A SHORT PROOF OF RADEMACHER'S FORMULA FOR k-COLOR PARTITIONS

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Abstract. This note points out that Rademacher's formula for k-color partitions, $1 \le k \le 24$ follows from the duality between nearly-holomorphic modular forms of weight $-k/2$ and $2 + k/2$.

1. Introduction

Let $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$, $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H} = \{x + iy : \, y > 0\}$ denote Dedekind's eta function, which is a modular form of weight $1/2$. More precisely, let

$$\tilde{\Gamma} = Mp_2(\mathbb{Z}) = \Big\{ (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}) : \quad a, b, c, d \in \mathbb{Z}, \; ad - bc = 1 \Big\}$$

be the metaplectic group, generated by $S = (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau})$ and $T = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1)$, where $\sqrt{\tau} \in \mathbb{H}$ for all $\tau \in \mathbb{H}$. Then η transforms by

$$\eta(M \cdot \tau) = \chi(M)\sqrt{c\tau + d} \cdot \eta(\tau), \quad M = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}) \in Mp_2(\mathbb{Z})$$

for the **eta character** $\chi : Mp_2(\mathbb{Z}) \to \mathbb{C}^{\times}$ which is determined by $\chi(S) = e^{-\pi i/4}$ and $\chi(T) = e^{\pi i/12}$. Closed formulas are known for $\chi(M)$ (see for example [3], section 6). $\eta(\tau)$ also has an interpretation as a vector-valued modular form for a Weil representation, as remarked in [6], section 3.2.

The Fourier coefficients of η^{-1} are very interesting:

$$\eta(\tau)^{-1} = q^{-1/24} \sum_{n=0}^{\infty} p(n) q^n,$$

where $p(n)$ is the **partition number** that counts the number of ways to write n as an unordered sum of positive integers (and $p(0) = 1$ by convention). More generally,

$$\eta(\tau)^{-k} = q^{-k/24} \sum_{n=0}^{\infty} p_k(n) q^n$$

where $p_k(n)$ counts **k-color partitions**. The modularity of η is a powerful tool in the study of partitions. For example, Poisson summation (and some work) shows that the series $q^{1/24} \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2 - k)/2}$ is modular of the same weight and character and Euler's pentagonal number theorem

$$\eta(\tau) = q^{1/24} \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2 - k)/2}$$

follows after comparing only the coefficients of $q^{1/24}$ on both sides, giving the famous recursive formula for $p(n)$. A considerably deeper result of Bruinier and Ono [6] finds a finite algebraic formula for $p(n)$; these are expressed as traces of singular moduli of a distinguished weak Maass form of level 6. For purposes of computation the most important result remains the Hardy-Ramanujan-Rademacher formula:

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Theorem 1 (Rademacher)**.**

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{c=1}^\infty \sqrt{c}A_c(n)\frac{d}{dn}\Big[\frac{1}{\sqrt{n-1/24}}\sinh\Big(\frac{\pi}{c}\sqrt{\frac{2}{3}(n-1/24)}\Big].$$

Here $A_c(n)$ is a Kloosterman sum (see below). This series converges rapidly and is the basis of modern algorithms for computing $p(n)$ (see e.g. [8], section 56.13).

Rademacher's original proof uses a sophisticated argument that involves integration over a contour made from Ford circles ([11], lectures 16-19). More recently a different point of view uses the observation of Hejhal ([9], appendix D) that negative-weight modular forms can be studied using real-analytic Maass-Poincaré series (see also [5], sect. 6.3). There are several real-analytic proofs: for example, [7] writes η^{-1} as a modified Poincaré series of weight $-1/2$, while [2] constructs a weight $5/2$ mock modular form with η^{-1} as its shadow.

In this note we point out a short, holomorphic proof of Rademacher's formula that uses the Fourier expansion of usual Poincaré series and the fact that any nearly-holomorphic weight two modular form has constant term 0. This is essentially Zagier duality [12] which is now a standard technique. The application to Rademacher's formula is not surprising and is likely known to experts, but it does not seem to have been written down explicitly and it may have some expository value.

2. Review of Poincaré series

The Fourier expansion of Poincaré series is a classical computation ([10], eqs. 10,11). For the reader's convenience we review this computation in the case of Poincaré series for the eta character χ.

