RANKIN-COHEN BRACKETS AND SERRE DERIVATIVES AS POINCARÉ SERIES

BRANDON WILLIAMS

Abstract. We give expressions for the Serre derivatives of Eisenstein and Poincaré series as well as their Rankin-Cohen brackets with arbitrary modular forms in terms of the Poincaré averaging construction, and derive several identities for the Ramanujan tau function as applications.

1. Introduction

Let $k \in 2\mathbb{Z}$, $k \geq 4$. To any q-series $\phi(q) = \phi(e^{2\pi i \tau}) = \sum_{n=0}^\infty a_n q^n$ on the upper half-plane $\tau \in \mathbb{H}$ whose coefficients grow slowly enough, one can construct a **Poincaré series**

$$\mathbb{P}_k(\phi;\tau) = \sum_{M \in \Gamma_\infty \backslash \Gamma} \phi|_k M(\tau) = \frac{1}{2} \sum_{c,d} \sum_{n=0}^\infty a_n (c\tau+d)^{-k} e^{2\pi i n \frac{a\tau+b}{c\tau+d}}$$

that converges absolutely and uniformly on compact subsets and defines a modular form of weight k. Here, the first sum is taken over cosets of $\Gamma = SL_2(\mathbb{Z})$ by the subgroup Γ_∞ generated by $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the second over all coprime integers $c, d \in \mathbb{Z}$. As usual, $|_k$ is the Petersson slash operator of weight k. (More generally one can also construct Poincaré series that are not holomorphic in this way; see [7], section 8.3 for some applications.)

It is easy to show that every modular form f (of weight $k \geq 4$) can be written as a Poincaré series $\mathbb{P}_k(\phi)$: because f can always be written as a linear combination of the Eisenstein series $E_k = \mathbb{P}_k(1)$ and the Poincaré series of exponential type $P_{k,N} = \mathbb{P}_k(q^N)$ of various indices N. However, expressions found by this argument tend to be messy because the coefficients of $P_{k,N}$ are complicated series over Kloosterman sums and special values of Bessel functions ([4], section 3.2). The most reliable way to produce Poincaré series with manageable Fourier coefficients seems to be to start with seed functions $\phi(\tau)$ that already behave in a manageable way under the action of $SL_2(\mathbb{Z})$.

Example 1. When $\phi = 1$ (a modular form of weight 0), we obtain the normalized Eisenstein series as mentioned above:

$$\mathbb{P}_k(1;\tau) = E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^\infty \sigma_{k-1}(n) q^n, \quad q = e^{2\pi i \tau}, \ \sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

where B_k is the k-th Bernoulli number. More generally, if ϕ is a modular form of any weight k then expanding formally yields

$$\begin{aligned} \mathbb{P}_{k+l}(\phi;\tau) &= \sum_{M \in \Gamma_\infty \backslash \Gamma} (c\tau+d)^{-k-l} \phi\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= \sum_{M \in \Gamma_\infty \backslash \Gamma} (c\tau+d)^{-k-l} (c\tau+d)^k \phi(\tau) \\ &= \phi(\tau) E_l(\tau), \end{aligned}$$

where M is the coset of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$; although the expression $\mathbb{P}_{k+l}(\phi)$ makes sense only when l is sufficiently large compared to the growth of the coefficients of ϕ. In recent work [8] the author has considered the Poincaré series $\mathbb{P}_k(\vartheta)$ constructed from what are essentially weight $1/2$ theta functions ϑ, which seem to be useful

2010 *Mathematics Subject Classification.* 11F11,11F25.

for computing with vector-valued modular forms for Weil representations; the details are somewhat more involved but this is related to the example above.

The motivation of this note was to consider the Poincaré series $\mathbb{P}_k(\phi)$ when ϕ is a **quasimodular form**, a more general class of functions which includes modular forms, their derivatives of all orders, and the series

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$$

(cf. [11], section 5.3). We find that one obtains Rankin-Cohen brackets and Serre derivatives (see section 2 below for their definitions) of Eisenstein series and Poincaré series essentially from such forms ϕ:

Theorem 2. *For any modular form $f \in M_k$ and $l \in 2\mathbb{N}$, $l \geq 4$, and $m, N \in \mathbb{N}_0$, with $l \geq k+2$ if f is not a cusp form, set*

$$\phi(\tau) = q^N \sum_{r=0}^{m} (-1)^r \binom{k+m-1}{m-r}\binom{l+m-1}{r} N^{m-r} D^r f(\tau);$$

then

$$[f, P_{l,N}]_m = \mathbb{P}_{k+l+2m}(\phi).$$

Here $D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$. Since $\mathbb{P}_{k+l+2m}(\phi)$ is modular by construction, and since $[-,-]_m$ is bilinear and $P_{l,N}$, $N \in \mathbb{N}_0$ span all modular forms, this gives another proof of the modularity of Rankin-Cohen brackets (at least for large l). This seed function ϕ is formally the Rankin-Cohen bracket $[f, q^N]_m$ where q^N is treated like a modular form of weight l, so by linearity we see that Rankin-Cohen brackets and Poincaré averaging "commute" in the following sense:

