COMPUTING ANTISYMMETRIC MODULAR FORMS AND THETA LIFTS

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ABSTRACT. We give coefficient formulas for antisymmetric vector-valued cusp forms with rational Fourier coefficients for the Weil representation associated to a finite quadratic module. The forms we construct always span all cusp forms in weight at least three. These formulas are useful for computing explicitly with theta lifts.

1. INTRODUCTION

This note is an extended version of chapter 7 of the author’s dissertation and is in some sense a continuation of [26]. Its purpose is to give formulas for a spanning set of vector-valued cusp forms with rational Fourier coefficients for the (dual) Weil representation $\rho^*$ attached to a finite quadratic module $(A, Q)$ which are antisymmetric under the action of $-I \in SL_2(\mathbb{Z})$. Equivalently the weight $k$ of these cusp forms is such that $k + \text{sig}(A, Q)/2$ is odd, where sig$(A, Q)$ is the signature of $(A, Q)$.

Bases of modular forms with rational coefficients are known to exist due to the work of McGraw [15]. On the other hand, all algorithms to compute such bases in the literature that the author is aware of (e.g. [19], [26]) assume that $k + \text{sig}(A, Q)/2$ is even. Computing antisymmetric modular forms has received less attention; the first effective formula to compute the space of Eisenstein series in antisymmetric weights for arbitrary $(A, Q)$ was given in [20]. The computation of cusp forms here complements this.

The most important application of antisymmetric vector-valued modular forms is that they are mapped to orthogonal modular forms under the theta lift (or Maass lift), as in [2], [10], [17], [18].

Our main results are the two theorems below. (The terms and notation are explained in section two.)

**Theorem 1.** Let $(A, Q)$ be a finite quadratic module, and let $k \geq 3$ be a weight for which $k + \text{sig}(A, Q)/2$ is an odd integer. For any $\beta \in A$ and $m \in \mathbb{Z} - Q(\beta)$, $m > 0$, let $R_{k, m, \beta}$ be the cusp form defined through the Petersson scalar product by

$$ (f, R_{k, m, \beta}) = 2 \cdot \frac{\Gamma(k - 1)}{(4\pi m)^{k-1}} L_{m, \beta}(f, 2k - 1) $$

where $L_{m, \beta}$ is essentially a rescaled symmetric square $L$-function:

$$ L_{m, \beta}(f, s) = \sum_{\lambda=1}^{\infty} \frac{c(\lambda^2 m, \lambda \beta)}{\lambda^s} \text{ if } f(\tau) = \sum_{\gamma \in \Lambda \mod Q(\gamma)} c(n, \gamma)q^n e^{\gamma}, \quad q = e^{2\pi i \tau}. $$

Then all $R_{k, m, \beta}$ have rational Fourier coefficients, and there is a finite collection of indices $(m, \beta)$ for which the forms $R_{k, m, \beta}$ span the entire cusp space $S_k(\rho^*)$.

**Theorem 2.** Let $(\Lambda, Q)$ be an even lattice which realizes the discriminant group $A = \Lambda' / \Lambda$, and let $k \geq 4$ be a weight for which $k + \text{sig}(\Lambda)/2$ is odd. For any $\beta \in \Lambda'$ and $m \in \mathbb{Z} - Q(\beta)$, $m > 0$, let $\Lambda_{m, \beta}$ denote the even lattice with underlying group $\Lambda \oplus \mathbb{Z}$ and quadratic form $Q_{m, \beta}(v, \lambda) = Q(v + \lambda \beta) + m\lambda^2$. Let $c_{m, \beta}(n, \gamma)$ denote the Fourier coefficients of the weight $k - 3/2$ Eisenstein series for the dual Weil representation attached to $\Lambda_{m, \beta}$ (as in [5],[14]), i.e.

$$ E_{k-3/2}(\tau; \Lambda_{m, \beta}) = \sum_{\gamma \in \Lambda'_{m, \beta} / \Lambda_{m, \beta}} \sum_{n \in \mathbb{Z} - Q_{m, \beta}(\gamma)} c_{m, \beta}(n, \gamma)q^n e^{\gamma}. $$
Then $R_{k,m,\beta}$ is given explicitly by the formula
\[
R_{k,m,\beta}(\tau) = \frac{1}{2m} \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} \left[ \sum_{r \in \mathbb{Z} - (\gamma, \beta)} r \cdot c_{m,\beta} \left( n, (\gamma - \frac{r}{2m}, \frac{r}{2m}) \right) \right] q^{n + r^2/4m} e_{\gamma}.
\]

This rest of this note is organized as follows. Sections 2 contains background material on vector-valued modular forms for Weil representations. Section 3 constructs the cusp forms $R_{k,m,\beta}$ and proves theorem 2. Sections 4 and 5 adapt the construction to small weights and complete the proof of theorem 1. Section 6 discusses the main application, i.e. the theta lift and orthogonal modular forms. Finally, sections 7 and 8 give two applications of these formulas: we compute Gundlach’s [11] well-known weight 5 Hilbert cusp form for $Q(\sqrt{5})$ (the product of 10 theta-constants) as a Doi-Naganuma lift, and we compute some identities among the Hurwitz class numbers of binary quadratic forms.

An implementation of these coefficient formulas in SAGE is available on the author’s university webpage.

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2. Finite quadratic modules and modular forms

This section reviews finite quadratic modules, their Weil representations, and vector-valued modular forms for those representations. There are better references in the literature (for example, chapter 14 of [6]); the main purpose of including this material is to fix some conventions and notation for the rest of the note.

A finite quadratic module $(A, Q)$ consists of a finite abelian group $A$ and a nondegenerate $(Q/\mathbb{Z})$-valued quadratic form $Q$ on it. (In other words the bilinear form $(x, y) = Q(x + y) - Q(x) - Q(y)$ is nondegenerate.) Given this data there is a unitary representation $\rho^* = \rho^*_A$ of the metaplectic group $\Gamma = Mp_2(\mathbb{Z})$ on the group ring $\mathbb{C}[A]$, through which the generators $S = ((0 \ 1 \ -1 \ 0), \sqrt{T})$ and $T = ((1 \ 1 \ 0 \ 1), 1)$ act by
\[
\rho^*(S)e_\gamma = \frac{1}{\sqrt{|A|}} e(\text{sig}(A, Q)/8) \sum_{\beta \in A} e(\langle \gamma, \beta \rangle) e_\beta, \quad \rho^*(T)e_\gamma = e(-Q(\gamma))e_\gamma, \quad \gamma \in A.
\]

Here we are using the notation $e(x) = e^{2\pi ix}$, and $e_\gamma, \gamma \in A$ denotes the canonical basis of $\mathbb{C}[A]$. In the most common convention $\rho^*$ is called the dual Weil representation associated to $(A, Q)$.

Let $\mathbb{H} = \{ \tau = x + iy : y > 0 \}$ be the upper half-plane. Modular forms for $\rho^*$ of weight $k \in \frac{1}{2}\mathbb{Z}$ are holomorphic functions $f = f(\tau) : \mathbb{H} \to \mathbb{C}[A]$ which remain bounded as $y = \text{im}(\tau)$ tends to $\infty$ and which satisfy
\[
f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k \rho^* \left( \frac{a}{c^2} \right) \sqrt{c\tau + d} f(\tau)
\]
for all $M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \sqrt{c\tau + d} \in \Gamma$. The space of modular forms of weight $k$ for $\rho^*$ will be denoted $M_k(\rho^*)$.

The transformation under $T$ implies that modular forms for $\rho^*$ have Fourier expansions
\[
f(\tau) = \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} c(n, \gamma) q^n e_\gamma, \quad c(n, \gamma) \in \mathbb{C}.
\]

Additionally, the element $Z = (-I, i) = S^2$ acts through $\rho^*$ by $\rho^*(Z)e_\gamma = (-1)^{\text{sig}(A, Q)/2} e_{-\gamma}$ and it acts trivially on $\mathbb{H}$, so the transformation under $Z$ implies that $M_k(\rho^*) = 0$ if $k + \text{sig}(A, Q)/2$ is not integral, and that the Fourier coefficients $c(n, \gamma)$ of any modular form $f(\tau)$ satisfy
\[
c(n, \gamma) = (-1)^{k + \text{sig}(A, Q)/2} c(n, -\gamma).
\]
Therefore it seems reasonable to refer to \( k \) as a **symmetric** or **antisymmetric weight** when \( k + \text{sig}(A, Q)/2 \) is respectively even or odd.

