OVERPARTITION $M2$-RANK DIFFERENCES, CLASS NUMBER RELATIONS, AND VECTOR-VALUED MOCK EISENSTEIN SERIES

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Abstract. We prove that the generating function of overpartition $M2$-rank differences is, up to coefficient signs, a component of the vector-valued mock Eisenstein series attached to a certain quadratic form. We use this to compute analogs of the class number relations for $M2$-rank differences. As applications we split the Kronecker-Hurwitz relation into its "even" and "odd" parts and calculate sums over Hurwitz class numbers of the form $\sum_{r\in\mathbb{Z}} H(n-2r^2)$.

1. Introduction and statement of results

An **overpartition** of $n \in \mathbb{N}_0$ is a partition in which the first occurance of a number may be overlined (distinguished). Equivalently, an overpartition of n is a decomposition $n = n_1 + n_2$, with $n_1, n_2 \geq 0$, together with a partition of n_1 into distinct parts (the overlined numbers) and an arbitrary partition of n_2. For example, the 8 overpartitions of $n = 3$ are

$$3 = \overline{3} = 2+1 = \overline{2}+1 = 2+\overline{1} = \overline{2}+\overline{1} = 1+1+1 = \overline{1}+1+1.$$

In [7], Lovejoy introduced the $M2$-rank statistic

$$M2\text{-rank}(\lambda) = \left\lceil \frac{\ell(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_0) - \chi(\lambda),$$

where $\ell(\lambda)$ denotes the largest part of the overpartition λ, $n(\lambda)$ is the number of parts, $n(\lambda_o)$ is the number of odd non-overlined parts, and

$$\chi(\lambda) = \begin{cases} 1: & \text{the largest part of } \lambda \text{ is odd and non-overlined;} \\ 0: & \text{otherwise.} \end{cases}$$

Let $M2_e(n)$ and $M2_o(n)$ denote the number of overpartitions of n with even $M2$-rank resp. odd $M2$-rank and define the $M2$-rank difference

$$\overline{\alpha}_2(n) = M2_e(n) - M2_o(n).$$

Bringmann and Lovejoy proved [2] that the generating function

$$\overline{f_2}(q) = \sum_{n=0}^{\infty} \overline{\alpha}_2(n) q^n = 1 + 2q + 4q^2 - 2q^4 + 8q^5 + 8q^6 + ...$$

is a mock modular form of weight 3/2, i.e. the holomorphic part of a harmonic weak Maass form.

It was proved in [11] that the generating function for the difference counts between overpartitions with even and odd (Dyson's) ranks is, up to the signs of the coefficients, the $\mathfrak{e}_0$-component of a vector valued mock Eisenstein series for the dual Weil representation attached to the quadratic form $q(x,y,z) = x^2 + y^2 - z^2$ on $\mathbb{Z}^3$. (It is also closely related to the mock Eisenstein series attached to the quadratic form $q(x) = 2x^2$.) It was observed there that the function $\overline{f_2}$ seemed to be similarly related to the mock Eisenstein series attached to the quadratic form $q_2(x,y,z) = 2x^2 - y^2 + 2z^2$ (or equivalently $2x^2 + 2y^2 - z^2$).

The purpose of this note is to prove the assertion above.

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Proposition 1. *The series*

$$1 - \sum_{n=1}^{\infty} |\overline{\alpha}_2(n)| q^n = 1 - 2q - 4q^2 - 2q^4 - 8q^5 - 8q^6 - ...$$

is the $\mathfrak{e}_0$-component of the mock Eisenstein series for the dual Weil representations attached to the quadratic form $q_2(x,y,z) = 2x^2 - y^2 + 2z^2$ and to the quadratic form $\tilde{q}_2(x,y,z) = 2x^2 - 2y^2 + z^2$.

As in the examples of [12], calculating the other components of the mock Eisenstein series leads to "class number relations" by comparing coefficients of certain vector-valued modular or quasimodular forms of weight 2. (This is very similar to the arguments of [1] that use mixed mock modular forms to derive relations among class numbers.) Let $n \in \mathbb{N}$; and let $\sigma_1(n) = \sum_{d|n} d$, $\lambda_1(n) = \frac{1}{2}\sum_{d|n} \min(d, n/d)$ and let $H(n)$ denote the Hurwitz class number. The identities we find in this case are:

Proposition 2. *Define*

$$\alpha(n) = \sum_{r \in \mathbb{Z}} |\overline{\alpha}_2(n - r^2)| + 4 \sum_{r \text{ odd}} H(4n - r^2) + 4\lambda_1(n) - \begin{cases} 4: & n = \square; \\ 0: & \text{otherwise.} \end{cases}$$

Then $\alpha(n)$ satisfies the identity

$$\alpha(n) = \begin{cases} 8\sigma_1(n/2) + 16\sigma_1(n/4): & n \equiv 0 \mod 4; \\ 24\sigma_1(n/2): & n \equiv 2 \mod 4; \\ 4\sigma_1(n): & n \equiv 1 \mod 2. \end{cases}$$

$\overline{\alpha}_2(n)$ can be expressed in terms of Hurwitz class numbers and representations counts by sums of three squares, using theorem 1.1 of [2]. We can use this to isolate the term $\sum_{r \text{ odd}} H(4n - r^2)$ in the identity above. This is a strengthening of theorem 1 of [3] which considers the case that n is prime.