Corollary 3. *Let f be a modular form of weight k and let ϕ be a q-series whose coefficients grow sufficiently slowly that $\mathbb{P}_l(\phi; \tau)$ is well-defined, and denote by $[f, \phi]_m$ the formal result of the m-th Rankin-Cohen bracket where ϕ is treated like a modular form of weight l (where $l \geq k+2$ if f is not a cusp form). Then*

$$[f, \mathbb{P}_l(\phi)]_m = \mathbb{P}_{k+l+2m}([f, \phi]_m).$$

This expression simplifies considerably for the Eisenstein series:

$$[f, E_l]_m = \mathbb{P}_{k+l+2m}(\phi)$$

for the function

$$\phi = (-1)^m \binom{l+m-1}{m} D^m f.$$

An equivalent result in this case has appeared in section 5 of [9] (in particular see Proposition 6). There may be particular interest in the case that f itself is an Eisenstein or Poincaré series as expressions of a different nature for the Rankin-Cohen brackets of two Poincaré series are known (e.g. [1], section 6).

Theorem 4. *For any $m, N \in \mathbb{N}_0$ and $l \in 2\mathbb{N}$ with $l \geq 2m+2$, set*

$$\phi(\tau) = q^N \sum_{r=0}^{m} \binom{m}{r} \frac{(l+m-1)!}{(l+m-r-1)!} (-E_2(\tau)/12)^r N^{m-r};$$

then the m-th order Serre derivative (in the sense of section 2) of $P_{l,N}$ is

$$\vartheta^{[m]} P_{l,N} = \mathbb{P}_{l+2m}(\phi).$$

Similarly, this seed function ϕ is formally the m-th Serre derivative of q^N if one pretends that q^N is a modular form of weight l; by linearity we find that Serre derivatives also commute with Poincaré averaging:

Corollary 5. *Let ϕ be a q-series whose coefficients grow sufficiently slowly that $\mathbb{P}_l(\phi; \tau)$ is well-defined, and denote by $\vartheta^{[m]}\phi$ the formal result of the m-th order Serre derivative where ϕ is treated like a modular form of weight l (where $l \geq 2m+2$). Then*

$$\vartheta^{[m]} \mathbb{P}_l(\phi) = \mathbb{P}_{l+2m}(\vartheta^{[m]}\phi).$$

As before, this simplifies for the Eisenstein series:

$$\vartheta^{[m]} E_l = \mathbb{P}_{l+2m}(\phi)$$

for the function

$$\phi = \frac{(l+m-1)!}{(-12)^m (l-1)!} E_2^m.$$

It is interesting to compare this to Theorem 2 which suggests that the Serre derivative (at least of the Eisenstein series) is analogous to a Rankin-Cohen bracket with E_2. Similar observations have been made before (e.g. [2], section 2).

By computing Rankin-Cohen brackets and Serre derivatives of $P_{l,N} = 0$ in weights $l \leq 10$ we can obtain new proofs of Kumar's identity ([6], eq. (14))

$$\tau(m) = -\frac{20m^{11}}{m-5/6} \sum_{n=1}^{\infty} \frac{\sigma_1(n)\tau(m+n)}{(m+n)^{11}}$$

and Herrero's identity ([3], eq. (1))

$$\tau(m) = -240 m^{11} \sum_{n=1}^{\infty} \frac{\sigma_3(n)\tau(m+n)}{(m+n)^{11}}$$

that express the Ramanujan tau function in terms of special values of a shifted L-series introduced by Kohnen [5], as well as four additional identities of this form. Namely we find

$$\begin{aligned} \tau(m) &= -\frac{14m^8}{m-7/12} \sum_{n=1}^{\infty} \frac{\sigma_1(n)\tau(m+n)}{(m+n)^8} \\ &= -\frac{16m^9}{m-2/3} \sum_{n=1}^{\infty} \frac{\sigma_1(n)\tau(m+n)}{(m+n)^9} \\ &= -\frac{18m^{10}}{m-3/4} \sum_{n=1}^{\infty} \frac{\sigma_1(n)\tau(m+n)}{(m+n)^{10}} \\ &= -240 m^{10} \sum_{n=1}^{\infty} \frac{\sigma_3(n)\tau(m+n)}{(m+n)^{10}}. \end{aligned}$$

Here $\tau(m)$ is Ramanujan's tau function, i.e. the coefficient of q^m in $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. We can also compute the values of these series with $m = 0$. Based on numerical computations it seems reasonable to guess that there are no other identities of this type. The details are worked out in section 5.

2. Background and notation

Let $\mathbb{H} = \{\tau = x + iy : y > 0\}$ be the upper half-plane and let Γ be the group $\Gamma = SL_2(\mathbb{Z})$, which acts on $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$. A **modular form of weight** k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which transforms under Γ by

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ \tau \in \mathbb{H}$$

and whose Fourier expansion involves only non-negative exponents: $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$, $q = e^{2\pi i \tau}$. We denote by M_k the space of modular forms of weight k and by S_k the subspace of cusp forms (which in this context means $a_0 = 0$).