The spaces \( M_k(\rho^\ast) \) of modular forms of weight \( k \) are always finite-dimensional and their dimensions can be calculated with the Riemann-Roch theorem. Applying the Riemann-Roch theorem directly leads to a formula which is not very efficient. However the special case of Riemann-Roch for Weil representations has been of particular interest and better results are available. The formula below for antisymmetric weights is derived from theorem 2.1 of [7]:

**Proposition 3.** Let \((A, Q)\) be a finite quadratic module and let \( k \) be a weight such that \( k > 2 \) and \( k + \text{sig}(A, Q)/2 \) is an odd integer. Let \( B(x) \) be the sawtooth function

\[
B(x) = x - \frac{|x| - \lfloor -x \rfloor}{2},
\]

and define the sums

\[
B_1 = \sum_{\gamma \in A} B(Q(\gamma)), \quad B_2 = \sum_{\gamma \in A} B(Q(\gamma)).
\]

Also let \( \tilde{\alpha}_4 \) denote the number of isotropic, antisymmetric pairs of elements of \( A \):

\[
\tilde{\alpha}_4 = \#\{ \gamma \in A : Q(\gamma) = 0 + \mathbb{Z}, \gamma \neq -\gamma \}/ \pm 1.
\]

Let \( d \) denote the number of pairs \( \{ \pm \gamma \}, \gamma \in A \) with \( \gamma \neq -\gamma \). Then

\[
\dim M_k(\rho^\ast) = \frac{d(k - 1)}{12} + \frac{1}{4\sqrt{|A|}} e^{\left( \frac{2(k + 1) + \text{sig}(A, Q)}{8} \right) \text{Im}[\Gamma(2, A)]} - \frac{1}{3\sqrt{3|A|}} \text{Re} \left[ e^{\left( \frac{4k + 3\text{sig}(A, Q) - 10}{24} \right) (G(1, A) - G(-3, A))} \right] + \frac{\tilde{\alpha}_4 + B_1 - B_2}{2},
\]

and \( \dim S_k(\rho^\ast) = \dim M_k(\rho^\ast) - \tilde{\alpha}_4 \).

In particular, calculating \( \dim M_k(\rho^\ast) \) only requires one summation over the discriminant group \( A \). Note that the formula above is only valid for antisymmetric weights; it has to be modified slightly when \( k + \text{sig}(A, Q)/2 \) is even. (That case can also be read off of theorem 2.1 of [7].)

The easiest way to produce modular forms is by averaging. Let \( \tilde{\Gamma}_\infty = \langle T, Z \rangle \leq \tilde{\Gamma} \) be the stabilizer of the constant function \( \epsilon_0 \). For any \( k > 2 \) and any smooth function \( \phi : \mathbb{H} \to \mathbb{C}[A] \) which satisfies \( \phi(\tau) = (-1)^k \rho^\ast(Z) \phi(\tau) \) and \( \phi(\tau + 1) = \rho^\ast(T) \phi(\tau) \), the **Poincaré series**, if it converges locally uniformly, is the series

\[
\mathbb{P}_k(\phi) = \sum_{M \in \tilde{\Gamma} \backslash \tilde{\Gamma}_\infty} \phi|_{k, \rho^\ast} M = \frac{1}{2} \sum_{c,d \in \mathbb{Z}, \gcd(c,d) = 1} (c\tau + d)^{-k} \rho^\ast \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \sqrt{c\tau + d} \right)^{-1} \phi \left( \frac{a\tau + b}{c\tau + d} \right).
\]

In the sum on the right, \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \sqrt{c\tau + d} \) is any element with bottom row \( c, d \). The most important case is the **Poincaré series of exponential type**: \( \beta \in A \) and \( m \in \mathbb{Z} - Q(\beta) \), we take the seed function \( \phi(\tau) = q^m e^{\frac{-i}{2}(1)^\beta \epsilon_{c,\beta}} \) and define \( P_{k,m,\beta} = \mathbb{P}_k(\phi) \). These converge normally when \( m > 0 \) and are cusp forms when \( m = 0 \). Moreover they are a spanning set of \( S_k(\rho^\ast) \) as \( \beta \) runs through \( A \) and as \( m \) runs through \( (\mathbb{Z} - Q(\beta))_{> 0} \) because, with respect to the Petersson scalar product

\[
(f, g) = \int_{SL_2(\mathbb{Z})/\mathbb{H}} (f(\tau), g(\tau)) y^{k-2} \, dx \, dy, \quad f, g \in S_k(\rho^\ast),
\]
these Poincaré series satisfy
\[(f, P_{k,m,\beta}) = (4\pi m)^{1-k} \Gamma(k-1)c(m,\beta)\]
for all \(f(\tau) = \sum_{\gamma \in \Lambda} \sum_{n \in \mathbb{Z} - Q(\gamma)} c(n,\gamma)q^n e_\gamma \in S_k(\rho^*)\).

(This is proved by the usual Rankin-Selberg unfolding argument.) In particular any cusp form orthogonal to all \(P_{k,m,\beta}\) is identically zero.

3. ANTISYMMETRIC Poincaré SERIES WITH RATIONAL COEFFICIENTS

Fix a discriminant form \((A, Q)\) and a weight \(k \geq 7/2\) which is antiasymmetric, i.e. \(k + \text{sig}(A, Q)/2\) is odd. For an index \((m, \beta)\) with \(\beta \in A\) and \(m \in \mathbb{Z} - Q(\beta)\), let \(P_{k,m,\beta}\) be the Poincaré series of exponential type of weight \(k\) as in section 2.

**Lemma 4.** The series \(R_{k,m,\beta} = \sum_{\lambda \in \mathbb{Z}} \lambda P_{k,\lambda^2 m,\lambda \beta} = 2 \sum_{\lambda = 1}^{\infty} \lambda P_{k,\lambda^2 m,\lambda \beta}\) converges in \(S_k(\rho^*)\).

**Proof.** \(S_k(\rho^*)\) is finite-dimensional so every reasonable notion of convergence (e.g. with respect to any norm) coincides with that of the weak topology with respect to the Petersson scalar product; in other words, it is enough to show that
\[
\sum_{\lambda = 1}^{\infty} \lambda(f, P_{k,\lambda^2 m,\lambda \beta}) = (4\pi m)^{1-k} \Gamma(1-k) \sum_{\lambda = 1}^{\infty} \frac{c(\lambda^2 m, \lambda \beta)}{\lambda^{2k-1}}
\]
converges for all cusp forms \(f(\tau) = \sum_{n,\gamma} c(n,\gamma)q^n e_\gamma\). For \(k \geq 7/2\) this follows from known coefficient bounds for cusp forms. See also the analogous argument in remark 10 of [26].

**Lemma 5.** The series \(R_{k,m,\beta}\) span \(S_k(\rho^*)\) as \(\beta\) runs through \(A\) and \(m\) runs through positive elements of \(\mathbb{Z} - Q(\beta)\).

**Proof.** Let \(\mu\) be the Möbius function. Möbius inversion implies
\[
P_{k,m,\beta} = \frac{1}{2} \sum_{\lambda = 1}^{\infty} \lambda \mu(\lambda) R_{k,\lambda^2 m,\lambda \beta}
\]
with convergence by the same argument as the previous lemma. Since \(S_k(\rho^*)\) is finite-dimensional it follows from this that all Poincaré series \(P_{k,m,\beta}\) lie in the span of \(R_{k,m,\beta}\).

One would expect the series \(R_{k,m,\beta}\) to have much nicer Fourier expansions than \(P_{k,m,\beta}\) because (ignoring convergence issues for now) they are the Poincaré series
\[
R_{k,m,\beta} = \mathbb{P}_k(\theta_{m,\beta}) = \sum_{M \in \mathbb{C} \setminus \Gamma} (c\tau + d)^{-k} \rho^*(M)^{-1} \theta_{m,\beta}((M \cdot \tau)
\]
whose seed functions are theta series \(\theta_{m,\beta}(\tau) = \sum_{\lambda \in \mathbb{Z}} \lambda q^{(\beta+\delta)\lambda^2} e_{\lambda \beta}\), and \(\theta_{m,\beta}\) already have a weight 3/2 modularity behavior. That behavior is not compatible with the Weil representation \(\rho^*\) so there are some difficulties making this argument explicit. The cleanest way to compute \(R_{k,m,\beta}\) is probably to use the Jacobi Eisenstein series of weight \(k - 1\). For the rest of this section we assume that \(k \geq 4\).

A Jacobi form of weight \(k\) and index \((m, \beta)\) is a holomorphic function of two variables
\[
\Phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}
\]
which satisfies the transformation laws
\[
\Phi\left(\frac{a\tau + b}{c\tau + d} ; \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\left(\frac{mez^2}{c\tau + d}\right)} \rho^*(M) \Phi(\tau, z) \text{ for all } M = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \sqrt{c\tau + d} \in \tilde{\Gamma}.
\]
and
\[
\Phi(\tau, z + \lambda \tau + \mu) = e(-\lambda \mu m) q^{-m\lambda^2} \zeta^{-2m\lambda} \sigma_\lambda^*(\lambda, \mu) \Phi(\tau, z) \text{ for all } \lambda, \mu \in \mathbb{Z},
\]
together with a vanishing condition on Fourier coefficients, where \(q = e^{2\pi i \tau}\) and \(\zeta = e^{2\pi i z}\) and where
\[
\sigma_\lambda^*(\lambda, \mu) e_\gamma = e(-\mu \langle \beta, \gamma \rangle + \lambda \mu Q(\beta)) e_{\gamma - \lambda \beta}.
\]
The Fourier expansion of Proposition 6. then we find:

\[ \Phi(\tau, z) = \sum_{\gamma \in \mathcal{A}} \sum_{n \in \mathbb{Z} - \{Q(\gamma)\}} \sum_{r \in \mathbb{Z} - \{\gamma, \beta\}} c(n, r, \gamma) q^n \zeta^r \epsilon_\gamma, \quad c(n, r, \gamma) \in \mathbb{C}, \]

and the vanishing condition on Fourier coefficients is that \( c(n, r, \gamma) = 0 \) whenever \( 4mn - r^2 < 0 \).

