Antisymmetric modular forms

1 Introduction

Let S be a Gram matrix of signature (b^+, b^-). The purpose of this SAGE worksheet is to extend the algorithms in the program "PSS" to compute modular forms for the (dual) Weil representation attached to S in weights k for which $2k + b^+ - b^- \equiv 2 \bmod 4$. It calculates the coefficients of the series

$$R_{k,m,\beta} = \sum_{\lambda \in \mathbb{Z}} \lambda P_{k,\lambda^2 m,\lambda\beta}$$

where $P_{k,m,\beta}$ is the Poincaré series, where $\beta \in S^{-1}\mathbb{Z}^{b^+ + b^-}$ and where $m \in \mathbb{Z} - Q(\beta)$ is a positive number. These modular forms always have rational Fourier coefficients and for $k \geq 3$ they always span the cusp space $S_k(\rho^*)$.

2 Functions

Let S be a Gram matrix with signature (b^+, b^-), let $e = b^+ + b^-$ and let k be a weight such that $2k + b^+ - b^- \equiv 2\,(4)$.

The function

```
DimensionFormula(k,S)
```

computes the dimension of $M_k(\rho^*)$ (modular forms) and $S_k(\rho^*)$ (cusp forms) and outputs them as a list $[\dim M_k(\rho^*), \dim S_k(\rho^*)]$.

The function

```
DiscriminantGroup(S)
```

computes a list of representatives γ of the discriminant group $S^{-1}\mathbb{Z}^e/\mathbb{Z}^e$ The function

```
ReducedDiscriminantGroup(S)
```

only outputs one representative from each pair $\pm\gamma$ with $Q(\gamma) \neq 0$.

The function

```
PSSd(g,b,m,n,k,S)
```

returns the coefficient $c(n, g)$ of $q^n \mathfrak{e}_g$ in the series $R_{k,m,b}$.

The function

```
CuspSpan(k,S)
```

returns a list of lists $[m, \beta]$ for which the series $R_{k,m,\beta}$ are a basis of $S_k(\rho^*)$.

3 Example: the Doi-Naganuma lift

We will compute some Doi-Naganuma lifts to $\mathbb{Q}(\sqrt{5})$. Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ are basically the same as orthogonal modular forms for the Gram matrix $S = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ of determinant -5.

Gundlach proved that the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ is generated by the Eisenstein series E_2, E_6 of weights 2 and 6 and by two cusp forms s_5, s_{15} of weights 5 and 15. The cusp form s_{15} is symmetric of odd weight so it is not a Doi-Naganuma lift. However s_5 is a Doi-Naganuma lift. We can compute its input function up to a constant multiple using the commands

```
S = -matrix([[2,1],[1,-2]])
b = vector([2/5,1/5])
m = 1/5
for g in ReducedDiscriminantGroup(S):
    offset = frac(g*S*g/2)
    for n in range(10):
        print [g,n-offset,PSSd(g,b,m,n-offset,5,S)]
```

which outputs the vector-valued input function up to precision $O(q^{10})$. By changing the 5 in PSSd(g,b,m,n-offset,5,S) to 7 or 9 we obtain input functions whose Doi-Naganuma lifts are $s_5 E_2$ and $s_5 E_2^2$, because the spaces of Hilbert modular forms of those weights are one-dimensional.