The **Rankin-Cohen brackets** are bilinear maps

$$[\cdot,\cdot]_n : M_k \times M_l \to M_{k+l+2n},$$

(1)
$$[f,g]_n = \sum_{j=0}^{n} (-1)^j \binom{k+n-1}{n-j}\binom{l+n-1}{j} D^j f D^{n-j} g,$$

where $D^j f(\tau) = \frac{1}{(2\pi i)^j}\frac{d^j}{d\tau^j}f(\tau) = \frac{1}{(2\pi i)^j}f^{(j)}(\tau)$. If $f(\tau) = \sum_{n=0}^\infty a_n q^n$ is the Fourier expansion of f then

$$D^j f(\tau) = \sum_{n=0}^\infty a_n n^j q^n;$$

and in particular, the Rankin-Cohen brackets preserve integrality of Fourier coefficients. For example, the first few brackets are

$$[f,g]_0 = fg, \;\; [f,g]_1 = kf\cdot Dg - lg\cdot Df,$$
$$[f,g]_2 = \frac{k(k+1)}{2}f\cdot D^2 g - (k+1)(l+1)Df\cdot Dg + \frac{l(l+1)}{2}D^2 f\cdot g.$$

These can be characterized as the unique (up to scale) bilinear differential operators of degree $2n$ that preserve modularity (see for example the second proof in section 1 of [10]).

The Serre derivatives $\vartheta^{[n]}$ following [11], section 5.1 are maps $M_k \to M_{k+2n}$ defined recursively by

$$\vartheta^{[0]}f = f, \;\; \vartheta^{[1]}f = \vartheta f = Df - \frac{k}{12}E_2 f,$$

and

$$\vartheta^{[n+1]}f = \vartheta\vartheta^{[n]}f - \frac{n(k+n-1)}{144}E_4 f, \;\; n \geq 1.$$

(In particular $\vartheta^{[n]}$ is *not* simply the n-th iterate of ϑ.) These functions are given in closed form by

$$\vartheta^{[n]}f(\tau) = \sum_{r=0}^n \binom{n}{r}\frac{(k+n-1)!}{(k+r-1)!}(-E_2(\tau)/12)^{n-r}D^r f(\tau), \tag{2}$$

as one can prove by induction or by inverting equation 65 of [11] (section 5.2).

3. Poincaré series

Remark 6. A sufficient criterion for the series

$$\mathbb{P}_k(\phi;\tau) = \sum_{c,d}(c\tau+d)^{-k}\phi\Big(\frac{a\tau+b}{c\tau+d}\Big), \;\; \phi(\tau) = \sum_{n=0}^\infty a_n q^n$$

to converge absolutely and locally uniformly is for the coefficients of ϕ to satisfy the bound $a_n = O(n^l)$ where $l = \frac{k}{2} - 2 - \varepsilon$ for some $\varepsilon > 0$. To see this, note that $\binom{n+l}{l}$ is also $O(n^l)$, so we can bound

$$\Big|\phi\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big| \ll \sum_{n=0}^\infty \binom{n+l}{l}e^{-2\pi n\frac{1}{|c\tau+d|^2}} = \Big(1 - e^{-\frac{2\pi}{|c\tau+d|^2}}\Big)^{-l-1}$$

up to a constant multiple. Since $(1-e^{-x})^{-1} < x^{-1-\delta}$ for any fixed (small enough) $\delta > 0$ and all small enough $x > 0$, we can then bound

$$\sum_{c,d}\Big|(c\tau+d)^{-k}\phi\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big| \ll \sum_{c,d}|c\tau+d|^{-k+2(l+1)(1+\delta)} < \sum_{c,d}|c\tau+d|^{-2}.$$

Remark 7. Given a q-series $\phi(\tau) = \sum_{n=0}^\infty a_n q^n$, one can also consider the series

$$\mathbb{P}_k'(\phi) = a_0 E_k + \sum_{n=1}^\infty a_n P_{k,n}$$

which generally has better convergence properties than the sum $\mathbb{P}_k(\phi)$ over cosets $\Gamma_\infty\backslash\Gamma$. Since S_k is finite-dimensional, the convergence of $\sum_{n=1}^\infty a_n P_{k,n}$ to a cusp form in any sense is equivalent to the convergence of the series

$$\sum_{n=1}^\infty a_n\langle f, P_{k,n}\rangle = \frac{(k-2)!}{(4\pi)^{k-1}}\sum_{n=1}^\infty \frac{a_n b_n}{n^{k-1}}$$

for every cusp form $f(\tau) = \sum_{n=1}^\infty b_n q^n \in S_k$. The Deligne bound $b_n = O(n^{(k-1)/2+\varepsilon})$ implies that this is satisfied when the slightly weaker bound $a_n = O(n^{k/2-3/2-\varepsilon})$ holds. It is clear that $\mathbb{P}_k'(\phi) = \mathbb{P}_k(\phi)$ whenever the latter series converges, so we will refer to both of these series by $\mathbb{P}_k(\phi;\tau)$ in what follows.