Corollary 3. *For any $n \in \mathbb{N}$,*

$$\sum_{r \text{ odd}} H(4n - r^2) = \begin{cases} (2/3)\sigma_1(n): & n \text{ odd}; \\ 4\sigma_1(n/2) - 2\lambda_1(n): & n \equiv 2 \,(\text{mod } 4); \\ (2/3)\sigma_1(n) + 2\sigma_1(n/2) - (8/3)\sigma_1(n/4) + 4\lambda_1(n/4) - 2\lambda_1(n): & n \equiv 0 \,(\text{mod } 4). \end{cases}$$

We can use similar arguments to give another "class number relation" for $\overline{\alpha}_2$ of a different sort:

Proposition 4. *Write $n \in \mathbb{N}$ in the form $n = 2^\nu m$ with m odd. Then*

$$\sum_{r \in \mathbb{Z}} |\overline{\alpha}_2(n - 2r^2)| = 2\sigma_1(n, \chi) \cdot \Big(2 + \frac{\chi(m)}{2^\nu}\Big) + 4 \sum_{N(\mathfrak{a}) = 2n} \Big(|b| - a\Big) + \begin{cases} 4: & 2n = \square; \\ 0: & \text{otherwise;} \end{cases}$$

where χ is the Dirichlet character modulo 8 given by $\chi(1) = \chi(7) = 1$ and $\chi(3) = \chi(5) = -1$; and $\sigma_1(n, \chi)$ is the twisted divisor sum

$$\sigma_1(n, \chi) = \sum_{d|n} \chi(n/d) d;$$

and $\mathfrak{a}$ runs through all ideals of $\mathbb{Z}[\sqrt{2}]$ having norm $2n$, and $a + b\sqrt{2} \in \mathfrak{a}$ is a generator with minimal trace $2a > 0$.

As before, this translates into an identity for Hurwitz class numbers:

Corollary 5. *(i) For any $n = 2^\nu m$, m odd,*

$$\sum_{r \in \mathbb{Z}} H(4n - 2r^2) = \frac{2}{3}\sigma_1(n, \chi) \cdot \Big(2 + \frac{\chi(m)}{2^{\nu+2}}\Big) + \frac{1}{2} \sum_{N(\mathfrak{a}) = 8n} \Big(|b| - a\Big).$$

(ii) If n is odd, then

$$\sum_{r \in \mathbb{Z}} H(2n - 2r^2) = \frac{4 + \chi(n)}{6} \sigma_1(n, \chi) + \frac{1}{2} \sum_{N(\mathfrak{a}) = 4n} \Big(|b| - a\Big).$$

(iii) If n is odd, then

$$\sum_{r \in \mathbb{Z}} H(n-2r^2) = \frac{2+\chi(n)}{6}\sigma_1(n,\chi) + \frac{1}{2}\sum_{N(\mathfrak{a})=2n} \Big(|b|-a\Big).$$

If there are no ideals of norm $2n$ then the error term $\frac{1}{2}\sum_{N(\mathfrak{a})=2n}(|b|-a)$ vanishes. For example, for primes p that remain inert in $\mathbb{Z}[\sqrt{2}]$ (i.e. $p \equiv 3$ or $p \equiv 5$ modulo 8), we get the formulas

$$\sum_{r\in\mathbb{Z}} H(4p-2r^2) = \frac{7}{6}(p-1); \quad \sum_{r\in\mathbb{Z}} H(2p-2r^2) = \frac{p-1}{2}; \quad \sum_{r\in\mathbb{Z}} H(p-2r^2) = \frac{p-1}{6}.$$

2. Proof of proposition 1

We first observe that $\overline{\alpha}_2(n)$ is positive when $n \equiv 1,2\,(\mathrm{mod}\,4)$ and $\overline{\alpha}_2(n)$ is negative or 0 when $n \equiv 0,3\,(\bmod\ 4)$ (with $n=0$ as an exception). This is a consequence of equation (1.5) of [2] which gives an exact expression for $\overline{\alpha}_2(n)$ in terms of Hurwitz class numbers and representation counts by sums of three squares. It follows that

$$f(\tau) := 1 - 2q - 4q^2 - 2q^4 - 8q^5 - 8q^6 - ... = \frac{1+i}{2} f_2(\tau+1/4) + \frac{1-i}{2} f_2(\tau - 1/4).$$

Theorem 2.1 of [2] shows that the real-analytic correction of $f_2(\tau)$ is the harmonic weak Maass form

$$(1) \qquad f_2(\tau) - \mathcal{N}_2(\tau) = f_2(\tau) + \frac{i}{\pi\sqrt{2}}\int_{-\overline{\tau}}^{i\infty} \frac{\Theta(t)}{(-i(\tau+t))^{3/2}}\,\mathrm{d}t = f_2(\tau) - \frac{\sqrt{i}}{\pi\sqrt{2}}\int_{-\overline{\tau}}^{\infty} \frac{\Theta(t)}{(\tau+t)^{3/2}}\,\mathrm{d}t$$

of level 16, where $\Theta(t)$ is the classical theta series $\Theta(\tau) = 1+2q+2q^4+2q^9+...$ It follows that the real-analytic correction of $f(\tau)$ is also

$$\begin{aligned} & f(\tau) - \frac{1+i}{2}\mathcal{N}_2(\tau+1/4) - \frac{1-i}{2}\mathcal{N}_2(\tau-1/4) \\ &= f(\tau) - \frac{\sqrt{i}}{\pi\sqrt{2}}\int_{-\overline{\tau}}^{i\infty} \frac{1}{(\tau+t)^{3/2}}\Big[\frac{1+i}{2}\Theta(t-1/4) + \frac{1-i}{2}\Theta(t+1/4)\Big]\,\mathrm{d}t \\ &= f(\tau) - \frac{\sqrt{i}}{\pi\sqrt{2}}\int_{-\overline{\tau}}^{i\infty} \frac{\Theta(t)}{(\tau+t)^{3/2}}\,\mathrm{d}t, \end{aligned}$$

using the fact that $\frac{1+i}{2}\Theta(t-1/4) + \frac{1-i}{2}\Theta(t+1/4) = \Theta(t)$ since the only nonzero Fourier coefficients of $\Theta(t)$ occur with exponents $n \equiv 0,1\,(\mathrm{mod}\,4)$.