The Jacobi Eisenstein series \( E_{k,m,\beta} \) of weight \( k \) and index \( (m, \beta) \) is obtained by averaging out the constant function \( \epsilon_0 \) to a Jacobi form:

\[
E_{k,m,\beta}(\tau, z) = \frac{1}{2} \sum_{c,d \in \mathbb{Z}} \sum_{\lambda \in \mathbb{Z}} (ct + d)^{-k} e \left( \frac{m\lambda^2(\alpha r + b) + 2m\lambda z - cmz^2}{ct + d} \right) \rho^* \left( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \sqrt{ct + d} \right)^{-1} \epsilon_{\lambda\beta},
\]

where \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) \) is any matrix with bottom row \( (c, d) \). This converges and defines a Jacobi form when \( k \geq 3 \) (and it is zero unless \( k \) is a symmetric weight for \( \rho^* \)).

When \( k \) is an antisymmetric weight that is at least 4 we swap the order of summation and find

\[
\frac{\partial}{\partial \zeta} \bigg|_{\zeta = 0} E_{k-1,m,\beta}(\tau, z) = \sum_{\lambda \in \mathbb{Z}} \sum_{c,d} (ct + d)^{1-k} \frac{\partial}{\partial \zeta} \bigg|_{\zeta = 0} e \left( \frac{m\lambda^2(\alpha r + b) + 2m\lambda z - cmz^2}{ct + d} \right) \rho^* \left( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \sqrt{ct + d} \right)^{-1} \epsilon_{\lambda\beta} = 4\pi mi \sum_{\lambda \in \mathbb{Z}} \sum_{c,d} \lambda (ct + d)^{-k} e \left( \frac{m\lambda^2\alpha r + b}{ct + d} \right) \rho^* \left( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \sqrt{ct + d} \right)^{-1} \epsilon_{\lambda\beta} = 4\pi mi R_{k,m,\beta}.
\]

Therefore if we write out the Fourier expansion of the Jacobi Eisenstein series as

\[
E_{k-1,m,\beta}(\tau, z) = \sum_{\gamma \in \mathcal{A}} \sum_{n \in \mathbb{Z} - \{Q(\gamma)\}} \sum_{r \in \mathbb{Z} - \{\gamma, \beta\}} c(n, r, \gamma) q^n \zeta^r \epsilon_\gamma,
\]

then we find:

**Proposition 6.** The Fourier expansion of \( R_{k,m,\beta}(\tau) \) is

\[
R_{k,m,\beta}(\tau) = \frac{1}{2\pi i} \sum_{\gamma \in \mathcal{A}} \sum_{n \in \mathbb{Z} - \{Q(\gamma)\}} \sum_{r \in \mathbb{Z} - \{\gamma, \beta\}} c(n, r, \gamma) q^n \zeta^r \frac{\partial}{\partial \zeta} \bigg|_{\zeta = 0} \epsilon_\gamma = \sum_{\gamma \in \mathcal{A}} \sum_{n \in \mathbb{Z} - \{Q(\gamma)\}} \left( \frac{1}{2m} \sum_{r \in \mathbb{Z} - \{\gamma, \beta\}} \frac{rc(n, r, \gamma)}{r^2 \leq 4mn} \right) q^n \epsilon_\gamma.
\]

To complete the proofs of theorems 1 and 2 of the introduction (in weights \( k \geq 4 \)), we use the theta decomposition (theorem 5.1 of [9]) to relate the coefficients of the Jacobi Eisenstein series to the usual (modular) Eisenstein series. In particular, one can show this way that the coefficients \( c(n, r, \gamma) \) are always rational. The proof of Eichler and Zagier [9] does not immediately apply to Jacobi forms of non-integral index, but a minor extension ([25], proposition 7) is sufficient to prove the formula claimed in theorem 2.

4. **Weights 5/2 and 7/2**

In weights \( k < 4 \) the arguments above break down. There is a standard workaround for this: instead of forming the Poincaré series with seed function \( \theta_{m,\beta}(\tau) = \sum_{\lambda \in \mathbb{Z}} \lambda q^m \lambda^2 \epsilon_{\lambda\beta} \) directly, we try to analytically
continue the series
\[ \sum_{M \in \Gamma \setminus \Gamma} \frac{1}{(c \tau + d)^k |c \tau + d|^{-2s}} \rho^*(M)^{-1} \theta_{m, \beta}(M \cdot \tau) \]
to \( s = 0 \). (This can usually be done by continuing each term in its Fourier expansion separately.) The result is generally not holomorphic but this turns out to be a rather minor problem; it satisfies good growth properties at the cusps so we can produce a cusp form, again denoted \( R_{k,m,\beta} \), by holomorphic projection (i.e., orthogonal projection into \( S_k(\rho^*) \) against the Petersson scalar product). This involves the continued Jacobi Eisenstein series as it appears for example in \([27]\):

\[
E^*_k-c, m, \beta(\tau, z; s) = \frac{1}{2} \sum_{c,d \in \mathbb{Z}} \sum_{\lambda \in \mathbb{Z}} \frac{y^s}{(c \tau + d)^{k+s-1}(c \tau + d)^s} e\left( \frac{m \lambda^2 (a \tau + b) + 2m \lambda z - cm z^2}{c \tau + d} \right) \rho^* \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \sqrt{c \tau + d} \right)^{-1} e_{\lambda \beta}.
\]

The zero-values \( E^*_k-c, m, \beta(\tau, z; 0) \) are somewhat more complicated when \( k = 3 \) than \( k = 5/2, 7/2 \) so we postpone the case \( k = 3 \) to the next section.

**Weight 5/2.** Suppose that \((A, Q)\) is a finite quadratic module of signature 1 mod 4, such that \( k = 5/2 \) is an antisymmetric weight. Section 5 of \([27]\) points out that the zero-values \( E^{3/2}_{k,m,\beta}(\tau, z; 0) \) are always holomorphic Jacobi forms which differ from the naive result \( E_{3/2,m,\beta}(\tau, z) \) of the coefficient formula of \([26]\) by a weight 1/2 theta series. In particular, no holomorphic projection is necessary: we simply define the cusp forms

\[
R_{5/2,m,\beta}(\tau) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \bigg|_{z=0} E^{3/2}_{3/2,m,\beta}(\tau, z; 0).
\]

It should be pointed out that, unlike weights \( k \geq 3 \), these series almost certainly do not span \( S_{5/2}(\rho^*) \) in general (although there are many cases where they do). A concrete example where they probably do not span is that of the finite quadratic module \( A = \mathbb{Z}/26\mathbb{Z}, Q(x) = (1/12)x^2 + \mathbb{Z} \). By theorem 5.1 of \([9]\), we can construct a weight 5/2 cusp form for \( \rho^* = \rho_{(A, Q)} \) by theta decomposition from the weight 3 Jacobi cusp form of index 13 (which is unique up to scalar multiple) whose Fourier expansion begins

\[
\phi_3(\tau, z) = (9 \sin(2\pi z) + 7 \sin(4\pi z) - 17 \sin(6\pi z) + 4 \sin(8\pi z) + 7 \sin(10\pi z) - 5 \sin(12\pi z) + \sin(14\pi z)) + O(q^2).
\]

The exact coefficients do not matter; the dimension bound of \([9]\) (corollary to theorem 9.2) shows that the space \( J_{5, 13} \) of Jacobi (cusp) forms is nonzero, which is enough. Numerical tests make it seem likely that all of the series \( R_{5/2,m,\beta} \) as defined above are identically zero. I do not have a proof of this.