Let $\Gamma_\infty \subseteq \tilde{\Gamma} = Mp_2(\mathbb{Z})$ denote the subgroup generated by T and $S^2 = (-I, i)$ and fix $k \in \frac{1}{2}\mathbb{Z}$, $k \geq 5/2$. The cosets $M \in \Gamma_\infty \backslash \tilde{\Gamma}$ correspond bijectively to the pairs of coprime integers $(0,1)$ and (c,d), $c > 0$ that make up the bottom row of a representative of M; since $S^4 \in \tilde{\Gamma}$ we can (and we do) always choose the branch of $\sqrt{c\tau+d}$ with $\mathrm{re}(\sqrt{c\tau+d}) > 0$ for all $\tau \in \mathbb{H}$. We use Petersson's slash notation

$$f|_{k,\chi^l} M(\tau) = (c\tau+d)^{-k}\chi(M)^{-l}f(M\cdot\tau), \;\; M = (M, \sqrt{c\tau+d}) \in \tilde{\Gamma}.$$

If $m \in \frac{1}{24}\mathbb{Z}$ satisfies $k - 12m \in 2\mathbb{Z}$, then $q^m = e^{2\pi i m\tau}$ is invariant under $|_{k,\chi^{24m}}T$ and $|_{k,\chi^{24m}}S^2$ and so the **Poincaré series** of index m,

$$P_{k,m}(\tau) = \sum_{M \in \Gamma_\infty\backslash\tilde{\Gamma}} q^m\Big|_{k,\chi^{24m}} M,$$

is well-defined and is modular of weight k with multiplier χ^{24m}. This transforms under T by

$$P_{k,m}(\tau+1) = \chi(T)^{24m}P_{k,m}(\tau) = e^{2\pi i m}P_{k,m}(\tau)$$

and therefore has a Fourier series representation $P_{k,m}(\tau) = \sum_{n\in\mathbb{Z}+m} a_n q^n$. We compute the coefficients a_n with some abuse of notation:

$$\begin{aligned}
a_n &= \int_0^1 P_{k,m}(\tau)e^{-2\pi i n\tau}\,\mathrm{d}x, \;\; (\tau = x+iy,\; y>0 \text{ fixed}) \\
&= \sum_{c,d}\chi(M)^{-24m}\int_0^1 (c\tau+d)^{-k}e^{2\pi i(m\frac{a\tau+b}{c\tau+d}-n\tau)}\,\mathrm{d}x \\
&= \underbrace{\delta_{m,n}}_{(c=0,d=1)} + \sum_{c=1}^\infty \sum_{\substack{d\in\mathbb{Z} \\ \gcd(c,d)=1}} \chi(M)^{-24m}\int_0^1 (c\tau+d)^{-k}e^{2\pi i(m\frac{a\tau+b}{c\tau+d}-n\tau)}\,\mathrm{d}x \\
&= \delta_{m,n} + \sum_{c=1}^\infty \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\chi(M)^{-24m}e^{2\pi i\frac{ma+nd}{c}}\int_{-\infty}^\infty (c\tau+d)^{-k}e^{2\pi i m(\frac{a\tau+b}{c\tau+d}-\frac{a}{c})-n(\tau+\frac{d}{c})}\,\mathrm{d}x \\
&= \delta_{m,n} + \sum_{c=1}^\infty c^{-k}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\chi(M)^{-24m}e^{2\pi i\frac{ma+nd}{c}}\int_{-\infty}^\infty \tau^{-k}e^{-2\pi i(n\tau+m/c^2\tau)}\,\mathrm{d}x \;\; (\tau\mapsto\tau-\frac{d}{c}).
\end{aligned}$$

Here $\chi(M)$ is not well-defined on cosets $M \in \Gamma_\infty \backslash \tilde{\Gamma}$, but $\chi(M)^{-24m}(c\tau+d)^{-k} e^{2\pi i m \frac{a\tau+b}{c\tau+d}}$ is well-defined; similarly, having fixed $c > 0$ and the branch of $\sqrt{c\tau+d}$, the expression $\chi(M)^{-24m} e^{2\pi i \frac{ma+nd}{c}}$ is well-defined and depends only on d mod c (as it remains the same after replacing M by either TM or MT).