Remark 6. This implies that

$$f_2(\tau) - f(\tau) = 2\sum_{n\equiv 1,2\,(4)} \overline{\alpha}_2(n)q^n = 4q + 8q^2 + 16q^5 + 16q^6 + ...$$

is a weight-3/2 modular form of level 16. This is true; for example, we can write this as a difference of theta series,

$$4q + 8q^2 + 16q^5 + 16q^6 + ... = \vartheta_A(\tau) - \vartheta_B(\tau)$$

for the matrices $A = \mathrm{diag}(2,2,2)$ and $B = \mathrm{diag}(4,4,2)$, which can be verified by computing finitely many coefficients. (This gives another proof that every number $n \equiv 0,3\,(4)$ has the same number of representations in the form $a^2+b^2+c^2$ as it does in the form $2a^2+2b^2+c^2$.)

We need to compare the weak Maass form above with the real-analytic correction of the mock Eisenstein series $E_{3/2}(\tau)$. Following remark 17 of [11], this is

$$E^*_{3/2}(\tau,0) = E_{3/2}(\tau) + \frac{\sqrt{2i}}{16\pi}\int_{-\overline{\tau}}^{i\infty} \frac{\vartheta(t)}{(t+\tau)^{3/2}}\,\mathrm{d}t,$$

where $\vartheta(t)$ is the shadow

$$\vartheta(\tau) = \sum_{\gamma\in\Lambda'/\Lambda}\ \sum_{\substack{n\in\mathbb{Z}-q_2(\gamma)\\ n\le 0}} a(n,\gamma)q^{-n}\mathfrak{e}_\gamma,$$

where Λ'/Λ is the discriminant group of the quadratic form $q_2(x,y,z) = 2x^2 - y^2 + 2z^2$ (and $\mathfrak{e}_\gamma$ are the corresponding basis elements of the group ring $\mathbb{C}[\Lambda'/\Lambda]$) and the coefficients $a(n,\gamma)$ are given by

$$a(n,\gamma) = -\frac{48\sqrt{2}}{\sqrt{|\Lambda'/\Lambda|}} \prod_{\text{bad } p} \lim_{s\to 0} \frac{1-p^{-2s}}{1+p^{-1}} L_p(n,\gamma,2+2s) \times \begin{cases} 1: & n<0; \\ 1/2: & n=0; \end{cases}$$

where the "bad primes" are $p=2$ and all primes dividing $|\Lambda'/\Lambda|$ or at which n has nonzero valuation, and L_p is the series

$$L_p(n,\gamma,s) = \sum_{\nu=0}^\infty p^{-\nu s} \# \Big\{ v \in (\mathbb{Z}/p^\nu \mathbb{Z})^3 : \; q_2(v-\gamma) + n \equiv 0 \pmod{p^\nu} \Big\}.$$

We only need to compute a small number of these coefficients to identify ϑ among modular forms of weight $1/2$ for the Weil representation (not its dual) attached to the quadratic form $\tilde{q}_2$. A general result of Skoruppa [10] implies that the space of such forms is spanned by the theta series

$$\begin{aligned}\vartheta_1(\tau) = &\Big(1 + 2q + 2q^4 + 2q^9 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/4,1/2,1/4)} + \mathfrak{e}_{(1/2,0,1/2)} + \mathfrak{e}_{(3/4,1/2,3/4)}) \\ &+ \Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(1/4,0,3/4)} + \mathfrak{e}_{(1/2,1/2,0)} + \mathfrak{e}_{(3/4,0,1/4)} + \mathfrak{e}_{(0,1/2,1/2)})\end{aligned}$$

and

$$\begin{aligned}\vartheta_2(\tau) = &\Big(1 + 2q + 2q^4 + 2q^9 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/4,1/2,3/4)} + \mathfrak{e}_{(1/2,0,1/2)} + \mathfrak{e}_{(3/4,1/2,1/4)}) \\ &+ \Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(1/4,0,1/4)} + \mathfrak{e}_{(1/2,1/2,0)} + \mathfrak{e}_{(3/4,0,3/4)} + \mathfrak{e}_{(0,1/2,1/2)}).\end{aligned}$$

Therefore, it is enough to compute the constant term of the shadow $\vartheta(\tau)$. The only bad prime at $n=0$ is $p=2$, and the corresponding local L-functions are

$$(2) \qquad L_2(0,\gamma,s) = \begin{cases} (2^{4s} - 2^{2s+3} + 2^7)(1-2^{2-s})^{-1}(2^{2s}-8)^{-1}: & \gamma = (0,0,0); \\ (1+2^{4-2s})(1-2^{2-s})^{-1}: & \gamma = (1/2,1/2,0); \\ (1-2^{2-s})^{-1}: & \text{otherwise;} \end{cases}$$

such that

$$\lim_{s\to 0} (1-2^{-2s}) L_2(0,\gamma,2+2s) = \begin{cases} 2: & \gamma = (0,0,0) \text{ or } \gamma = (1/2,1/2,0); \\ 1: & \text{otherwise.} \end{cases}$$

This implies that the shadow is

$$\begin{aligned}\vartheta(\tau) &= -8(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/2,1/2,0)}) - 4(\mathfrak{e}_{(1/4,1/4,1/2)} + \mathfrak{e}_{(3/4,3/4,1/2)} + \mathfrak{e}_{(1/4,3/4,1/2)} + \mathfrak{e}_{(3/4,1/4,1/2)}) + O(q^{1/4}) \\ &= -4\vartheta_1(\tau) - 4\vartheta_2(\tau)\end{aligned}$$

with $\mathfrak{e}_0$-component $-8\Theta(\tau)$, so the $\mathfrak{e}_0$-component of $E^*_{3/2}(\tau,0)$ is

$$E^*_{3/2}(\tau,0)_0 = E_{3/2}(\tau)_0 - \frac{\sqrt{i}}{\pi\sqrt{2}} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta(t)}{(t+\tau)^{3/2}} \, \mathrm{d}t$$

has the same real-analytic part as $f(\tau)$. It follows that $E^*_{3/2}(\tau,0)_0 - f(\tau)$ is a modular form of level 16; we can easily verify that it is identically 0 by comparing Fourier coefficients up to the Sturm bound.

