Divisors and the Riemann-Roch theorem
Brandon Williams
Math 274 final paper

This is an expository paper submitted for Bernd Sturmfels’ Math 274 course in Spring 2016. It is mostly based on the exposition in [5].

Recall the Riemann-Roch theorem for Riemann surfaces:

**Theorem 1.** Let $X$ be a Riemann surface of genus $g$ with canonical divisor $K$, and let $D$ be any divisor on $X$. Then

$$\ell(D) - \ell(K - D) = \deg(D) - (g - 1).$$

Here, a divisor is a formal sum of points of $X$. $K$ is the divisor of any meromorphic one-form (since $X$ is a complex curve, all meromorphic one-forms are equal up to multiplying by a function); i.e. $K = \sum_{p \in X} n_p(p)$, where $n_p$ is the order of the one-form in $p$. The genus $g$ is the dimension of the space of holomorphic one-forms. $\ell(D)$ is the dimension of the space of meromorphic functions where the order of poles does not exceed the value prescribed by $D$. The degree $\deg(D)$ is the sum of all coefficients of $D$.

For example, on the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, the meromorphic one-form $dz$ has a pole of order 2 in $\infty$, so we can take $K = 2(\infty)$. There are no holomorphic one-forms, so $g = 0$. If $D$ denotes the divisor $(\infty) - (0)$, then $\ell(D)$ is the dimension of the space of holomorphic functions with at most a pole of order 1 in $\infty$ and a zero in 0; this is one-dimensional, spanned by the function $f(z) = z$.

**Divisors on Tropical Curves**

**Definition 1.** A **tropical curve** is a finite graph $G$ together with a metric

$$d: G \times G \to [0, \infty],$$

where the distance between two points of $G$ is allowed to be infinite if and only if one of the points is a leaf.

More precisely, it is a graph $G$ with a topology and with fixed homeomorphisms between each edge and a closed interval $(0, a]$, $a \in [0, \infty]$, such that only leaves of $G$ are allowed to correspond to $\infty$. The interval $[0, \infty]$ is understood as the one-point compactification of the nonnegative reals $\mathbb{R}_{\geq 0}$.

Any two graph structures on the same space with the same topology are understood to be the same tropical curve; in particular, subdividing edges does not change the tropical curve.

**Definition 2.** Let $G$ be a tropical curve. A **divisor** on $G$ is a formal $\mathbb{Z}$-linear combination of points. The group of divisors is denoted $\text{Div}(G)$.

For any divisor $D = \sum_{P \in G} n_P(P)$, the sum $\sum_{P \in G} n_P$ is the **degree** $\deg(D)$.
Definition 3. Let $G$ be a tropical curve. A **rational function** on $G$ is a continuous map $f : G \rightarrow \mathbb{R} \cup \{\infty\}$ that is piecewise-linear with integer slopes on each edge. The **divisor** of $f$ is
\[
(f) = \sum_{P \in G} f_P(P),
\]
where $f_P$ is the sum of all slopes of $f$ in directions coming out of $P$.

Any divisor on $G$ of the form $(f)$ is a **principal divisor**; the **class group**, or **Picard group**, of $G$ is the quotient
\[
\text{Cl}(G) = \text{Div}(G)/\text{Prin}(G)
\]
of divisors by principal divisors.

By subdividing $G$, we may assume that rational functions are linear between the vertices of $G$.

**Proposition 4.** Every principal divisor has degree 0.

*Proof.* This is because the sum on every edge of the graph is zero: if $f$ is a rational function that is linear between points $P$ and $Q$ on an edge of the graph, then the slope exiting $P$ cancels with the slope exiting $Q$. \qed

Generalizing this argument slightly proves that every divisor class is represented by a divisor supported on only finite vertices, since the infinite vertices can only occur on leaves. From now on, we will only need to consider true metric graphs.

Definition 5. Let $G$ be a tropical curve. The **canonical divisor** is the divisor
\[
K = \sum_{P \in \text{Vert}(G)} K_P(P),
\]
where $K_P$ is the number of rays incident to $P$ in $G$ minus 2.