Suppose $m \neq 0$. If $n \leq 0$ then the integral $\int_{\mathbb{R}+\alpha} \tau^{-k} e^{-2\pi i(n\tau + m/c^2\tau)}\, \mathrm{d}\tau$ is holomorphic as a function of α on the upper half-plane and is constant by the identity principle (it is constant as α varies on horizontal lines). Setting $\alpha = i\infty$ shows that it is zero and therefore $a_n = \delta_{m,n}$. If $n > 0$ then one can deform $\mathbb{R} + iy$ to a keyhole contour $-\gamma$ encircling the negative real axis (the contour will be oriented negatively) and use Schläfli's integral

$$J_{\nu-1}(z) = \frac{1}{2\pi i} \int_\gamma w^{-\nu} e^{(z/2)(w - w^{-1})}\, \mathrm{d}w$$

for the Bessel J-function (set $t = \frac{wz}{2}$ in [1, Eq. 10.9.19]) to see that

$$\int_{-\infty}^{\infty} \tau^{-k} e^{-2\pi i (n\tau + m/c^2\tau)}\, \mathrm{d}x = -(2\pi i) c^{k-1} (-i\sqrt{n/m})^{k-1} J_{k-1}(4\pi\sqrt{mn}/c).$$

(Here we use the convention $(-i)^{1/2} = e^{-\pi i/4}$.) Finally, since the modified Bessel functions are given by $I_{k-1}(w) = i^{1-k} J_{k-1}(iw)$, we conclude:

Lemma 2. *If $m > 0$ and $k - 12m \in 2\mathbb{Z}$, then $P_{k,m}$ has the Fourier series*

$$P_{k,m}(\tau) = q^m + \sum_{n \in (\mathbb{Z}+m)_{>0}} a_n q^n, \;\; a_n = 2\pi(-i)^k (n/m)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} \frac{A(m,n,c)}{c} J_{k-1}(4\pi\sqrt{mn}/c)$$

and if $k + 12m \in 2\mathbb{Z}$, then $P_{k,-m}$ has the Fourier series

$$P_{k,-m}(\tau) = q^{-m} + \sum_{n \in (\mathbb{Z}-m)_{>0}} a_n q^n, \;\; a_n = 2\pi(-i)^k (n/m)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} \frac{A(-m,n,c)}{c} I_{k-1}(4\pi\sqrt{mn}/c),$$

*where we use $A(m,n,c)$ to denote the **Kloosterman sum***

$$A(m,n,c) = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(M)^{-24m} e^{2\pi i \frac{ma+nd}{c}}, \;\; c \in \mathbb{N},\; m,n \in \frac{1}{24}\mathbb{Z} \text{ with } n \in \mathbb{Z}+m.$$

The Kloosterman sum $A_c(n)$ of theorem 1 is $e^{-\pi i/4} A(-n+1/24, 1/24, c)$ in this notation. Using the closed formula for $\chi(M)$ one can also express $A_c(n)$ in terms of Dedekind sums (cf. [7], eqs. 3,4).

3. Proof of Rademacher's formula

Theorem 3. *Let $k \leq 24$. Then the number of k-color partitions of $m \in \mathbb{N}$ is*

$$p_k(m) = -a_{k/24},$$

where $a_{k/24}$ is the coefficient of $q^{k/24}$ in the Poincaré series $P_{k/2+2,-m+k/24}$ as computed in lemma 2.

Proof. This is essentially the proof of Zagier duality attributed to Kaneko in [12]. $P_{k/2+2,-m+k/24} \cdot \eta^{-k}$ is a meromorphic modular form of weight two, level one and trivial multiplier with a unique pole at ∞. By subtracting off derivatives of the Hecke images $J|T_n$ where $J(\tau) = j(\tau) - 744 = q^{-1} + O(q)$ is the level one Hauptmodul, one can remove the principal part of $P_{k/2+2,-m+k/24} \cdot \eta^{-k}$. This leaves a holomorphic modular form of weight two which is therefore zero. Since the derivatives $(J|T_n)'$ have constant term zero,

$$P_{k/2+2,-m+k/24}(\tau) \cdot \eta(\tau)^{-k} = \Big(q^{-m+k/24} + a_{k/24} q^{k/24} + ...\Big)\Big(\sum_{n=0}^{\infty} p_k(n) q^{n-k/24}\Big)$$

also has constant term $0 = p_k(m) + a_{k/24}$. $\square$

Alternatively, the vanishing of the constant term above follows from the residue theorem on $\Gamma\backslash\mathbb{H}$ applied to the invariant differential $P_{k/2+2,-m+k/24} \cdot \eta^{-k}\, \mathrm{d}\tau$. This use of the residue theorem can also be interpreted as the Serre duality pairing ([4], section 3).

The traditional form of Rademacher's formula as in theorem 1 follows after expressing modified Bessel functions of half-integer order by elementary functions, here $I_{3/2}(x) = \sqrt{\frac{2x}{\pi}} \frac{d}{dx}(\sinh(x)/x)$.

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Note added v2: I have learned that this proof was obtained previously by W. Pribitkin and was presented in an AMS Special Session at JMM 2017, Atlanta.

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