The case of the quadratic form $\tilde{q}_2(x,y,z) = 2x^2 - 2y^2 + z^2$ is similar; here, the space of weight-1/2 modular forms is spanned by the theta functions

$$\begin{aligned}\vartheta_1(\tau) = &\Big(1 + 2q + 2q^4 + 2q^9 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/4,1/4,0)} + \mathfrak{e}_{(1/2,1/2,0)} + \mathfrak{e}_{(3/4,3/4,0)}) \\ &+ \Big(2q^{1/4} + 2q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(0,0,1/2)} + \mathfrak{e}_{(1/4,1/4,1/2)} + \mathfrak{e}_{(1/2,1/2,1/2)} + \mathfrak{e}_{(3/4,3/4,1/2)})\end{aligned}$$

and

$$\vartheta_2(\tau) = \Big(1 + 2q + 2q^4 + 2q^9 + ...\Big)(\mathfrak{e}_{(0,0,0)} + \mathfrak{e}_{(1/4,3/4,0)} + \mathfrak{e}_{(1/2,1/2,0)} + \mathfrak{e}_{(3/4,1/4,0)})$$
$$+ \Big(2q^{1/4} + q^{9/4} + 2q^{25/4} + ...\Big)(\mathfrak{e}_{(0,0,1/2)} + \mathfrak{e}_{(1/4,3/4,1/2)} + \mathfrak{e}_{(1/2,1/2,1/2)} + \mathfrak{e}_{(3/4,1/4,1/2)}),$$

and the local L-functions for $n = 0$ are exactly the same as those of equation (2) (although "otherwise" now stands for a different set of elements γ). It follows that $\mathfrak{e}_0$ component of $E^*_{3/2}(\tau, 0)$ for $\tilde{q}_2$ is the same as the $\mathfrak{e}_0$-component of $E^*_{3/2}(\tau, 0)$ for q_2, and therefore also equals $f(\tau)$.

3. Proof of proposition 2

The real-analytic Eisenstein series $E^*_{3/2}(\tau, 0)$ for $\tilde{q}_2$ corresponds via the theta decomposition to a real-analytic Jacobi form $E^*_{2,1,0}(\tau, z, 0)$ for the dual Weil representation ρ^* attached to the quadratic form $2x^2 - 2y^2$, and we obtain a quasimodular form $Q_{2,1,0}$ (the "Poincaré square series of index 1") by projecting $E^*_2(\tau, 0, 0)$ orthogonally to the cusp space $S_2(\rho^*)$ and then adding to it the quasimodular Eisenstein series E_2. Since $S_2(\rho^*) = \{0\}$, it follows that $Q_{2,1,0}(\tau) = E_2(\tau)$.

The $\mathfrak{e}_0$-component of the series $E_2(\tau)$ for the quadratic form $2x^2 - 2y^2$ is

$$E_2(\tau)_0 = 1 - 4q - 24q^2 - 16q^3 - 40q^4 - 24q^5 - ...$$

and using the structure of quasimodular forms of higher level (proposition 1(b) of [6]) this can be identified as

$$1 - 4q - 24q^2 - 16q^3 - 40q^4 - ... = \frac{2}{3}E_2(2\tau) + \frac{2}{3}E_2(4\tau) - \frac{1}{3}E_2(2\tau + 1/2) - 4\sum_{n=0}^{\infty} \sigma(2n+1)q^{2n+1},$$

in which the coefficient of q^n is

$$\alpha(n) = \begin{cases} -4\sigma_1(n): & n \equiv 1\,(2); \\ -24\sigma_1(n/2): & n \equiv 2\,(4); \\ -8\sigma_1(n/2) - 16\sigma_1(n/4): & n \equiv 0\,(4). \end{cases}$$

If Λ is an even e-dimensional lattice of signature $\sigma \equiv 0 \bmod 4$ with quadratic form q, then define the lattice $\tilde{\Lambda} = \Lambda \oplus \mathbb{Z}$ with quadratic form $\tilde{q}(v, \lambda) = q(v) + \lambda^2$. Suppose the weight-3/2 mock Eisenstein series for $\tilde{\Lambda}$ has coefficients denoted

$$E_{3/2}(\tau) = \sum_{\tilde{\gamma} \in \tilde{\Lambda}'/\tilde{\Lambda}} \sum_{n \in \mathbb{Z} - \tilde{q}(\tilde{\gamma})} c(n, \tilde{\gamma}) q^n \mathfrak{e}_{\tilde{\gamma}}$$

with shadow

$$\vartheta(\tau) = \sum_{\gamma \in \tilde{\Lambda}'/\tilde{\Lambda}} \sum_{\substack{n \in \mathbb{Z} - \tilde{q}(\tilde{\gamma}) \\ n \leq 0}} a(n, \tilde{\gamma}) q^{-n} \mathfrak{e}_{\tilde{\gamma}}.$$