**Weight 7/2.** Suppose that \((A, Q)\) is a finite quadratic module of signature 3 mod 4, so \( k = 7/2 \) is an antisymmetric weight. The series \( R_{7/2,m,\beta} = \sum \lambda \rho_{k, \lambda^2, m, \lambda, \beta} \) converges with no need for tricks such as analytic continuation so it is amusing that the formula (theorem 2) to compute it is incorrect. (The problem is that when understood as a triple series over tuples \((\lambda, c, d) \in \mathbb{Z}^3 \) with \( c, d \) coprime, \( R_{7/2,m,\beta} \) converges conditionally but not absolutely and some manipulations are not valid.) As in section 7 of \([26]\) the continued Jacobi Eisenstein series to \( s = 0 \) can be decomposed in the form

\[
E^*_{7/2,m,\beta}(\tau, z; 0) = E_{5/2,m,\beta}(\tau, z) + y^{-1} \vartheta(\tau, z),
\]

where \( E_{5/2,m,\beta} \) and \( \vartheta \) are holomorphic and \( \vartheta \) is a Jacobi form of weight 1/2 and index \((m, \beta)\) for \( \rho^* \) (and in particular \( \vartheta \) is a theta function). The holomorphic part \( E_{5/2,m,\beta} \) has the Fourier expansion

\[
E_{5/2,m,\beta}(\tau, z) = \sum_{\gamma, n, r} c(n, r, \gamma) q^n \zeta^r e_\gamma,
\]

where \( c(n, r, \gamma) \) is given by the coefficient formula of \([26]\) naively evaluated at \( k = 5/2 \).

**Proposition 7.** The cusp form \( R_{7/2,m,\beta} \) is given by

\[
R_{7/2,m,\beta}(\tau) = \frac{1}{4\pi i} \frac{\partial}{\partial z} \bigg|_{z=0} E_{5/2,m,\beta}(\tau, z) - \frac{1}{3\pi i} \frac{\partial^2}{\partial \tau \partial z} \bigg|_{z=0} \vartheta(\tau, z).
\]
Proof. If we write out the Fourier expansions

\[ E_{5/2,m,\beta}(\tau, z) = \sum_{\gamma,n,r} c(n, r, \gamma)q^n \zeta^r e_{\gamma}, \quad \vartheta(\tau, z) = \sum_{\gamma,n,r} a(n, r, \gamma)q^n \zeta^r e_{\gamma} \]

then the coefficients of \( R_{7/2,m,\beta}(\tau) = \sum_{n,\gamma} b(n, \gamma)q^n e_{\gamma} \) can be found by the Rankin-Selberg method:

\[
b(n, \gamma) = \frac{(4\pi n)^{5/2}}{\Gamma(5/2)} (R_{7/2,m,\beta}, P_{7/2,n,\gamma})
\]

\[
= \frac{(4\pi n)^{5/2}}{\Gamma(5/2)} \left( \frac{1}{2\pi i} \frac{\partial}{\partial z} \right)_{z=0} E^*_{5/2,m,\beta}(\tau, z; 0), P_{7/2,n,\gamma} \right)
\]

\[
= 64\pi^2 n^{5/2} \sum_{r \in \mathbb{Z}-(\gamma,\beta)} \int_{0}^{\infty} r \cdot \left( (c(n, r, \gamma) + a(n, r, \gamma)/y) e^{-4\pi n y^{3/2}} dy \right.
\]

\[
= \sum_{r \in \mathbb{Z}-(\gamma,\beta)} r \cdot \left( \frac{1}{2m} c(n, r, \gamma) + \frac{4\pi n}{3m} a(n, r, \gamma) \right) .
\]

\[\square\]

5. Weight 3

Suppose that \((A, Q)\) is a finite quadratic module of signature 0 mod 4, such that \(k = 3\) is an antisymmetric weight. By \([27]\) the continued Jacobi Eisenstein series to \(s = 0\) can be decomposed in the form

\[ E_{2,m,\beta}(\tau, z; 0) = E_{2,m,\beta}(\tau, z) + y^{-1/2} \sum_{r \in \mathbb{Z}-(\gamma,\beta)} \sum_{n \geq 0} \sum_{r \geq 2mn} a(n, r, \gamma) \beta \left( \frac{\pi y(r^2 - 4mn)}{m} \right) q^n \zeta^r e_{\gamma}, \]

where \(E_{2,m,\beta}(\tau, z)\) is the result of the formula \([26]\) for the Jacobi Eisenstein series evaluated naively at \(k = 2\); where \(\beta(t) = \frac{1}{16\pi} \int_{0}^{\infty} u^{-3/2} e^{-4u} du\) is an incomplete Gamma function (using the notation of \([30]\), which must be distinguished from the index component \(\beta \in A\) by context), and where \(a(n, r, \gamma)\) are certain numbers for which \(\sqrt{m|A|} \cdot a(n, r, \gamma)\) is rational and which are zero unless \((r^2 - 4mn)|A|\) is a rational square.

Holomorphic projection of \(\frac{1}{2\pi i} \frac{\partial}{\partial z} \bigg|_{z=0} E^*_{2,m,\beta}(\tau, z; 0)\) gives the Fourier expansion \(R_{3,m,\beta}(\tau) = \sum_{n,\gamma} b(n, \gamma)q^n e_{\gamma}\)

where

\[
b(n, \gamma) = \frac{(4\pi n)^2}{\Gamma(2)} (R_{3,m,\beta}, P_{3,n,\gamma})
\]

\[
= 16\pi^2 n^2 \left( \frac{1}{2\pi i} \frac{\partial}{\partial z} \right)_{z=0} E^*_{2,m,\beta}(\tau, z; 0), P_{3,n,\gamma} \right)
\]

\[
= 16\pi^2 n^2 \sum_{r \in \mathbb{Z}-(\gamma,\beta)} \int_{0}^{\infty} r \left( c(n, r, \gamma) + y^{-1/2} a(n, r, \gamma) \beta \left( \frac{\pi y(r^2 - 4mn)}{m} \right) \right) e^{-4\pi n y} dy
\]

\[
= \frac{1}{2m} \sum_{r} rc(n, r, \gamma) + (4\pi n)^2 \sum_{r} ra(n, r, \gamma) \int_{0}^{\infty} e^{-4\pi n y} \beta \left( \frac{\pi y(r^2 - 4mn)}{m} \right) y^{1/2} dy.
\]

The integral that remains above can be computed by swapping the order of integration:

\[
\int_{0}^{\infty} e^{-4\pi n y} \beta \left( \frac{\pi y(r^2 - 4mn)}{m} \right) y^{1/2} dy = \frac{1}{16\pi} \int_{0}^{\infty} \int_{1}^{\infty} u^{-3/2} y^{1/2} e^{4\pi ny(u-1) - \pi r^2 yu/m} du dy \]

\[
= \frac{1}{16\pi} \int_{1}^{\infty} u^{-3/2} [(r^2/m - 4n)u + 4n]^{-3/2} du
\]

\[
= \frac{1}{16\pi |r|} ((|r| - \sqrt{r^2 - 4mn})^2,
\]

so altogether we find

\[
b(n, \gamma) = \frac{1}{2m} \sum_{r \in \mathbb{Z}-(\gamma,\beta)} rc(n, r, \gamma) + \frac{1}{32m^{3/2}} \sum_{r \in \mathbb{Z}-(\gamma,\beta)} sgn(r)a(n, r, \gamma)((|r| - \sqrt{r^2 - 4mn})^2.\]
The term on the right over the numbers \( a(n, r, \gamma) \) is in most cases an infinite sum and computing it in exact form requires a little work. We proceed similarly to section 7 of [27]. If the discriminant \(|A|\) is square then this is a finite series (because there are only finitely many \( r \in \mathbb{Z} - \langle \gamma, \beta \rangle \) which make \( r^2 - 4mn \) a perfect square), and the series can be summed directly. Otherwise, let \( d_\beta, d_\gamma \) be the denominators of \( \beta \) and \( \gamma \); i.e. the smallest positive integers with \( d_\beta \beta, d_\gamma \gamma = 0 \) in \( A \), and let \( K = \mathbb{Q}(\sqrt{|A|}) \) with ring of integers \( O_K \). The point of section 7 of [27] is that there are finitely many algebraic integers \( \mu_i, i = 1, \ldots, N \) of norm \( N_{K/Q}(\mu_i) = 4d_\beta^2d_\gamma^2mn \) and finitely many units \( \varepsilon_i \in O_K \) such that, as \( r \) runs through numbers \( \mathbb{Z} \pm \langle \gamma, \beta \rangle \) for which \( (r^2 - 4mn)|A| \) is square, \( d_\beta d_\gamma (|r| - \sqrt{r^2 - 4mn}) \) runs through the set

\[
\{ \mu_i, \mu_i \varepsilon_i^{-n}, \mu_i \varepsilon_i^{-n} : i = 1, \ldots, N, n \in \mathbb{N} \}
\]

exactly once, with two possible exceptions: that \( r^2 - 4mn = 0 \) has a solution (in which case it takes the value \(|r|\) twice) or that \( \mu_i'/\mu_i \in O_K \) (in which case, including \( \mu_i'\varepsilon_i^{-n} \) causes the result to be doubled). Here \( \mu_i' \) is the conjugate of \( \mu_i \) in \( K \). Moreover, the modified coefficient

\[
a_i = a(n, r, \gamma) \begin{cases} 1 : & r^2 \neq 4mn; \\
2 : & r^2 = 4mn;
\end{cases}
\]

depends only on the index \( i \) of \(|r| - \sqrt{r^2 - 4mn}\) in that set and the sign \( \text{sgn}(r) \) equals \((-1)^n\) if \(|r| - \sqrt{r^2 - 4mn} \in \{ \mu_i \varepsilon_i^{-n}, \mu_i \varepsilon_i^{-n} \}\).