4. Proofs

Proof of Theorem 2. The coefficients a_n of any modular form of weight k satisfy the bound $a_n = O(n^{k-1+\varepsilon})$ for any $\varepsilon > 0$, while cusp forms satisfy the Deligne bound $a_n = O(n^{(k-1)/2+\varepsilon})$. In particular, the coefficients of

$$\phi(\tau) = q^N \sum_{r=0}^{m} (-1)^r \binom{k+m-1}{n-r} \binom{l+m-1}{r} N^{m-r} D^r f(\tau)$$

always satisfy the bound $O(n^{k+m-1+\varepsilon})$, and our growth condition (of Remark 7),

$$k+m-1+\varepsilon \leq \frac{k+l+2m}{2} - 3/2 - \varepsilon$$

becomes $k \leq l-1-2\varepsilon$ and therefore (since $k, l \in 2\mathbb{Z}$) $l \geq k+2$; while for cusp forms we instead require

$$\frac{k-1}{2} + m + \varepsilon \leq \frac{k+l+2m}{2} - 3/2 - \varepsilon,$$

or equivalently $2 \leq l - 2\varepsilon$ which is always satisfied for small enough ε.

Suppose first that the series $\mathbb{P}(\phi;\tau)$ over cosets $\Gamma_\infty \backslash \Gamma$ converges normally. Repeatedly differentiating the equation

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k f(\tau)$$

yields

$$f^{(m)}\Big(\frac{a\tau+b}{c\tau+d}\Big) = \sum_{r=0}^{m} \binom{m}{r} \frac{(k+m-1)!}{(k+r-1)!} c^{m-r} (c\tau+d)^{k+m+r} f^{(r)}(\tau),$$

as one can prove by induction or derive directly by considering the action of $SL_2(\mathbb{Z})$ on τ in the generating series

$$\sum_{m=0}^{\infty} f^{(m)}(\tau) \frac{w^m}{m!} = f(\tau+w),$$

for $|w|$ sufficiently small. By another induction argument one finds the similar formula

$$\frac{d^m}{d\tau^m}\Big((c\tau+d)^{-k} e^{2\pi i N\tau}\Big) = \sum_{r=0}^{m} \binom{m}{r} \frac{(k+m-1)!}{(k+r-1)!} (-c)^{m-r} (2\pi i N)^r (c\tau+d)^{-k-m-r} e^{2\pi i N\tau} \tag{3}$$

for any $N \in \mathbb{N}_0$.

Let $(a)_m = \frac{(a+m-1)!}{(a-1)!} = a \cdot (a+1) \cdot \ldots \cdot (a+m-1)$ denote the Pochhammer symbol. Then

$$\begin{aligned}
&\sum_{M \in \Gamma_\infty \backslash \Gamma} \Big[q^N \sum_{r=0}^{m} (-1)^r \binom{k+m-1}{m-r} \binom{l+m-1}{r} N^{m-r} D^r f(\tau)\Big]\Big|_{k+l+2m} M(\tau) \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} \sum_{r=0}^{m} \sum_{j=0}^{r} \Big[\binom{k+m-1}{m-r} \binom{l+m-1}{r} \binom{r}{j} N^{m-r} \times \\
&\quad = (-2\pi i)^{-r} c^{r-j} (k+j)_{r-j} (c\tau+d)^{j+r-l-2m} e^{2\pi i N \frac{a\tau+b}{c\tau+d}} f^{(j)}(\tau)\Big] \\
&= (-1)^m \sum_{M \in \Gamma_\infty \backslash \Gamma} \sum_{j=0}^{m} (-1)^j f^{(j)}(\tau) \sum_{r=0}^{m-j} \Big[(2\pi i N)^r \binom{k+m-1}{r} \binom{l+m-1}{m-r} \binom{m-r}{j} (k+j)_{m-j-r} \times \\
&\qquad \times (-c)^{m-r-j} (c\tau+d)^{j-l-m-r} e^{2\pi i N \frac{a\tau+b}{c\tau+d}}\Big],
\end{aligned}$$

where we have replaced r by $m-r$ in the second equality. Since

$$\begin{aligned}
&\binom{k+m-1}{r}\binom{l+m-r}{m-r}\binom{m-r}{j}(k+j)_{m-j-r} \\
&= \frac{(k+m-1)!(l+m-1)!(m-r)!(k+m-r-1)!}{r!(k+m-r-1)!(m-r)!(l+r-1)!j!(m-r-j)!(k+j-1)!} \\
&= \binom{k+m-1}{m-j}\binom{l+m-1}{j}\binom{m-j}{r}(l+r)_{m-j-r},
\end{aligned}$$

as we see by replacing $\frac{(m-r)!(k+m-r-1)!}{(m-r)!(k+m-r-1)!}$ by $\frac{(m-j)!(l+m-j-1)!}{(m-j)!(l+m-j-1)!}$ in the above expression, this equals