The results of [12] imply that the coefficient formula for

$$Q_{2,1,0}(\tau) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - q(\gamma)} b(n, \gamma) q^n \mathfrak{e}_\gamma$$

is

$$b(n, \gamma) = \sum_{r \in \mathbb{Z}} c(n - r^2/4, (\gamma, r/2)) + \frac{1}{8} \sum_{r \in \mathbb{Z}} a(n - r^2/4, (\gamma, r/2)) \Big(|r| - \sqrt{r^2 - 4n}\Big).$$

In particular, the formula for $b(n, 0)$ involves knowing both the components of $\mathfrak{e}_{(0,0,0)}$ and $\mathfrak{e}_{(0,0,1/2)}$ in the mock Eisenstein series $E_{3/2}(\tau)$. This is the series

$$E_{3/2}(\tau)_{(0,0,1/2)} = -4q^{7/4} - 8q^{15/4} - 12q^{23/4} - 12q^{31/4} - ...$$

Lemma 7. *Let $H(n)$ denote the Hurwitz class number of $n \in \mathbb{N}_0$; then*

$$E_{3/2}(\tau)_{(0,0,1/2)} = -4 \sum_{n \equiv 7\,(8)} H(n) q^{n/4}.$$

Proof. In general, the formula of [4] for the weight-3/2 mock Eisenstein series for an even lattice of signature (b^+, b^-) and dimension $e = b^+ - b^-$ (under the assumption $3 + b^+ - b^- \equiv 0\,(4)$) implies that the coefficient $c(n, \gamma)$ of $E_{3/2}(\tau)$ is

$$\begin{aligned} &\frac{24(-1)^{(3+b^+-b^-)/4} L(1, \chi_{\mathcal{D}})}{\sqrt{\frac{1}{2n}|\Lambda'/\Lambda|}\pi} \cdot \Big[\sum_{d|f} \mu(d) \chi_{\mathcal{D}}(d) d^{-1} \sigma_{-1}(f/d) \Big] \times \\ (3) \qquad &\times \prod_{p | (2|\Lambda'/\Lambda|)} \Big[\lim_{s \to 0} \frac{1 - p^{(e-3)/2 - 2s}}{1 - p^{-2}} L_p(n, \gamma, 1/2 + e/2 + 2s) \Big], \end{aligned}$$

where $\mathcal{D}$ is a discriminant defined in theorem 4.5 of [4], and $\chi_{\mathcal{D}}(d) = \left(\frac{\mathcal{D}}{d}\right)$ is the Kronecker symbol, and $\sigma_{-1}(n) = \sum_{d|n} d^{-1}$, and f^2 is the largest square dividing n but coprime to $2 \cdot |\Lambda'/\Lambda|$. Finally, recall that $L_p(n, \gamma, s)$ denotes the series

$$L_p(n, \gamma, s) = \sum_{\nu=0}^{\infty} p^{-\nu s} \# \Big\{ v \in (\mathbb{Z}/p^\nu \mathbb{Z})^e : \ q(v - \gamma) + n \equiv 0 \,(\mathrm{mod}\, p^\nu) \Big\}.$$

For a fixed lattice, the series $L_p(n, \gamma, s)$ can always be evaluated in closed form in both n and s (for example, using the p-adic generating functions of [5]) although the result tends to be messy (particularly for $p = 2$).

In our case, most of these terms are the same between the lattice $\mathbb{Z}$ with quadratic form x^2 (in which case, $E_{3/2}$ is essentially the Zagier Eisenstein series and $c(n, \gamma)$ is $-12H(4n)$) and the lattice $\mathbb{Z}^3$ with quadratic form $2x^2 - 2y^2 + z^2$. The only differences are the order of the discriminant group and the local L-functions.

For $q(x) = x^2$, the result is that

$$\lim_{s \to 0} (1 - 2^{-2s}) L_2\Big(\frac{8n+7}{4}, \frac{1}{2}, 1 + 2s\Big) = \lim_{s \to 0} \frac{(1 - 2^{-2s})(2^{1+2s} + 1)}{2^{1+2s} - 1} = \frac{3}{2},$$

while for $q(x, y, z) = 2x^2 - 2y^2 + z^2$ the result is that

$$\lim_{s \to 0} (1 - 2^{-2s}) L_2\Big(\frac{8n+7}{4}, \frac{1}{2}, 2 + 2s\Big) = \lim_{s \to 0} \frac{(1 - 2^{-2s})(2^{2+2s} + 4)}{2^{2+2s} - 4} = 2$$

and

$$\lim_{s \to 0} (1 - 2^{-2s}) \underbrace{L_2\Big(\frac{8n+3}{4}, \frac{1}{2}, 2 + 2s\Big)}_{=1} = 0.$$

Since the discriminant group of $2x^2 - 2y^2 + z^2$ is 16 times the size of that of x^2, we conclude that the coefficient of $q^{(4n+7)/4}$ in the series $E_{3/2}(\tau)_{(0,0,1/2)}$ is

$$\frac{1}{4} \cdot \frac{2}{3/2} \cdot \begin{cases} -12H(4n+7) : & n \equiv 0\,(2); \\ 0 : & n \equiv 1\,(2); \end{cases}$$

which implies the claim. □

Finally, the coefficients $a(n - r^2/4, (0, r/2))$ of the shadow $\vartheta(\tau)$ are

$$a(n - r^2/4, (0, r/2)) = \begin{cases} -8 : & n - r^2/4 = 0; \\ -16 : & r^2 - 4n \text{ is a square}; \\ 0 : & \text{otherwise}. \end{cases}$$

Comparing coefficients in $Q_{2,1,0} = E_2(\tau)_0$ gives the formula:

