Antisymmetric modular forms

1 Introduction

Let \( S \) be a Gram matrix of signature \((b^+, b^-)\). The purpose of this SAGE worksheet is to extend the algorithms in the program “PSS” to compute modular forms for the (dual) Weil representation attached to \( S \) in weights \( k \) for which \( 2k + b^+ - b^- \equiv 2 \mod 4 \). It calculates the coefficients of the series

\[
R_{k,m,\beta} = \sum_{\lambda \in \mathbb{Z}} \lambda P_{k, \lambda^2 m, \lambda \beta}
\]

where \( P_{k,m,\beta} \) is the Poincaré series, where \( \beta \in S^{-1}\mathbb{Z}^{b^+ + b^-} \) and where \( m \in \mathbb{Z} - Q(\beta) \) is a positive number. These modular forms always have rational Fourier coefficients and for \( k \geq 3 \) they always span the cusp space \( S_k(\rho^*) \).

2 Functions

Let \( S \) be a Gram matrix with signature \((b^+, b^-)\), let \( e = b^+ + b^- \) and let \( k \) be a weight such that \( 2k + b^+ - b^- \equiv 2 \mod 4 \).

The function

\texttt{DimensionFormula}(k,S)

computes the dimension of \( M_k(\rho^*) \) (modular forms) and \( S_k(\rho^*) \) (cusp forms) and outputs them as a list \([\dim M_k(\rho^*), \dim S_k(\rho^*)]\).

The function

\texttt{DiscriminantGroup}(S)

computes a list of representatives \( \gamma \) of the discriminant group \( S^{-1}\mathbb{Z}^e/\mathbb{Z}^e \). The function

\texttt{ReducedDiscriminantGroup}(S)

only outputs one representative from each pair \( \pm \gamma \) with \( Q(\gamma) \neq 0 \).

The function

\texttt{PSSd}(g,b,m,n,k,S)

returns the coefficient \( c(n, g) \) of \( q^n e_g \) in the series \( R_{k,m,\beta} \).

The function

\texttt{CuspSpan}(k,S)

returns a list of lists \([m, \beta]\) for which the series \( R_{k,m,\beta} \) are a basis of \( S_k(\rho^*) \).
3 Example: the Doi-Naganuma lift

We will compute some Doi-Naganuma lifts to $\mathbb{Q}(\sqrt{5})$. Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ are basically the same as orthogonal modular forms for the Gram matrix $S = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ of determinant $-5$.

Gundlach proved that the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ is generated by the Eisenstein series $E_2, E_6$ of weights 2 and 6 and by two cusp forms $s_5, s_{15}$ of weights 5 and 15. The cusp form $s_{15}$ is symmetric of odd weight so it is not a Doi-Naganuma lift. However $s_5$ is a Doi-Naganuma lift. We can compute its input function up to a constant multiple using the commands

```plaintext
S = -matrix([[2,1],[1,-2]])
b = vector([2/5,1/5])
m = 1/5
for g in ReducedDiscriminantGroup(S):
    offset = frac(g*S*g/2)
    for n in range(10):
        print [g,n-offset,PSSd(g,b,m,n-offset,5,S)]
```

which outputs the vector-valued input function up to precision $O(q^{10})$. By changing the 5 in PSSd(g,b,m,n-offset,5,S) to 7 or 9 we obtain input functions whose Doi-Naganuma lifts are $s_5E_2$ and $s_5E_2^2$, because the spaces of Hilbert modular forms of those weights are one-dimensional.