With that in mind we compute

\[
\begin{align*}
\sum_{r \in \mathbb{Z} - \langle \pm \gamma, \beta \rangle} \text{sgn}(r)a(n, r, \pm \gamma)(|r| - \sqrt{r^2 - 4mn})^2 &= \sum_{i=1}^{N} \frac{a_i}{d_\beta d_\gamma N_{K/Q}(1 + \varepsilon_i^2)} \left( \mu_i^2 + (\mu_i')^2 \right) \left( \varepsilon_i^{-2n} \right) \times \begin{cases} 1 : & \mu_i'/\mu_i \notin O_K; \\
2/1 : & \mu_i'/\mu_i \in O_K
\end{cases} \\
&= \sum_{i=1}^{N} \frac{a_i}{d_\beta d_\gamma N_{K/Q}(1 + \varepsilon_i^2)} \left( \mu_i^2 - (\mu_i')^2 \right) \left( \varepsilon_i^{-2n} \mu_i\varepsilon_i^{-2} \right) \times \begin{cases} 1 : & \mu_i'/\mu_i \notin O_K; \\
2/1 : & \mu_i'/\mu_i \in O_K
\end{cases}
\end{align*}
\]

Since each \( a_i \) lies in \( \sqrt{|a|}\cdot \mathbb{Q} \) and since each \( \mu_i^2 - (\mu_i')^2 \) and \( (\mu_i\varepsilon_i)^2 - (\mu_i'\varepsilon_i')^2 \) lies in \( \sqrt{|a|}\cdot \mathbb{Q} \), it follows that

\[
\frac{1}{32m^{3/2}} \sum_{r \in \mathbb{Z} - \langle \pm \gamma, \beta \rangle} \text{sgn}(r)a(n, r, \pm \gamma)(|r| - \sqrt{r^2 - 4mn})^2
\]

is rational. This is twice the actual correction term

\[
\frac{1}{32m^{3/2}} \sum_{r \in \mathbb{Z} - \langle \gamma, \beta \rangle} \text{sgn}(r)a(n, r, \gamma)(|r| - \sqrt{r^2 - 4mn})^2
\]

in the formula of theorem 2 by antisymmetry of \( R_{3, m, \beta} \), as we can see by replacing \( r \) by \(-r\). Therefore \( R_{3, m, \beta} \) again has rational Fourier coefficients.

6. Orthogonal modular forms and the theta lift

The modular forms we constructed in this note are mapped under the theta lift to orthogonal modular forms with explicitly computable Fourier expansions. Since this is the main application of these coefficient formulas, it seems appropriate at this point to review how the theta lift works. Orthogonal modular forms are automorphic forms on the Hermitian symmetric spaces acted upon by the special orthogonal groups of quadratic forms of signature \((2, \ell)\), and the theta lift produces orthogonal modular forms from elliptic modular form inputs by integration against a theta kernel. The “classical” approach to the theta lift was given by Gritsenko [10] in the equivalent language of Jacobi forms. We mostly follow Borcherds [2] below with a few changes to the exposition.

We simplify things slightly and consider only lattices from which a unimodular hyperbolic plane can be split off. In this way we associate automorphic forms to a Lorentzian matrix \( S \) which describes the quadratic
form on the orthogonal complement of that plane. In many cases of interest this restriction is either already satisfied, or it becomes satisfied after replacing the lattice by a rescaling of its dual lattice which leaves the orthogonal group the same (the Atkin-Lehner involutions in the sense of section 3.6 of [1]).

Fix a symmetric, integral matrix $S$ of signature $(1, \ell - 1)$ with even diagonal. Let $Q(x) = x^T S x / 2$ and $\langle x, y \rangle = x^T S y$ denote the quadratic and bilinear forms defined by $S$. Also fix a positive cone $C$ of $S$, i.e. a connected component of those vectors $v \in \mathbb{R}^\ell$ with $v^T S v > 0$. The orthogonal upper half-space is

$$\mathbb{H}_S = \{ z = u + iv \in \mathbb{C}^\ell : v \in C \}.$$  

Also let $\tilde{S}$ denote the block matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and let $I$ be the identity matrix of size $\ell + 2$. The orthogonal modular group is

$$\Gamma_S = \left\{ M \in SL_{\ell+2}(\mathbb{Z}) : M^T \tilde{S} M = \tilde{S}, M \equiv I \bmod \tilde{S}, M \text{ preserves } \mathbb{H}_S \right\}.$$  

(“$M \equiv I \bmod \tilde{S}$” means that $(M - I)\tilde{S}^{-1}$ is integral. What “$M$ preserves $\mathbb{H}_S$” means will be explained in the next paragraph.)

$\Gamma_S$ acts on $\mathbb{H}_S$ by Möbius transformations in the following sense. For $M \in \Gamma_S$ and $z \in \mathbb{H}_S$, write $\tilde{z} = \begin{pmatrix} -Q(z) \\ 1 \end{pmatrix}$; then one defines

$$M \cdot z = w \text{ if and only if } M\tilde{z} = j(M;z)\tilde{w} \text{ for some } j(M;z) \in \mathbb{C}.$$  

Here $M\tilde{z}$ is the usual matrix-vector multiplication. Such a factor $j(M;z)$ does not exist for arbitrary $M \in SO(\tilde{S})$ because $M$ may swap the connected components of positive-norm vectors. The condition that $j(M;z)$ does exist is the last part of the definition of $\Gamma_S$; in this case, $j(M;z)$ is unique.

**Definition 8.** An orthogonal modular form $f(z)$ of weight $k$ is a holomorphic function $f : \mathbb{H}_S \to \mathbb{C}$ which transforms by $f(M \cdot z) = j(M;z)^k f(z)$ for all $M \in \Gamma_S$ and in whose Fourier expansion

$$f(z) = \sum_{r \in \mathbb{S}^{-1}\mathbb{Z}^\ell} a(r)q^r, \quad q^r = e^{2\pi i (r,z)} = e^{2\pi i r^T S z},$$

the coefficient $a(r)$ is zero unless $r$ lies in the closure of the positive cone $C$ (or equivalently $r^T S v \geq 0$ for all $z = u + iv \in \mathbb{H}_S$). It is a *cusp form* if its Fourier coefficients are supported only on the positive cone itself rather than its closure.

Such a Fourier expansion exists because $\Gamma_S$ contains all translations by lattice vectors:

$$T_b = \begin{pmatrix} 1 & -b^T S & -Q(b) \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad T_b \cdot z = z + b, \ b \in \mathbb{Z}^\ell.$$  

The vanishing condition is redundant when $\ell \geq 3$ (and in many cases when $\ell = 2$) by Koecher’s principle.

**Proposition 9.** Let $k \in \mathbb{N}, k \geq 2$. Define a finite quadratic module $(A, \overline{Q})$ by $A = S^{-1}\mathbb{Z}^\ell / \mathbb{Z}^\ell$ and for $\gamma \in A$, $\overline{Q}(\gamma) = -\gamma^T S^{-1} \gamma + \mathbb{Z}$ and let

$$F(\tau) = \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - \overline{Q}(\gamma)} c(n, \gamma)q^n \epsilon_{\gamma}$$

be a cusp form of weight $k + 1 - \ell/2$ for the Weil representation $\rho^* = \rho^*_A(\overline{Q})$. Then the theta lift

$$\Phi_F(z) = \sum_{\lambda \in S^{-1}\mathbb{Z}^\ell \cap C} \sum_{n=1}^{\infty} c(Q(\lambda), \lambda)n^{k-1}q^{n\lambda}$$

of $F$ is an orthogonal cusp form of weight $k$.

**Proof sketch.** We sketch the argument in [2]. $\Phi_F$ is the Petersson scalar product of $F$ with a theta kernel:

$$\Phi_F(z) = \frac{1}{2} (2iQ(v))^{-k} \int_{SL_2(\mathbb{Z}) \setminus \mathbb{H}} \langle F(\tau), y^{\ell/2} \Theta_k(\tau, z) \rangle y^{k-1-\ell/2} \, dx \, dy,$$
where

\[ \Theta_k(\tau, z) = \sum_{a,c \in \mathbb{Z}} \sum_{b \in \mathbb{Z}^n} (aQ(z) + b^Tz + c)^k e^{-\frac{\pi i}{4}aQ(z) + b^Tz + c + 2\pi i b^T \frac{z}{2} - a} \epsilon_{S^{-1}b}, \]

for \( \tau = x + iy \in \mathbb{H} \) and \( z = u + iv \in \mathbb{H}_S \). This theta integral is well-defined because \( y^{\ell/2} \Theta_k \) transforms like a modular form of weight \( k + 1 - \ell/2 \) in the \( \tau \)-variable by the usual Poisson summation argument. (If \( F \) has weight less than two then this integral diverges and would need to be regularized if this were meant to be a rigorous argument.)