$$\begin{aligned}
&(2\pi i)^{-m}\sum_{j=0}^{m}\Big[(-1)^j f^{(j)}(\tau)\binom{k+m-1}{m-j}\binom{l+m-1}{j}\times \\
&\qquad\times\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{r=0}^{m-j}(2\pi iN)^r\binom{m-j}{r}(l+r)_{m-j-r}(-c)^{m-r-j}(c\tau+d)^{j-l-m-r}e^{2\pi iN\frac{a\tau+b}{c\tau+d}}\Big] \\
&= \sum_{j=0}^{m}(-1)^j D^j f(\tau)\binom{k+m-1}{m-j}\binom{l+m-1}{j}\sum_{M\in\Gamma_\infty\backslash\Gamma}D^{m-j}\Big((c\tau+d)^{-l}e^{2\pi iN\frac{a\tau+b}{c\tau+d}}\Big) \\
&= \sum_{j=0}^{m}(-1)^j\binom{k+m-1}{m-j}\binom{l+m-1}{j}D^j f(\tau)D^{m-j}P_{l,N}(\tau) \\
&= [f,P_{l,N}]_m(\tau),
\end{aligned}$$

the last equality by definition (equation (1)), and the third-to-last equality using equation (3).

When ϕ satisfies the weaker growth condition, we can include a convergence factor $(c\overline{\tau}+d)^{-s}$ into the argument above (which is ignored by the operator D) to see that, if $\phi(\tau)=a_0+a_1q+a_2q^2+...$, then

$$\begin{aligned}
&a_0E_k(\tau;s)+a_1P_{k,1}(\tau;s)+a_2P_{k,2}(\tau;s)+... \\
&= \sum_{M\in\Gamma_\infty\backslash\Gamma}(c\overline{\tau}+d)^{-s}\times\phi(\tau)\Big|_{k+l+2m}M \\
&= \sum_{j=0}^{m}(-1)^j\binom{k+m-1}{m-j}\binom{l+m-1}{j}D^j f(\tau)D^{m-j}\Big(\sum_{M\in\Gamma_\infty\backslash\Gamma}(c\overline{\tau}+d)^{-s}\times q^N\Big|_l M\Big)
\end{aligned}$$

when $\mathrm{Re}[s]$ is sufficiently large, and $E_k(\tau;s)$ and $P_{k,N}(\tau;s)$ denote the deformed series

$$E_k(\tau;s)=\frac{1}{2}\sum_{c,d}\frac{1}{(c\tau+d)^k(c\overline{\tau}+d)^s},\quad P_{k,N}(\tau;s)=\frac{1}{2}\sum_{c,d}\frac{e^{2\pi iN\frac{a\tau+b}{c\tau+d}}}{(c\tau+d)^k(c\overline{\tau}+d)^s}.$$

The claim follows by analytic continuation to $s=0$.

□

Proof of Theorem 4. The condition $l\geq 2m+2$ makes the Fourier coefficients of ϕ grow sufficiently slowly: the n-th coefficient of E_2^m is $O(n^{2m-1+\varepsilon})$ for any $\varepsilon>0$, so the growth condition

$$2m-1+\varepsilon\leq\frac{l+2m}{2}-3/2-\varepsilon$$

of Remark 7 is satisfied for all $l\geq 2m+2$.

Using the transformation law

$$E_2\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^2E_2(\tau)+\frac{6}{\pi i}c(c\tau+d),$$

it follows that

$$E_2\Big(\frac{a\tau+b}{c\tau+d}\Big)^m = \sum_{r=0}^m \binom{m}{r}(c\tau+d)^{m+r}c^{m-r}\Big(\frac{12}{2\pi i}\Big)^{m-r}E_2(\tau)^r$$

for all $m \in \mathbb{N}$. Therefore, with

$$\phi(\tau) = q^N \sum_{r=0}^m \binom{m}{r}\frac{(l+m-1)!}{(l+m-r-1)!}(-E_2(\tau)/12)^r N^{m-r},$$

ignoring convergence issues for now, we find

$$\begin{aligned}
&\sum_{M \in \Gamma_\infty \backslash \Gamma} \phi\Big|_{l+2m} M(\tau) \\
&= \sum_{r=0}^m (-12)^{-r}N^{m-r}\binom{m}{r}\frac{(l+m-1)!}{(l+m-r-1)!}\sum_M \sum_{j=0}^r \binom{r}{j} c^{r-j}(c\tau+d)^{r+j}(12/2\pi i)^{r-j}E_2(\tau)^j e^{2\pi i N\frac{a\tau+b}{c\tau+d}} \\
&= \sum_{j=0}^m \sum_{r=j}^m (-12)^{-r}N^{m-r}\binom{m}{r}\frac{(l+m-1)!}{(l+m-r-1)!}\binom{r}{j}(12/2\pi i)^{r-j}E_2(\tau)^j \sum_M \Big[c^{r-j}(c\tau+d)^{r+j-l-2m}e^{2\pi i N\frac{a\tau+b}{c\tau+d}}\Big] \\
&= \sum_{j=0}^m E_2(\tau)^j \sum_{r=0}^{m-j}(-12)^{r-m}N^r\binom{m}{r}\frac{(l+m-1)!}{(l+r-1)!}\binom{m-r}{j}(12/2\pi i)^{m-r-j}\sum_M c^{m-r-j}(c\tau+d)^{j-l-m-r}e^{2\pi i N\frac{a\tau+b}{c\tau+d}},
\end{aligned}$$

