertex on the edge between (1) and (5). We choose an arbitrary vertex with negative coefficient; for example, (5); and this is the second lowest element. Continuing this procedure, we can find an order $<$ such that $D_< - A$ is effective; one example is

$$2 > 3 > 4 > 6 > 5 > 1$$

with order divisor $(2)+(4)-(1)$.

Proposition 11 (Riemann-Roch theorem)**.** *Let A be any divisor on a tropical curve G. Then*

$$r(A) - r(K-A) = \deg(A) - (g-1).$$

Proof. Choose an effective divisor E with $\deg(E) = r(A)+1$ and $|A-E| = \emptyset$. Then there is a rational function f and a vertex order $<$ such that

$$F := D_< - (A-E) - (f)$$

is effective. Then

$$(K - A) - F = (K - D_<) - E + (f) = D_{\overline{<}} - E + (f),$$

where $D_{\overline{<}}$ denotes the divisor for the opposite order. Therefore $|(K - A) - F| = \emptyset$, i.e.

$$r(K - A) \leq \deg(F) - 1 = g - 2 + \deg(E) - \deg(A) = g - 1 + r(A) - \deg(A),$$

so

$$r(A) - r(K - A) \leq \deg(A) + 1 - g.$$

Replacing A with $K - A$ implies

$$r(K - A) - r(A) \leq \deg(K) - \deg(A) + 1 - g = -(\deg(D) + 1 - g),$$

so this is an equality. □

Summary

With the appropriate definition of $r(D)$, the Riemann-Roch theorem translates almost word-for-word to the tropical setting. This result is interesting in its own right, and it also gives information about classical curves over a field with valuation through the *specialization lemma* of [1], which is an inequality relating the rank of a divisor on a classical curve and the rank of its specialization to a tropical curve.

The classical Riemann-Roch theorem has generalizations to higher dimensions (namely, the Hirzebruch-Riemann-Roch theorem). Tropical curves have also been generalized to tropical varieties of arbitrary dimension, as in the book [4]. It is an open problem to prove (or even to conjecture) a corresponding generalization of the tropical Riemann-Roch theorem.

References

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