To compute the integral one reduces from \( \tilde{S} \) to \( S \) as in section 5 of [2] to express \( \Theta_k \) as a Poincaré series in the variable \( \tau \) whose seed function is essentially a theta function for \( S \). Attached to \( S \) one would like to define the theta function

\[ \vartheta_k(\tau, z) = \sum_{b \in \mathbb{Z}^n} (b^Tv)^k e^{-\frac{\pi i}{4} b^T v + \pi i b^T S^{-1} b} \epsilon_{S^{-1}b}, \]

however, for \( k \geq 2 \) these series transform in \( \tau \) under \( Mp_2(\mathbb{Z}) \) like a quasimodular form [12] rather than a true modular form because the polynomial \((b^Tv)^k\) is not harmonic in \( b \). This should be resolved by identifying \( \vartheta_k \) as the “constant term” in a polynomial \( \vartheta \) in \((1/4\pi y)\) whose coefficients are theta functions and which does transform correctly under \( SL_2(\mathbb{Z}) \). For \( n \in \mathbb{Z} \) one defines

\[ \vartheta_k(\tau, z, n) = \sum_{j=0}^{[k/2]} \sum_{b \in \mathbb{Z}^n} (-Q(v)/4\pi y)^j \frac{k!}{(k - 2j)!} (b^Tv)^{k - 2j} e^{-\frac{\pi i}{4} b^T v + \pi i b^T S^{-1} b - 2\pi inb^T u} \epsilon_{S^{-1}b}, \]

and the special case \( \vartheta_k(\tau, z) = \vartheta_k(\tau, z, 0) \).

The special case of theorem 5.2 and theorem 7.1 of [2] considered here becomes

\[ y^{\ell/2} \Theta_k(\tau, z) = i^k \sqrt{Q(v)} y^{(\ell - 1)/2} \vartheta_k(\tau, z) + 2k \sum_{n=0}^{\infty} \sum_{h=0}^{k} Q(v)^{k + 1/2} n^h \binom{k}{h} \varphi_{k+1-\ell/2} \left( e^{-\pi n^2 Q(v)/y} y^{(\ell - 1)/2 - h} \vartheta_{k-h}(\tau, z, n) \right). \]

The integral of \( F \) against the first term \( i^k \sqrt{Q(v)} y^{(\ell - 1)/2} \vartheta_k(\tau, z) \) vanishes by section 9 of [2] and the integral against the Poincaré series can be computed by the usual Rankin-Selberg unfolding method. \( \square \)

**Remark 10.** Oda [17] observed that the easiest way to show that \( \Phi_F \) is an orthogonal modular form is (in large enough weight) to check this when the input function \( F \) is a Poincaré series \( P_{k+1-\ell/2, m, \beta} \) by unfolding the theta integral against \( P_{k+1-\ell/2, m, \beta} \) rather than the kernel. This is a short computation:

\[ \Phi_F(z) = \frac{1}{2} (2iQ(v))^{-k} \sum_{a,c \in \mathbb{Z}} \sum_{b \in \mathbb{Z}(2^l + \beta)} \frac{aQ(z) + b^T z + c^k}{b^T S^{-1}b - 2 - ac = m} \int_0^\infty e^{-\pi y|aQ(z) + b^T z + c|/Q(v)} y^{k-1} dy \]

\[ = \frac{1}{2} (2iQ(v))^{-k} \sum_{a,b,c} aQ(z) + b^T z + c^k \frac{(k - 1)!Q(v)}{\pi^k|aQ(z) + b^T z + c|2^k} \]

\[ = \frac{(k - 1)!}{2} (2\pi i)^{-k} \sum_{a,b,c} aQ(z) + b^T z + c^{-k}, \]

which is holomorphic and whose behavior under \( \Gamma_S \) can be checked directly. The principle of holomorphic projection implies that the above forms arise as Fourier coefficients in a different kernel function for the same lift (see e.g. [13] or [29]). We also mention that Borcherds ([2], section 14) defined a singular theta lift that maps nearly-holomorphic modular forms (poles at cusps being allowed) to meromorphic orthogonal modular forms whose singularities are all poles of order exactly \( k \) along rational quadratic divisors. In large weights Borcherds’ singular theta lift is equivalent to this remark continuing to hold for \( m < 0 \).
Example 11. The motivation of the theta lift was originally to generalize the Shimura lift [21]. The vector-valued version above corresponds to Kohnen’s refinement of the Shimura lift [13] and it may be worthwhile to review how. Let $N \in \mathbb{N}$ and set $S = (2N)$ and $\tilde{S} = \left( \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right)$. As positive cone we take $C = \{ y > 0 \}$ so that $\mathbb{H}_S$ is just the usual upper half-plane $\mathbb{H}$. There is an embedding

\[ \Psi : \Gamma_0(N)/\{ \pm I \} \to \Gamma_S, \quad \left( \begin{array}{cc} a & b \\ Nc & d \end{array} \right) \mapsto \left( \begin{array}{cc} a^2 & -2Nab - Nbc^2 \\ -ac & 2Nbc + 1 \\ -Nc^2 & 2Nd \end{array} \right) \]

which comes from the action of $\Gamma_0(N)$ by conjugation on symmetric matrices. Since

\[ \Psi(M) \left( -\frac{Nz^2}{1} \right) = (Ncz + d)^2 \left( -N(az + b)(Ncz + d)^2 \right), \]

the orthogonal action of $\Gamma_0(N)$ on $\mathbb{H}_S$ through $\Psi$ is the usual action by Möbius transformations and the factor of automorphy is $j(M; z) = (Ncz + d)^2$. Therefore orthogonal modular forms of weight $k$ for $S$ are also elliptic modular forms of level $\Gamma_0(N)$ and weight $2k$.

If $N = 1$ or $N$ is prime then vector-valued modular forms of weight $k \in 1/2 + 2\mathbb{Z}$ for $\rho^\ast_{(A,\overline{Q})}$ can be identified bijectively with modular forms of level $\Gamma_0(4N)$ that satisfy Kohnen’s vanishing condition on Fourier coefficients [13]. This works by identifying $f(\tau) = \sum_{\gamma \in A} f(\gamma) e_\gamma$, with the scalar-valued form $\sum_{\gamma \in A} f_\gamma(4N\tau)$. Through this identification the theta kernel becomes the result of projecting the theta kernel of Niwa and Shintani [16,22] into the Kohnen plus space:

\[ \Theta_k(\tau, z) = \sum_{a,b,c \in \mathbb{Z}} \sum_{d \equiv 0 (N)} (az^2 + bz + c)e^{\frac{-\pi i z}{N}}|az^2 + bz + c|^2 + 2\pi i(\tau - 4ac), \quad \tau = x + iy, \quad z = u + iv. \]

It does not make sense to consider weights $k \in 3/2 + 2\mathbb{Z}$ from this point of view because forms $f \in S_k(\rho^\ast)$ are antisymmetric and therefore the sums $\sum_{\gamma \in A} f_\gamma(4N\tau)$ are always zero. On the other hand, proposition 9 for vector-valued input forms continues to make sense.

This is vacuous in level $N = 1$ because there are no nonzero forms that are antisymmetric. In level $N = 2$, where the finite quadratic module is $A = \mathbb{Z}/4\mathbb{Z}$, $\overline{Q} = -\gamma^2/8 \in \mathbb{Q}/\mathbb{Z}$, the input forms in weights $k \in 3/2 + 2\mathbb{Z}$ can be identified with scalar valued modular forms of level $1$ by the isomorphism

\[ M_{k-3/2} \sim \rightarrow S_k(\rho^\ast), \quad f(\tau) \mapsto f(\tau)\eta(\tau)^3(\epsilon_{1/4} - \epsilon_{3/4}), \]

where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is Dedekind’s eta function. (Showing that this is an isomorphism amounts to comparing the action of $S = ((0 1 \mathbb{Z}), \sqrt{7})$ through $\rho^\ast_{(A,\overline{Q})}$ with Dedekind’s functional equation for $\eta$.) For example, let $E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \in M_4$. Writing

\[ E_4(\tau) \eta(\tau)^3(\epsilon_{1/4} - \epsilon_{3/4}) = (q^{1/8} + 237q^{9/8} + 1440q^{17/8} + 245q^{25/8} + \ldots)(\epsilon_{1/4} - \epsilon_{3/4}) \]

as $\sum_{n=1}^{\infty} c(n)q^n(\epsilon_{1/4} - \epsilon_{3/4})$ we find that its Shimura lift (theta lift) is

\[ \sum_{n=1}^{\infty} \left( \sum_{d|n} \left( \begin{array}{c} -1 \\ d \end{array} \right) c(d)q^n(n/d)^4 \right) q^n = q + 2q^2 + (3q^4 - 237)q^3 + 4q^4 + (5q^4 + 245)q^5 + \ldots \]

\[ = q + 16q^2 - 156q^3 + 256q^4 + 870q^5 + \ldots \]

\[ = \eta(\tau)^6 \eta(2\tau)^8 (2E_2(2\tau) - E_2(\tau)) \in S_{10}(\Gamma_0(2)), \]

where $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n$, $\sigma_1(n) = \sum_{d|n} d$. While these input functions can always be interpreted as (skew-holomorphic) Jacobi forms as in [23], it seems difficult in general to use scalar-valued modular forms to determine the input function space in the way that we did here for $N = 2$. On the other hand, the formula in theorem 2 makes computing this space (for all $N$) quite easy; for example, we recover the input function above with

\[ R_{11/2,1/8,3}(\tau) = E_4(\tau)\eta(\tau)^3(\epsilon_{1/4} - \epsilon_{3/4}). \]
7. Example: the Doi-Naganuma lift