where in the last line we replaced r by $m-r$. Since

$$\begin{aligned}
&(-12)^{r-m}N^r\binom{m}{r}\frac{(l+m-1)!}{(l+r-1)!}\binom{m-r}{j}(-12/2\pi i)^{m-r-j} \\
&= (2\pi i)^{-m}\binom{m}{j}\frac{(l+m-1)!}{(l+m-j-1)!}(-2\pi i/12)^j\binom{m-j}{r}\frac{(l+m-j-1)!}{(l+r-1)!}(2\pi i N)^r,
\end{aligned}$$

as one can see by expanding both sides of this equation, the expression above equals

$$\begin{aligned}
&(2\pi i)^{-m}\sum_{j=0}^m E_2(\tau)^j \sum_{r=0}^{m-j}\Big[\binom{m}{j}\frac{(l+m-1)!}{(l+m-j-1)!}(-2\pi i/12)^j\binom{m-j}{r}\frac{(l+m-j-1)!}{(l+r-1)!}\times \\
&\qquad \times \sum_M (-c)^{m-j-r}(c\tau+d)^{j-l-m-r}(2\pi i N)^r e^{2\pi i N\frac{a\tau+b}{c\tau+d}}\Big] \\
&= \sum_{j=0}^m \binom{m}{j}\frac{(l+m-1)!}{(l+m-j-1)!}\Big(-E_2(\tau)/12\Big)^j D^{m-j}P_{l,N}(\tau) \\
&= \vartheta^{[m]}P_{l,N}(\tau),
\end{aligned}$$

using equation (2) from section 2. Convergence issues can be resolved by including the factor $(c\overline{\tau}+d)^{-s}$ as in the proof of Theorem 2. ☐

5. Examples involving Ramanujan's tau function

In weight 12, the space S_k of cusp forms is one-dimensional and therefore all Poincaré series are multiples of the discriminant $\Delta(\tau) = \sum_{n=1}^\infty \tau(n)q^n$; we find this multiple by writing $P_{k,m} = \lambda_m \Delta$ and using $\lambda_m\langle \Delta, \Delta\rangle = \langle \Delta, P_{k,m}\rangle = \tau(m)\frac{10!}{(4\pi m)^{11}}$, such that

$$P_{k,m} = \frac{10!\cdot \tau(m)}{(4\pi m)^{11}\langle \Delta,\Delta\rangle}\Delta.$$

We can form the Poincaré series $\mathbb{P}_{12}(\phi)$ from any q-series $\phi(\tau) = \sum_{n=0}^\infty a_n q^n$ with $a_n = O(n^{9/2-\varepsilon})$. This includes the q-series E_2 and E_4 and some of their derivatives. Applying Theorems 2 and 4 together with the vanishing of cusp forms in weight ≤ 10 gives identities involving $\tau(n)$. (Similar arguments can be used to derive identities for the coefficients of the normalized cusp forms of weights $16, 18, 20, 22, 26$.)

Example 8. By Theorem 4,

$$0 = \vartheta P_{10,m} = \mathbb{P}_{12}\Big[q^m\Big(m - \frac{5}{6}E_2\Big)\Big] = (m-5/6)P_{12,m} + (-5/6)\cdot(-24)\sum_{n=1}^{\infty}\sigma_1(n)P_{12,m+n},$$

so we recover Kumar's identity

$$\tau(m) = -\frac{20m^{11}}{m-5/6}\sum_{n=1}^{\infty}\frac{\tau(m+n)\sigma_1(n)}{(m+n)^{11}}.$$

Example 9. By Theorem 2,

$$0 = P_{8,m}E_4 = \mathbb{P}_{12}(q^m E_4) = P_{12,m} - 240\sum_{n=1}^{\infty}\sigma_3(n)P_{12,m+n},$$

which yields Herrero's identity

$$\tau(m) = -240m^{11}\sum_{n=1}^{\infty}\frac{\sigma_3(n)\tau(m+n)}{(m+n)^{11}}.$$

Example 10. By Theorem 2,

$$0 = [E_4, P_{6,m}]_1 = \mathbb{P}_{12}\Big(4mE_4 + 6DE_4\Big) = 4mP_{12,m} + 240\sum_{n=1}^{\infty}(4m+6n)\sigma_3(n)P_{12,m+n},$$

which implies

$$\tau(m) = -60m^{10}\sum_{n=1}^{\infty}\frac{(4m+6n)\sigma_3(n)\tau(m+n)}{(m+n)^{11}}.$$

Together with the previous identity this implies

$$\tau(m) = -240m^{10}\sum_{n=1}^{\infty}\frac{(m+n)\sigma_3(n)\tau(m+n)}{(m+n)^{11}} = -240m^{10}\sum_{n=1}^{\infty}\frac{\sigma_3(n)\tau(m+n)}{(m+n)^{10}}.$$