Definition 6. Let $G$ be a tropical curve.
(i) A divisor $D = \sum_{P \in G} n_P(P)$ is **effective** if $n_P \geq 0$ for all $P$.
(ii) Let $D$ be a divisor. $|D|$ is the set of all effective divisors in the class of $D$.
(iii) Let $D$ be a divisor. Its **rank** is
\[
r(D) = \min\{\deg(E) : E \text{ effective with } |D - E| = \emptyset\} - 1.
\]

Finding the correct definition of rank is probably the most difficult part of tropicalizing the Riemann-Roch theorem. The direct definition of $\ell(D)$ from the case of Riemann surfaces does not seem to have a natural analogy. Another approach is noticing that, for Riemann surfaces, $|D|$ is in bijection with the projective space $\left(\Gamma(X, \mathcal{O}(D))\setminus\{0\}\right)/\mathbb{C}^\times$ (see [3], Proposition II.7.7, for example), so $\ell(D)$ can be recovered as $\dim |D| + 1$. This also leads to difficulties: in fact, for tropical graphs, $|D|$ is naturally a polyhedral complex but not of a pure dimension, and the naive dimension (dimension of the largest cell) does not make the Riemann-Roch theorem true. For example, in the graph
with canonical divisor \( (1) + (2) \), the space \(|K|\) consists of a two-dimensional cell \( (K) \) is linearly equivalent to any divisor \( (p) + (q) \) with points \( p, q \) on the line from 1 to 2) and two one-dimensional cells \( (K) \) is only linearly equivalent to a divisor \( (p) + (q) \) with \( p, q \) on one of the loops if \( p \) and \( q \) are at an equal distance to their nearest vertex). Since \(|0| = \{0\}\) consists of a single point, the natural definition is \( r(0) = 0; \) then

\[
r(K) - r(0) = \deg(K) - (g - 1) = 2 - 1 = 1
\]

implies that 1 is the correct ‘dimension’ for \(|K|\).

The definition of \( r(D) \) presented above is constructed in a way that forces the Riemann-Roch theorem to be true. It also behaves nicely with respect to specialization from curves over the Puiseux series field \( \mathbb{C}\{t\}\) to tropical curves; see [1].

Vertex orders and the proof of Riemann-Roch

**Definition 7.** Let \( G \) be a tropical curve, and let \( < \) be an order of the vertices of \( G \). Orient all edges of \( G \) from the higher to lower vertices. The \textbf{order divisor} of \( < \) is

\[
D_\prec = \sum_{P \in G} n_P(P), \quad n_P = \#\{\text{edges oriented out of } P\} - 1.
\]

For example, consider the following graph with six vertices (edge lengths have been omitted, since these are irrelevant for order divisors):

```
2--4--5
|   |
3--6--1
```

where the vertices are ordered 1 > 2 > 3 > 4 > 5 > 6. The corresponding directed graph is

```
2->4->5
|   |
3<6<1
```

and the order divisor is

\[
D_\prec = (1) + (2) + (4) - (5) - (6).
\]

We can also consider the reverse order \( \prec \), in which the vertices are ordered

\[
6\geq5\geq4\geq3\geq2\geq1.
\]

The directed graph is
with order divisor

\[ D_\prec = (5) + 2(6) - (2) - (1). \]

It is clear that the sum of \( D_\prec \) and \( D_\succ \) will always be the canonical divisor: the number of rays incident to any vertex \( P \) is always the number of rays entering plus the number of rays exiting with respect to any orientation of edges. In this case, we can verify that the canonical divisor is

\[ K = (4) + (6) = D_\prec + D_\succ. \]

**Proposition 8.** Let \( \prec \) be a vertex order on a tropical curve \( G \). Then \( |D_\prec| = \emptyset \) and \( \deg D_\prec = g - 1 \).

**Proof.** The degree of \( D_\prec \) is

\[ \sum_{P \in \text{Vert}(G)} n_P = \#E - \#V = g - 1. \]

\( |D_\prec| = 0 \), because: for any rational function \( f \), consider the vertices that maximize \( f \), and choose a vertex \( P \) among these that is minimal with respect to the order. Then the coefficient of \( P \) in \( D_\prec + (f) \) is negative, because: every outward-oriented edge incident to \( P \) leads to a vertex on which \( f \) has a strictly lesser value, so \( f \) has a slope less than or equal to \(-1\) on that edge.

**Definition 9.** Let \( G \) be a tropical curve and let \( P \) be a point on \( G \).

(i) Let \( A, B \) be divisors on \( G \). \( A \) is nearer to \( P \) than \( B \) if there is an open neighborhood \( P \in U \subseteq G \) such that for any \( Q \in U \), the coefficient of \( Q \) in \( A \) is greater than or equal to the coefficient of \( Q \) in \( B \), and at least one point has strictly larger coefficient.

(ii) A divisor \( A \) on \( G \) is tight at \( P \) if it is effective away from \( P \), and no divisor of the form \( A + (f) \) that is effective away from \( P \) is nearer to \( P \).

Remember that we have assumed \( G \) is a metric graph (with finite edge lengths) - otherwise, this definition does not make sense.

**Proposition 10.** A divisor \( A \) satisfies \( |A| = \emptyset \) if and only if there is a vertex order \( \prec \) and a rational function \( f \) such that \( D_\prec - A - (f) \) is effective.