Let $K$ be a real-quadratic number field of discriminant $D \equiv 1 \mod 4$ and ring of integers $O_K = \mathbb{Z} [\omega]$, where $\omega = \frac{1 + \sqrt{D}}{2}$. Recall that Hilbert modular forms of weight $k \in \mathbb{N}_0$ for $O_K$ are holomorphic functions $f(\tau_1, \tau_2)$ on the product $\mathbb{H} \times \mathbb{H}$ of two upper half-planes that transform under $PSL_2(O_K)$ through

$$f \left( \frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a'\tau_2 + b'}{c'\tau_2 + d'} \right) = (c\tau_1 + d)^k(c'\tau_2 + d')^k f(\tau_1, \tau_2)$$

(where $a'$ is the conjugate of $K$ of $a$), and which satisfy a growth condition at cusps. Let $S$ denote the Lorentzian Gram matrix $\left( \frac{1}{2}, \frac{1}{2} \right)$ and $Q(v) = v^T S v/2$, such that $Q(v_1, v_2) = N_{K/\mathbb{Q}}(v_1 + v_2 \omega)$ for all $v_1, v_2 \in \mathbb{Z}$. Fix the positive cone $C$ containing the vector $(1, 0)$. Then the orthogonal upper half-space $\mathbb{H}_S$ can be identified with $\mathbb{H} \times \mathbb{H}$ via

$$(\frac{z_1}{z_2}) \in \mathbb{H}_S \leftrightarrow (\frac{\tau_1}{\tau_2}) = \left( \frac{1}{\omega}, \frac{z_1}{z_2} \right) \in \mathbb{H} \times \mathbb{H},$$

such that $Q(z_1, z_2) = \tau_1 \tau_2$. There is an embedding $\Psi : PSL_2(O_K) \rightarrow \Gamma_S$ induced by the action of $PSL_2(O_K)$ by conjugation on the lattice of conjugate-symmetric matrices with entries in $O_K$. The easiest way to describe $\Psi$ explicitly is to observe that $PSL_2(O_K)$ is generated by the translations $T_\nu = (\begin{array}{cc} 1 & \nu \\ 0 & 1 \end{array})$ and by $S = (\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array})$ [24],[29], and that $\Psi$ maps the translation $T_\nu, \nu = a + b\omega$ to the orthogonal translation $T_\mu = (\begin{array}{cc} 1 - \mu^T S & -Q(\mu) \\ 0 & \mu \end{array})$ by $\mu = (\frac{a}{b})$ and it maps $S$ to the matrix $\Psi(S) = (\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{array}) \in \Gamma_S$. From

$$\Psi(S) \left( -Q(z_1, z_2) \begin{array}{c} z_1 \\ z_2 \\ 1 \end{array} \right) = -Q(z_1, z_2) \left( \begin{array}{c} -1/Q(z_1, z_2) \\ (z_1 + z_2)/Q(z_1, z_2) \\ -z_2/Q(z_1, z_2) \end{array} \right),$$

we see that the orthogonal action of $S$ on $\mathbb{H}_S$ is $(z_1, z_2) \mapsto (\frac{1}{Q(z_1, z_2)})(z_1 + z_2, -z_2)$ with factor of automorphy $Q(z_1, z_2)$. In terms of $\tau_1 = z_1 + z_2 \omega$ and $\tau_2 = z_1 + z_2 \omega'$, this sends $(\tau_1, \tau_2)$ to $(-1/\tau_1, -1/\tau_2)$ with factor of automorphy $\tau_1 \tau_2$. Finally, the translations in the sense of orthogonal and Hilbert modular forms have trivial factor of automorphy. Therefore the actions of $PSL_2(O_K)$ and $\Gamma_S$ are compatible and this gives the interpretation of orthogonal modular forms for $S$ as Hilbert modular forms for $O_K$ of the same weight.

(Such an interpretation also makes sense when $K$ has even discriminant. There is an ambiguity in this case with two natural, inequivalent definitions of Hilbert modular forms for $O_K$, both of which can be constructed via orthogonal modular forms.)

When $K$ has prime discriminant $p$ and the weight $k$ is even, [4] gives an equivalence between vector-valued modular forms for the Weil representation attached to the finite quadratic module $S$ and scalar-valued modular forms for $\Gamma_0(p)$ with the quadratic Nebentypus $\chi(n) = (\frac{n}{p})$ in which the coefficient of $q^n$ vanishes if $\chi(n) = -1$. These are exactly the input functions in most treatments of the Doi-Naganuma lift (i.e. the theta lift) e.g. [8],[17],[29]. This equivalence involves summing the components of the vector-valued modular forms and in particular there seems to be no such interpretation of these input functions in odd (antisymmetric) weights in the literature, where those component sums are always zero.

We give an example in the simplest case $K = \mathbb{Q}(\sqrt{5})$. There is a cusp form $s_5$ of weight 5 which was constructed by Gundlach [11] as the product of ten theta constants and whose divisor consists of a simple zero exactly on the orbit of the diagonal $\{(\tau, \tau) : \tau \in \mathbb{H}\}$. The Fourier expansion of the form $s_5$ is somewhat easier to compute as a Borcherds product ([4], eq. 17) and is much easier to compute as a Doi-Naganuma lift. From the formula of theorem 2 we find the input form

$$R_{5,1/5,(2/5,1/5)}(\tau) = (q^{1/5} + 42q^{6/5} - 108q^{11/5} - 4q^{16/5} - 378q^{21/5} + 1512q^{26/5} \pm ...) (\epsilon(2,1/5) - \epsilon(3,5/4,5)/5))$$

$$+ (-26q^{4/5} - 39q^{9/5} + 378q^{14/5} - 140q^{19/5} - 420q^{24/5} \pm ...) (\epsilon(1,5/3,5) - \epsilon(4,5/2,5)/5),$$

which spans $S_5(\rho^*)$ and whose theta lift spans the one-dimensional space of Hilbert cusp forms of weight 5, so it equals $s_5$ up to a scalar multiple.
Remark 12. The map $\Psi : PSL_2(\mathcal{O}_K) \to \Gamma_S$ considered above is never surjective. An explicit matrix which is not contained in the image of $\Psi$ is 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]
which maps $(z_1, z_2) \mapsto (z_1 + z_2, -z_2)$ or in the picture of Hilbert modular forms $(\tau_1, \tau_2) \mapsto (\tau_2, \tau_1)$. Its factor of automorphy is $(-1)$. In particular, Doi-Naganuma lifts $f(\tau_1, \tau_2)$ of weight $k \in \mathbb{N}$ always satisfy the (anti)symmetry $f(\tau_2, \tau_1) = (-1)^k f(\tau_1, \tau_2)$.

8. EXAMPLE: CLASS NUMBER SUMS

Let $(A, Q)$ be the cyclic quadratic module $A = \frac{1}{2} \mathbb{Z}/\mathbb{Z}$ with $Q(x) = -Nx^2 + Z$ for some odd number $N \in \mathbb{N}$. (If $N$ is even then $Q$ is degenerate because 1/2 is orthogonal to all of $A$.) A special case of [3] shows that the signature of $(A, Q)$ is
\[
\text{sig}(A, Q) \mod 8 = \begin{cases} 0 : & N \equiv 1 \pmod{4}, \\
6 : & N \equiv 3 \pmod{4}.
\end{cases}
\]
In particular, if $N \equiv 1 \pmod{4}$ then we can form the Jacobi Eisenstein series of weight 2.

Let $H(d)$ denote the Hurwitz class number: the number of equivalence classes under $SL_2(\mathbb{Z})$ of binary quadratic forms of discriminant $d$, each quadratic form weighted by $2/w$ where $w$ is the size of its automorphism group. It was pointed out in [28] that the Jacobi Eisenstein series of index $(m, \beta) = (1/N, 1/N)$ for the cyclic quadratic module $(A, Q)$ above has the Fourier expansion
\[
E^*_{2,1/N,1/N}(\tau, z) = -12 \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} \sum_{r \in \mathbb{Z} - (-\gamma, 1/N)} H(4n - Nr^2)q^n \zeta^r \epsilon_\gamma,
\]
\[
+ y^{-1/2} \sum_{\gamma \in A} \sum_{n \in \mathbb{Z} - Q(\gamma)} \sum_{r \in \mathbb{Z} - (-\gamma, 1/N)} A(n, r, \gamma) \beta(\pi y(Nr^2 - 4n))q^n \zeta^r \epsilon_\gamma,
\]
with $\beta(x) = \frac{1}{8\pi} \int_1^\infty u^{-3/2}e^{-xu} du$ as before and with coefficients $A(n, r, \gamma)$ determined by $A(n, r, \gamma) = -24$ if $Nr^2 - 4n = 0$; $A(n, r, \gamma) = -48$ if $Nr^2 - 4n$ is a nonzero square; and $A(n, r, \gamma) = 0$ otherwise. (Note that $4n - Nr^2$ is always integral.)