Lemma 8. *If* $n \equiv 0 \mod 4$*, then*

$$8\sigma_1(n/2)+16\sigma_1(n/4) = \sum_{r \in \mathbb{Z}} |\overline{\alpha}_2(n-r^2)|+4\sum_{r \text{ odd}} H(4n-r^2)+2\sum_{r^2-4n=\square}\Big(|r|-\sqrt{r^2-4n}\Big)-\begin{cases} 4(\sqrt{n}+1): & n=\square; \\ 0: & \text{otherwise.}\end{cases}$$

If $n \equiv 2 \mod 4$*, then*

$$24\sigma_1(n/2) = \sum_{r \in \mathbb{Z}} |\overline{\alpha}_2(n-r^2)|+4\sum_{r \text{ odd}} H(4n-r^2)+2\sum_{r^2-4n=\square}\Big(|r|-\sqrt{r^2-4n}\Big).$$

If $n \equiv 1 \mod 2$*, then*

$$4\sigma_1(n) = \sum_{r \in \mathbb{Z}} |\overline{\alpha}_2(n-r^2)|+2\sum_{r \in \mathbb{Z}}\Big(|r|-\sqrt{r^2-4n}\Big)-\begin{cases} 4(\sqrt{n}+1): & n=\square; \\ 0: & \text{otherwise;}\end{cases}$$

Since

$$\sum_{r^2-4n=\square}\Big(|r|-\sqrt{r^2-4n}\Big) = 2\sum_{d|n}\min(d,n/d)+\begin{cases} 4\sqrt{n}: & n=\square; \\ 0: & \text{otherwise;}\end{cases}$$

this implies proposition 2.

Corollary 3 follows from this and the identity

$$(4) \qquad |\overline{\alpha}_2(n)| = \begin{cases} r_3(n)/3: & n \equiv 1,2\,(4); \\ 8H(n)-r_3(n)/3: & n \equiv 0,3\,(4);\end{cases}$$

where $r_3(n)$ is the representation count of n as a sum of three squares, using the identities

$$\sum_r r_3(n-r^2) = 8\sigma_1(n)-32\sigma_1(n/4)$$

(Jacobi's formula), and

$$\sum_r (-1)^r r_3(n-r^2) = (-1)^{(n-1)/2}\cdot 4\sigma_1(n), \;\; n \text{ odd},$$

(which follows from Jacobi's formula) and the class number relations

$$\sum_{r \in \mathbb{Z}} H(4n-r^2) = 2\sigma_1(n)-2\lambda_1(n)$$

and

$$\sum_{r \in \mathbb{Z}} H(n-r^2) = \frac{1}{3}\sigma_1(n)-\lambda_1(n), \;\; n \text{ odd},$$

the latter of which is due to Eichler. (See also equation 1.3 of [9].) We leave the details to the reader.

4. Proof of proposition 4

This identity is derived analogously to proposition 2, but it arises from the equality $Q_{2,2,0} = E_2$ for the dual Weil representation attached to the quadratic form $2x^2-y^2$. The discriminant group of this quadratic form has nonsquare order 8, so $E_2(\tau)$ is a true modular form; its $\mathfrak{e}_0$-component is

$$E_2(\tau)_0 = 1-18q-34q^2-28q^3-66q^4-56q^5-60q^6-... \in M_2(\Gamma_1(8))$$

in which the coefficient of q^n is

$$-8\sigma_1(n,\chi)\cdot\Big(2+\frac{\chi(m)}{2^{\nu+2}}\Big), \;\; n=2^\nu m, \; m \text{ odd},$$

where χ is the Dirichlet character modulo 8 given by $\chi(1)=\chi(7)=1$, $\chi(3)=\chi(5)=-1$ and where $\sigma_1(n,\chi)=\sum_{d|n}\chi(n/d)d$. (This can be calculated using the coefficient formula of [4], for example.)

The relevant components of the weight-3/2 Eisenstein series for $2x^2 - y^2 + 2z^2$ are

$$E_{3/2}(\tau)_{(0,0,0)} = 1 - 2q - 4q^2 - 4q^4 - 8q^5 - ...$$
$$E_{3/2}(\tau)_{(0,0,1/2)} = -2q^{1/2} - 4q^{3/2} - 4q^{5/2} - 8q^{7/2} - 6q^{9/2} - ...$$
$$E_{3/2}(\tau)_{(0,0,1/4)} = E_{3/2}(\tau)_{(0,0,3/4)} = -4q^{7/8} - 4q^{15/8} - 4q^{23/8} - 8q^{31/8} - 4q^{39/8} - ...$$

Lemma 9. *(i) The coefficient of $q^{n/2}$ in $E_{3/2}(\tau)_{(0,0,1/2)}$ is $(-1/2)$ times the number of representations of $2n$ by the quadratic form $4a^2 + b^2 + c^2$.*
(ii) The coefficient of $q^{n/8}$ in $E_{3/2}(\tau)_{(0,0,1/4)}$ is $(-1/2)$ times the number of representations of n by the quadratic form $4a^2 + 2b^2 + c^2$.