**Proof.** Any divisor of the form \( A = D_\prec + (f) - E \) for some effective divisor \( E \) satisfies \( |A| = \emptyset \), because: if \( g \) were a rational function such that \( A + (g) \) is effective, then

\[ D_\prec + (f + g) = A + (g) + E \]

is effective; contradiction to the previous proposition.

On the other hand, assume that \( |A| = \emptyset \), and assume \( G \) is subdivided such that every
point appearing in $A$ is a vertex of $G$. The finite vertices of $G$ are ordered as follows:

(i) Choose a rational function $f$ such that $D + (f)$ is tight at some vertex $P$. $P$ will be the smallest vertex.

(ii) Assume that a set of vertices $S$ has already been ordered and define $T = \text{Vert}(G) \setminus S$. Choose the shortest length $\ell$ between a vertex from $S$ and a vertex not from $S$, and define a rational function that is constant 0 on $S$, constant $\ell$ on $T$, and has slope $-1$ on the part of every edge leaving $T$ toward $S$. Since $A + (f) + (g)$ is nearer to $P$ than $A + (f)$, it must not be effective, so there is a vertex $Q$ with negative coefficient; let this be the next vertex in the order.

Then $D_\prec - A - (f)$ is effective, because: by construction, each vertex other than $P$ has at least as many edges oriented towards it than the coefficient of $A + (f)$. Also, since $A + (f)$ is not effective, its coefficient at $P$ is at most $-1$; this is exactly the coefficient of $D_\prec$ at $P$, so the coefficient of $D_\prec - A - (f)$ is nonnegative.

For example, consider the graph from before, with edge lengths:

```
2 2 4 1
3 3 4 6 2
5 5
```

and the divisor

$$A = (2) - 2(1).$$

Certainly, $|A| = \emptyset$ because $A$ has negative degree. It is also tight at (1): if $f$ is a rational function such that $A + (f)$ is effective away from 1, then the sum of slopes of $f$ exiting (2) can be at most 1. However, if the sum of slopes were 1, this would contradict continuity of $f$ on the square $(2) - (4) - (6) - (3)$ (since $f$ is not allowed to decrease from any other vertex); so $f$ must be constant going into 2, and therefore constant everywhere.

This means that we can let (1) be the lowest element in our order $\prec$. In the second step, we consider the divisor $A + (g) = (2) - (6) - (5) + (x)$, where $x$ is a new vertex on the edge between (1) and (5). We choose an arbitrary vertex with negative coefficient; for example, (5); and this is the second lowest element. Continuing this procedure, we can find an order $\prec$ such that $D_\prec - A$ is effective; one example is

$$2 > 3 > 4 > 6 > 5 > 1$$

with order divisor $(2) + (4) - (1)$.

**Proposition 11** (Riemann-Roch theorem). Let $A$ be any divisor on a tropical curve $G$. Then

$$r(A) - r(K - A) = \deg(A) - (g - 1).$$

**Proof.** Choose an effective divisor $E$ with $\deg(E) = r(A) + 1$ and $|A - E| = \emptyset$. Then there is a rational function $f$ and a vertex order $\prec$ such that

$$F := D_\prec - (A - E) - (f)$$

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$$F := D_\prec - (A - E) - (f)$$
is effective. Then
\[(K - A) - F = (K - D_<) - E + (f) = D_< - E + (f),\]
where \(D_<\) denotes the divisor for the opposite order. Therefore \(|(K - A) - F| = \emptyset\), i.e.
\[r(K - A) \leq \deg(F) - 1 = g - 2 + \deg(E) - \deg(A) = g - 1 + r(A) - \deg(A),\]
so
\[r(A) - r(K - A) \leq \deg(A) + 1 - g.\]
Replacing \(A\) with \(K - A\) implies
\[r(K - A) - r(A) \leq \deg(K) - \deg(A) + 1 - g = -(\deg(D) + 1 - g),\]
so this is an equality. \(\square\)

Summary

With the appropriate definition of \(r(D)\), the Riemann-Roch theorem translates almost
word-for-word to the tropical setting. This result is interesting in its own right, and it also
provides information about classical curves over a field with valuation through the specialization
lemma of [1], which is an inequality relating the rank of a divisor on a classical curve and
the rank of its specialization to a tropical curve.

The classical Riemann-Roch theorem has generalizations to higher dimensions (namely,
the Hirzebruch-Riemann-Roch theorem). Tropical curves have also been generalized to
tropical varieties of arbitrary dimension, as in the book [4]. It is an open problem to
prove (or even to conjecture) a corresponding generalization of the tropical Riemann-Roch
theorem.

References

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