Example 11. By Theorem 4,

$$0 = \vartheta^{[2]}P_{8,m} = \mathbb{P}_{12}\Big(q^m(m^2 - (3/2)mE_2(\tau) + (1/2)E_2(\tau)^2)\Big),$$

where by Ramanujan's equation $DE_2 = \frac{1}{12}(E_2^2 - E_4)$ the coefficient of q^n in $E_2(\tau)^2$ is $240\sigma_3(n) - 288n\sigma_1(n)$. Therefore we find

$$0 = \left(m^2 - \frac{3}{2}m + \frac{1}{2}\right)P_{12,m} + \sum_{n=1}^{\infty}\Big(36m\sigma_1(n) + 120\sigma_3(n) - 144n\sigma_1(n)\Big)P_{12,m+n}$$

and therefore

$$(2m^2 - 3m + 1)\tau(m) = -24m^{11}\sum_{n=1}^{\infty}\frac{((3m-12n)\sigma_1(n) + 10\sigma_3(n))\tau(m+n)}{(m+n)^{11}}, \quad m \in \mathbb{N}.$$

Combining this with the previous examples, we find

$$\tau(m) = -180m^9\sum_{n=1}^{\infty}\frac{n\sigma_1(n)\tau(m+n)}{(m+n)^{11}}$$

and therefore

$$\tau(m) = -\frac{18m^{10}}{m-3/4}\sum_{n=1}^{\infty}\frac{\sigma_1(n)\tau(m+n)}{(m+n)^{10}}.$$

Example 12. It is not valid to form the Poincaré series $\mathbb{P}_{12}(\phi)$ with either $\phi = E_2^3$ or E_6, because their Fourier coefficients grow too quickly; however, their difference $E_2^3 - E_6 = 9DE_4 + 72D^2E_2$ has coefficients that satisfy the required bound $O(n^{9/2-\varepsilon})$. We use

$$\begin{aligned} 0 &= \vartheta^{[3]} P_{6,m} + \frac{7}{36} P_{6,m} E_6 \\ &= \mathbb{P}_{12}\Big(q^m(m^3 - 2m^2 E_2 + (7/6)mE_2^2 - (7/36)(E_2^3 - E_6))\Big) \\ &= \left(m^3 - 2m^2 + \frac{7}{6}m\right) P_{12,m} + \sum_{n=1}^\infty \Big[(48m^2 - 336mn + 336n^2)\sigma_1(n) + (280m - 420n)\Big] P_{12,m+n} \end{aligned}$$

to obtain

$$\tau(m) = -720m^8 \sum_{n=1}^\infty \frac{n^2 \sigma_1(n)\tau(m+n)}{(m+n)^{11}},$$

and combining this with the previous examples,

$$\tau(m) = -\frac{16m^9}{m - 2/3} \sum_{n=1}^\infty \frac{\sigma_1(n)\tau(m+n)}{(m+n)^9}.$$

Similarly, by expressing $D^3 E_2$ in terms of powers of E_2 and derivatives of modular forms one obtains the formula

$$\tau(m) = -\frac{14m^8}{m - 7/12} \sum_{n=1}^\infty \frac{\sigma_1(n)\tau(m+n)}{(m+n)^8}.$$

Remark 13. In particular, for any $m \in \mathbb{N}$ the values of the L-series $\sum_{n=1}^\infty \frac{\sigma_1(n)\tau(m+n)}{(m+n)^s}$ at $s = 8, 9, 10, 11$ and of $\sum_{n=1}^\infty \frac{\sigma_3(n)\tau(m+n)}{(m+n)^s}$ at $s = 10, 11$ are rational numbers, and Lehmer's conjecture that $\tau(n)$ is never zero is equivalent to the non-vanishing of any of these L-values. Computing these L-series at other integers s numerically does not seem to yield rational numbers. In any case, the methods of this note do not apply to other values of s.

We can also evaluate the values of these L-series with $m = 0$ by a similar argument. Comparing

$$\vartheta E_{10} = -\frac{5}{6} - 24q - ... = -\frac{5}{6} E_{12} + \frac{38016}{691}\Delta$$

with the result of Theorem 4,

$$\vartheta E_{10} = -\frac{5}{6} E_{12} + 20 \sum_{n=1}^\infty \sigma_1(n) P_{12,n}$$

we find

$$\tau(m) = \frac{20 \cdot 691}{38016} \sum_{n=1}^\infty \tau(n)\tau(m)\sigma_1(n) \cdot \frac{10!}{\langle \Delta, \Delta \rangle \cdot (4\pi n)^{11}},$$

i.e.

$$\sum_{n=1}^\infty \frac{\tau(n)\sigma_1(n)}{n^{11}} = \frac{2^{19} \cdot 11}{3 \cdot 5^3 \cdot 7 \cdot 691} \pi^{11} \langle \Delta, \Delta \rangle \approx 0.968.$$

Here, the Petersson norm-square of Δ to 18 decimal places is

$$\langle \Delta, \Delta \rangle \approx 1.03536205680 \times 10^{-6}$$

which can be computed using PARI/GP.