Proof. The components $E_{3/2}(\tau)_{(0,0,1/2)}$ and $E_{3/2}(\tau)_{(0,0,1/4)}$ are modular forms because the components $\mathfrak{e}_{(0,0,1/2)}$ and $\mathfrak{e}_{(0,0,1/4)}$ do not appear in the shadow $\vartheta(\tau)$. Once an equality between two modular forms has been conjectured (here, the components of $E_{3/2}(\tau)$ and two theta series), it can always be proved by comparing a finite number of coefficients. In principle this could also be proven directly via the same argument as lemma 7. ☐

The coefficient formula for the $\mathfrak{e}_0$-component of the index-2 series

$$Q_{2,2,0}(\tau)_0 = \sum_{n=0}^{\infty} b(n)q^n$$

is now

$$b(n) = \sum_{r \in \mathbb{Z}} c(n - r^2/8, (0,0,r/4)) + \frac{1}{8\sqrt{2}} \sum_{r \in \mathbb{Z}} a(n - r^2/8, (0,0,r/4))\Big(|r| - \sqrt{r^2 - 8n}\Big),$$

where $c(n,\gamma)$ is the coefficient of $q^n \mathfrak{e}_\gamma$ in the mock Eisenstein series above and $a(n,\gamma)$ is the coefficient of $q^{-n}\mathfrak{e}_\gamma$ in its shadow. Here,

$$\sum_{r \in \mathbb{Z}} c(n - r^2/8, (0,0,r/4)) = -\sum_{r \in \mathbb{Z}} |\overline{\alpha}_2(n - 2r^2)| - \frac{1}{2} \sum_{r \text{ odd}} \Big(r_A(4n - 2r^2) + r_B(8n - r^2)\Big) + \begin{cases} 4: & 2n = \square; \\ 0: & \text{otherwise.} \end{cases}$$

Here, $r_A(n)$ is the representation count of n by $4a^2 + b^2 + c^2$ and $r_B(n)$ is the representation count of n by $4a^2 + 2b^2 + c^2$.

Remark 10. The generating function of the coefficients $\sum_{r \text{ odd}} r_A(n - 2r^2)$ is the difference of theta functions for the quadratic forms $4a^2 + b^2 + c^2 + 2d^2$ and $4a^2 + b^2 + c^2 + 4d^2$. In particular, $\sum_{n=0}^{\infty} \sum_{r \text{ odd}} r_A(4n - 2r^2)q^n$ is a modular form of weight 2; we can identify it as the eta product

$$\sum_{n=0}^{\infty} \sum_{r \text{ odd}} r_A(4n - 2r^2)q^n = 8q + 16q^2 + 16q^3 + 32q^4 + ... = \frac{8\eta(2\tau)^3\eta(4\tau)\eta(8\tau)^2}{\eta(\tau)^2}.$$

Similarly,

$$\sum_{n=0}^{\infty} \sum_{r \text{ odd}} r_B(8n - r^2)q^n = 16q + 32q^2 + 32q^3 + 64q^4 + ... = \frac{16\eta(2\tau)^3\eta(4\tau)\eta(8\tau)^2}{\eta(\tau)^2}.$$

The eta product $\frac{\eta(2\tau)^3\eta(4\tau)\eta(8\tau)^2}{\eta(\tau)^2}$ is one of the few such products with multiplicative coefficients, as classified by Martin [8], and its coefficient of q^n is the twisted divisor sum $\sigma_1(n,\chi) = \sum_{d|n} \chi(n/d)d$ for the character $\chi(1) = \chi(7) = 1$, $\chi(3) = \chi(5) = -1$ mod 8 that we consider throughout. Therefore, we can simplify the above sum to

$$\sum_{r \in \mathbb{Z}} c(n - r^2/8, (0,0,r/4)) = -\sum_{r \in \mathbb{Z}} |\overline{\alpha}_2(n - 2r^2)| - 12\sigma_1(n,\chi) + \begin{cases} 4: & 2n = \square; \\ 0: & \text{otherwise.} \end{cases}$$

The correction term

$$\frac{1}{8\sqrt{2}}\sum_{r\in\mathbb{Z}} a(n-r^2/8,(0,0,r/4))\Big(|r|-\sqrt{r^2-8n}\Big) = -\sqrt{2}\sum_{\substack{r\in\mathbb{Z}\\ 2(r^2-8n)=\square}}\Big(|r|-\sqrt{r^2-8n}\Big)\times\begin{cases}1: & r^2\neq 8n;\\ 1/2: & r^2=8n;\end{cases}$$

is more difficult to calculate than the corresponding term in the proof of proposition 2 because the discriminant order $|\Lambda'/\Lambda| = 8$ is not square. Following section 7 of [12], this term can be calculated by finding minimal solutions to the Pell-type equation $a^2 - 8b^2 = -64n$.

The true Pell equation $a^2 - 8b^2 = 1$ has fundamental solution $a = 3, b = 1$. We let $\mu_i = a_i + b_i\sqrt{8}$, $i \in \{1,...,N\}$ denote the representatives of orbits of elements in $\mathbb{Z}[\sqrt{2}]$ up to conjugation having norm $2n$ and minimal positive trace; then

$$\frac{1}{\sqrt{2}}\sum_{2(r^2-8n)=\square}\left[\Big(|r|-\sqrt{r^2-8n}\Big)\times\begin{cases}1: & r^2-8n\neq 0;\\ 1/2: & r^2-8n=0;\end{cases}\right] = -4\sum_{i=1}^{N}\Big(|b_i|-\frac{a_i}{2}\Big)\times\begin{cases}2: & \overline{\mu_i}/\mu_i\notin\mathbb{Z}[\sqrt{2}];\\ 1: & \overline{\mu_i}/\mu_i\in\mathbb{Z}[\sqrt{2}].\end{cases}$$

Since $\mathbb{Q}(\sqrt{2})$ has class number one, these orbits correspond to the ideals of $\mathbb{Z}[\sqrt{2}]$ with ideal norm $2n$ and we can write

$$-\sqrt{2}\sum_{2(r^2-8n)=\square}\left[\Big(|r|-\sqrt{r^2-8n}\Big)\times\begin{cases}1: & r^2-8n\neq 0;\\ 1/2: & r^2-8n=0;\end{cases}\right] = 4\sum_{N(\mathfrak{a})=2n}\Big(|b|-a\Big),$$

where $\mathfrak{a}$ runs through the ideals of $\mathbb{Z}[\sqrt{2}]$ of norm $2n$ and $a+b\sqrt{2}\in\mathfrak{a}$ is a generator with minimal $a>0$.