Some identities among class numbers arise by projecting the development coefficients of $E^*_{2,1/N,1/N}$ to obtain cusp forms. The zero-value $E^*_{2,1/N,1/N}(\tau, 0)$ (i.e. the 0th development coefficient) was considered in [28]. When $N = 5, 9$ there are no nonzero cusp forms of weight three so the identity we obtain from $R_{3,1/N,1/N}$ (i.e. from the first development coefficient) is particularly interesting.

Example 13. Let $N = 5$. The term $\frac{1}{x^m} \sum_{r \in \mathbb{Z} - (-\gamma, \beta)} r c(n - Q(\gamma), r, \gamma)$, $n \in \mathbb{N}$ in the formula for $R_{3,1/5,1/5}$ takes the form
\[
-30 \sum_{r \in \mathbb{Z}} (r + 4/5)H(4n - 5r^2 - 8r - 4), \quad \text{or} \quad -30 \sum_{r \in \mathbb{Z}} (r - 2/5)H(4n - 5r^2 + 4r - 4)
\]
when $\gamma$ equals 1/5 or 2/5, respectively (and their negatives when $\gamma$ equals 4/5 or 3/5).

Let $K = \mathbb{Q}(\sqrt{5})$ with ring of integers $\mathcal{O}_K$. One can calculate that the units $\epsilon_i$ in section 5 are always $\epsilon = \frac{7 + 3\sqrt{5}}{2}$; and that for $\mu = a + b\sqrt{5} \in \mathcal{O}_K$,
\[
\mu^2 - (\mu')^2 + (\mu\epsilon)^2 - (\mu'\epsilon')^2 = -3\sqrt{5}(7a^2 - 30ab + 35b^2).
\]
The correction term
\[
C(n) = \frac{1}{32(1/5)^{3/2}} \sum_{\gamma \in A} \text{sgn}(r) a(n, r, \gamma) \left( |r| - \sqrt{r^2 - 4n/5} \right)^2, \quad \gamma \in (1/5, 2/5), \quad n \in \mathbb{Z} - Q(\gamma),
\]
is somewhat easier to compute in terms of ideals of $\mathcal{O}_K$ of norm $5n \in \mathbb{Z}$:

(1) For any ideal $a$ of norm $N(a) = 5Z + 4$, let $a + b\sqrt{5}, c + d\sqrt{5} \in a$ be generators with minimal positive trace $2a > 0$, $2c > 0$ for which $2a \equiv 1 \pmod{5}$ and $2c \equiv 4 \pmod{5}$. Then we find
\[
C(n) = \frac{1}{5} \sum_{N(a) = 5n} \left( 7(c^2 - a^2) - 30(|cd| - |ab|) + 35(d^2 - b^2) \right), \quad n \in 4/5 + \mathbb{Z}.
\]
(2) For any ideal \(a\) of norm \(N(a) \in 5\mathbb{Z} + 1\), let \(a + b\sqrt{5}, c + d\sqrt{5} \in a\) be generators with minimal positive trace \(2a > 0, 2c > 0\) for which \(2a \equiv 2 \pmod{5}\) and \(2c \equiv 3 \pmod{5}\). Then we again find
\[
C(n) = \frac{1}{5} \sum_{N(a)=5n} \left(7(c^2 - a^2) - 30(|cd| - |ab|) + 35(d^2 - b^2)\right), \quad n \in 1/5 + \mathbb{Z}.
\]

Finally, comparing coefficients of \(n + 4/5\) and \(n + 1/5\) in \(R_{3,1/5,1/5} = 0\) leaves us with two bizarre identities:

**Proposition 14.** (i) For any \(n \in \mathbb{N}_0\),
\[
\sum_{r \in \mathbb{Z}} (r + 4/5)H(4n - 5r^2 - 8r) = -\frac{1}{150} \sum_{N(a)=5n+4} \left(7(c^2 - a^2) - 30(|cd| - |ab|) + 35(d^2 - b^2)\right).
\]

(ii) For any \(n \in \mathbb{N}_0\),
\[
\sum_{r \in \mathbb{Z}} (r - 2/5)H(4n - 5r^2 + 4r) = -\frac{1}{150} \sum_{N(a)=5n+1} \left(7(c^2 - a^2) - 30(|cd| - |ab|) + 35(d^2 - b^2)\right).
\]

In both cases, \(a\) runs through all ideals of \(\mathcal{O}_K = \mathbb{Z}[1 + \sqrt{5}]\) of the given norm and for each \(a, a,b,c,d\) are defined according to (1) and (2) in the paragraph above.

**Example 15.** Let \(n = 3\). The sum in (i) is
\[
\sum_{r \in \mathbb{Z}} (r + 4/5)H(4n - 5r^2 - 8r) = -\frac{6}{5}H(8) - \frac{1}{5}H(15) + \frac{4}{5}H(12) = -\frac{6}{5} \cdot 1 - \frac{1}{5} \cdot 2 + \frac{4}{5} \cdot \frac{4}{3} = -\frac{8}{15}.
\]
The ideals of norm 19 are \((2\sqrt{5} - 1)\) and \((2\sqrt{5} + 1)\), and one can choose minimal-trace generators
\[
a + b\sqrt{5} = 8 + 3\sqrt{5}, \quad c + d\sqrt{5} = \frac{9}{2} + \frac{1}{2}\sqrt{5} \in (2\sqrt{5} - 1),
\]
\[
a + b\sqrt{5} = 8 - 3\sqrt{5}, \quad c + d\sqrt{5} = \frac{9}{2} - \frac{1}{2}\sqrt{5} \in (2\sqrt{5} + 1)
\]
satisfying the congruence conditions. In particular both ideals contribute the same to the sum on the right which is
\[
-\frac{2}{150} \left(7 \cdot (81/4 - 64) - 30 \cdot (9/4 - 24) + 35 \cdot (1/4 - 9)\right) = -\frac{2}{150} \cdot 40 = -\frac{8}{15}.
\]
The sum in (ii) for \(n = 3\) is
\[
-\frac{7}{5}H(3) - \frac{2}{5}H(12) + \frac{3}{5}H(11) + \frac{8}{5}H(0) = -\frac{7}{5} \cdot \frac{1}{3} - \frac{2}{5} \cdot \frac{4}{3} + \frac{3}{5} \cdot 1 + \frac{8}{5} \cdot \frac{1}{12} = -\frac{8}{15}.
\]
The only ideal of norm 16 in \(\mathcal{O}_K\) is \(a = (4)\), and one can choose minimal-trace generators
\[
a + b\sqrt{5} = 6 + 2\sqrt{5}, \quad c + d\sqrt{5} = 4 \in a.
\]
Therefore the sum on the right is
\[
-\frac{1}{150} \left(7 \cdot (4^2 - 6^2) - 30 \cdot (0 - 12) + 35 \cdot (0 - 2^2)\right) = -\frac{1}{150} \cdot 80 = -\frac{8}{15}.
\]

**Remark 16.** A similar computation is possible for the Jacobi Eisenstein series attached to the cyclic quadratic module of discriminant \(N = 9\) and index \((1/9, 1/9)\) considered above. We find a particularly nice result by considering the components attached to the isotropic elements \(1/3, 2/3 \in \mathbb{A}\). Comparing coefficients in \(R_{3,1/9,1/9} = 0\) yields the identities
\[
\sum_{r \equiv 1(3)} rH(4n - r^2) = \varepsilon(n) \sum_{d|n} \left(\frac{d}{3}\right) \min(d, n/d^2), \quad \sum_{r \equiv 2(3)} rH(4n - r^2) = -\varepsilon(n) \sum_{d|n} \left(\frac{d}{3}\right) \min(d, n/d^2),
\]
where \(\left(\frac{d}{3}\right)\) is the Kronecker symbol and where \(\varepsilon(n) = -1\) if 3 divides \(n\), and \(\varepsilon(n) = 1/2\) otherwise. The derivation is easier than the case \(N = 5\) given above so we omit it. (The point is that if \(r^2 - 4n/9\) is integral then \(d = (3/2)(|r| - \sqrt{r^2 - 4n/9})\) is always integral, and this runs through certain divisors of \(n\) which are always less than or equal to \(n/d = (3/2)(|r| + \sqrt{r^2 - 4n/9})\); this is where the \(\min(d, n/d^2)\) term comes from.)


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