Similarly, comparing $E_8 E_4 = 1 + 720q + ... = E_{12} + \frac{432000}{691}\Delta$ with

$$E_8 E_4 = \mathbb{P}_{12}(E_4) = E_{12} + 240 \sum_{n=1}^\infty \sigma_3(n) P_{12,n},$$

we find

$$\tau(m) = \frac{240 \cdot 691}{432000} \sum_{n=1}^{\infty} \tau(n)\tau(m)\sigma_3(n) \cdot \frac{10!}{\langle \Delta, \Delta \rangle \cdot (4\pi n)^{11}},$$

i.e.

$$\sum_{n=1}^{\infty} \frac{\tau(n)\sigma_3(n)}{n^{11}} = \frac{2^{17}}{3^2 \cdot 7 \cdot 691} \pi^{11} \langle \Delta, \Delta \rangle \approx 0.917.$$

With similar arguments applied to

$$\begin{aligned} -3456\Delta &= [E_4, E_6]_1 = -6\mathbb{P}_{12}(DE_4), \\ \frac{1}{2}E_{12} - \frac{49344}{691}\Delta &= \vartheta^{[2]}E_8 = \frac{1}{2}\mathbb{P}_{12}(E_2^2), \\ -168\Delta &= \vartheta^{[3]}E_6 + \frac{7}{36}E_6^2 = \frac{7}{36}\mathbb{P}(E_6 - E_2^3), \end{aligned}$$

and

$$-600\Delta = \vartheta^{[4]}E_4 - \frac{35}{864}E_4E_8 - \frac{7}{40}[E_4, E_4]_2 + \frac{35}{432}[E_6, E_4]_1 = \frac{35}{3}\mathbb{P}_{12}(D^3E_2),$$

one can compute the values

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\tau(n)\sigma_3(n)}{n^{10}} &= \frac{2^{16}}{3^3 \cdot 5^3 \cdot 7} \pi^{11} \langle \Delta, \Delta \rangle \approx 0.845, \\ \sum_{n=1}^{\infty} \frac{\tau(n)\sigma_1(n)}{n^{10}} &= \frac{2^{17}}{3^5 \cdot 5^2 \cdot 7} \pi^{11} \langle \Delta, \Delta \rangle \approx 0.939, \\ \sum_{n=1}^{\infty} \frac{\tau(n)\sigma_1(n)}{n^{9}} &= \frac{2^{13}}{3^4 \cdot 5 \cdot 7} \pi^{11} \langle \Delta, \Delta \rangle \approx 0.880, \\ \sum_{n=1}^{\infty} \frac{\tau(n)\sigma_1(n)}{n^{8}} &= \frac{2^{14}}{3^3 \cdot 5 \cdot 7^2} \pi^{11} \langle \Delta, \Delta \rangle \approx 0.754. \end{aligned}$$

Unlike the L-values of examples 8 through 12, none of these are expected to be rational.

Acknowledgments: I thank the reviewers for pointing out a mistake in Remark 6 in an earlier version of this note, and also for several suggestions that improved the exposition.

References

[1] Nikolaos Diamantis and Cormac O'Sullivan. Kernels for products of L-functions. *Algebra Number Theory*, 7(8):1883–1917, 2013. ISSN 1937-0652. URL `https://doi.org/10.2140/ant.2013.7.1883`.

[2] Fernando Gouvêa. Non-ordinary primes: a story. *Experiment. Math.*, 6(3):195–205, 1997. ISSN 1058-6458. URL `http://projecteuclid.org/euclid.em/1047920420`.

[3] Sebastián Daniel Herrero. The adjoint of some linear maps constructed with the Rankin-Cohen brackets. *Ramanujan J.*, 36(3):529–536, 2015. ISSN 1382-4090. URL `https://doi.org/10.1007/s11139-013-9536-5`.

[4] Henryk Iwaniec. *Topics in classical automorphic forms*, volume 17 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. ISBN 0-8218-0777-3. URL `https://doi.org/10.1090/gsm/017`.

[5] Winfried Kohnen. Cusp forms and special values of certain Dirichlet series. *Math. Z.*, 207(4):657–660, 1991. ISSN 0025-5874. URL `https://doi.org/10.1007/BF02571414`.

[6] Arvind Kumar. The adjoint map of the Serre derivative and special values of shifted Dirichlet series. *J. Number Theory*, 177:516–527, 2017. ISSN 0022-314X. URL `https://doi.org/10.1016/j.jnt.2017.01.011`.

[7] Ken Ono. Unearthing the visions of a master: harmonic Maass forms and number theory. In *Current developments in mathematics, 2008*, pages 347–454. Int. Press, Somerville, MA, 2009.

[8] Brandon Williams. Poincaré square series for the Weil representation. *Ramanujan J.*, (in press). doi: 10.1007/s11139-017-9986-2. URL `https://doi.org/10.1007/s11139-017-9986-2`.
[9] Don Zagier. Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. pages 105–169. Lecture Notes in Math., Vol. 627, 1977.
[10] Don Zagier. Modular forms and differential operators. *Proc. Indian Acad. Sci. Math. Sci.*, 104(1):57–75, 1994. ISSN 0253-4142. URL `https://doi.org/10.1007/BF02830874`. K. G. Ramanathan memorial issue.
[11] Don Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 1–103. Springer, Berlin, 2008. URL `https://doi.org/10.1007/978-3-540-74119-0_1`.

Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: `btw@math.berkeley.edu`