Comparing coefficients between $Q_{2,2,0}(\tau)_0$ and $E_2(\tau)_0$ results in the identity

$$\sum_{r\in\mathbb{Z}}|\overline{\alpha}_2(n-2r^2)| = 2\sigma_1(n,\chi)\cdot\Big(2+\frac{\chi(m)}{2^\nu}\Big)+4\sum_{N(\mathfrak{a})=2n}\Big(|b|-a\Big)+\begin{cases}4: & 2n=\square;\\ 0: & \text{otherwise};\end{cases}$$

as claimed, where $a+b\sqrt{2}\in\mathfrak{a}$ is a generator with minimal $a>0$.

Example 11. Let $n=7$. The ideals of $\mathbb{Z}[\sqrt{2}]$ of norm 14 are $(4\pm\sqrt{2})$ and the trace 8 is minimal within both ideals. The left side of lemma 11 is

$$\sum_{r\in\mathbb{Z}}|\overline{\alpha}_2(7-2r^2)| = |\overline{\alpha}_2(7)|+2\cdot|\overline{\alpha}_2(5)| = 24,$$

while the right side is

$$2\sigma_1(7,\chi)(2+\chi(7))+4(1-4)+4(1-4) = 48-12-12 = 24.$$

Remark 12. When $n=p$ is a prime that remains inert in $\mathbb{Z}[\sqrt{2}]$ (i.e. $\chi(p)=-1$) this identity simplifies to

$$\sum_{r\in\mathbb{Z}}|\overline{\alpha}_2(p-2r^2)| = 2(p-1).$$

To prove the corollary, we again use equation (4). In the first case,

$$\sum_{r}|\overline{\alpha}_2(4n-2r^2)| = 8\sum_{r\in\mathbb{Z}}H(4n-2r^2)-\frac{1}{3}\sum_{r\in\mathbb{Z}}(-1)^r r_3(4n-2r^2).$$

The generating function

$$\sum_{n=0}^{\infty}\sum_{r\in\mathbb{Z}}(-1)^r r_3(4n-2r^2)q^n = 1-18q-34q^2-28q^3-66q^4-56q^5-60q^6-...$$

is a difference of theta functions and therefore a modular form of level 8; and we identify it as $E_2(\tau)_0$, giving the identity

$$\sum_{r\in\mathbb{Z}}(-1)^r r_3(4n-2r^2) = -8\sigma_1(n,\chi)\cdot\Big(2+\frac{\chi(m)}{2^{\nu+2}}\Big),\ \ n=2^\nu m,\ m\text{ odd}.$$

Therefore,

$$\sum_{r\in\mathbb{Z}} H(4n-2r^2) = -\frac{1}{3}\sigma_1(n,\chi)\cdot\Big(2+\frac{\chi(m)}{2^{\nu+2}}\Big) + \frac{1}{4}\sigma_1(4n,\chi)\cdot\Big(2+\frac{\chi(m)}{2^{\nu+2}}\Big) + \frac{1}{2}\sum_{N(\mathfrak{a})=8n}\Big(|b|-a\Big)$$
$$= \frac{2}{3}\sigma_1(n,\chi)\cdot\Big(2+\frac{\chi(m)}{2^{\nu+2}}\Big) + \frac{1}{2}\sum_{N(\mathfrak{a})=8n}\Big(|b|-a\Big).$$

In the second case, for odd n,

$$\sum_r |\overline{\alpha}_2(2n-2r^2)| = 8\sum_{r\in\mathbb{Z}} H(2n-2r^2) + \frac{1}{3}\sum_{r\in\mathbb{Z}}(-1)^r r_3(2n-2r^2).$$

Here,

$$\sum_{r\in\mathbb{Z}}(-1)^r r_3(2n-2r^2) = (8+2\chi(n))\cdot\sigma_1(n,\chi), \quad n \text{ odd},$$

so

$$\sum_{r\in\mathbb{Z}} H(2n-2r^2) = -\frac{8+2\chi(n)}{24}\sigma_1(n,\chi) + \frac{1}{4}\sigma_1(2n,\chi)\cdot\Big(2+\frac{\chi(n)}{2}\Big) + \frac{1}{2}\sum_{N(\mathfrak{a})=4n}\Big(|b|-a\Big)$$
$$= \frac{4+\chi(n)}{6}\sigma_1(n,\chi) + \frac{1}{2}\sum_{N(\mathfrak{a})=4n}\Big(|b|-a\Big).$$

In the third case, for odd n,

$$\sum_r |\overline{\alpha}_2(n-2r^2)| = 8\sum_{r\in\mathbb{Z}} H(n-2r^2) - \frac{\chi(n)}{3}\sum_{r\in\mathbb{Z}}(-1)^r r_3(n-2r^2),$$

where

$$\sum_{r\in\mathbb{Z}}(-1)^r r_3(n-2r^2) = \begin{cases} 6\sigma_1(n,\chi): & n\equiv 1\,(8); \\ -2\sigma_1(n,\chi): & n\equiv 3\,(8); \\ 2\sigma_1(n,\chi): & n\equiv 5\,(8); \\ -6\sigma_1(n,\chi): & n\equiv 7\,(8); \end{cases}$$

and therefore

$$\sum_{r\in\mathbb{Z}} H(n-2r^2) = \frac{2+\chi(n)}{6}\sigma_1(n,\chi) + \frac{1}{2}\sum_{N(\mathfrak{a})=2n}\Big(|b|-a\Big).$$

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