NOTES ON DIFFERENTIAL EQUATIONS

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Contents

1. Some Definitions

Differential equations arise in myriad fields, from physics to engineering. Our journey into this theory begins with constant coefficient homogeneous linear second order differential equations. These are differential equations of the form:

$$ay''(t) + by'(t) + cy(t) = 0, \quad a, b, c \in \mathbf{R},\ a \neq 0. \tag{1.1}$$

We are looking for functions $y(t)$ from the real numbers to the real numbers that solve the above equation. Let's go over each of the key words in the somewhat protracted name for these things:

- **Constant coefficient**: The coefficients in front of $y''(t)$, $y'(t)$ and $y(t)$ in (1.1) are constant real numbers a, b, c that do not depend on t.
- **Homogeneous**: The right hand side of (1.1) is 0; this is precisely the way we used this term in linear algebra.

- **Linear**: The solutions to (1.1), also called the *solution space*, form a vector space. Indeed, suppose that $y_1(t)$ and $y_2(t)$ both solve (1.1). For constants $c_1, c_2 \in \mathbf{R}$, one can verify using properties of the derivative that
$$a(c_1y_1(t) + c_2y_2(t))'' + b(c_1y_1(t) + c_2y_2(t))' + c(c_1y_1(t) + c_2y_2(t)) = 0.$$
In other words, $c_1y_1(t) + c_2y_2(t)$ solves (1.1) as well. The solution space to (1.1) contains the zero function $y(t) \equiv 0$ and is also closed under linear combinations. We can thus think of it as a vector space.[1]
- **Second order**: The highest derivative in (1.1) is the second derivative.

Before we talk about how to solve (1.1), we describe the structure of the solution space. Recall that two functions $y_1(t)$ and $y_2(t)$ are linearly independent when neither of them is a constant multiple of the other.

Theorem 1.1. *The solution space to (1.1) is a two-dimensional real vector space.*

The proof is omitted. But why should we expect this to be the case? It lies in the fact that we are solving a *second* order differential equation; we then have *two* degrees of freedom. We will return to this point later when discussing intial value problems.

Theorem 1.1 tells us that if we find *any* two linearly independent solutions of (1.1) $y_1(t)$ and $y_2(t)$, then these two functions form a basis of the solutions space and all solutions to the differential equation are of the form $cy_1(t) + cy_2(t)$. This is a ***general solution*** to (1.1). We will now describe a strategy for finding these general solutions.

2. Finding General Solutions in the Homogeneous Case

Here is the algorithm for finding general solutions in the homogeneous case. Almost no motivation will be given. Let's start with (1.1), where we no longer write the "(t)" for brevity:
$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbf{R},\ a \neq 0.$$
The key idea is to form the ***auxiliary equation***. This is a polynomial equation formed by converting y'' to r^2, y' to r, and y to 1, i.e., we convert the n^{th} derivative to the n^{th} power for $n = 0, 1, 2$. So the auxiliary equation is:
$$ar^2 + br + c = 0. \tag{2.1}$$
We solve (2.1) using the quadratic formula to obtain two roots
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$
Depending on the discriminant $b^2 - 4ac$, there are three cases to consider:

(1) **Two Distinct Real Roots** ($b^2 - 4ac > 0$): There are two distinct real roots r_1 and r_2. We can say that r_1 and r_2 are *simple* roots.
(2) **One Real Root** ($b^2 - 4ac = 0$): In this case $r_1 = r_2$ and we can say that we have a *double* root.

[1]You might have realized that we've only verified the subspace criterion, so what is the larger space? One can take it to be the space of infinitely differentiable functions form $\mathbf{R}$ to $\mathbf{R}$, labelled $C^\infty(\mathbf{R})$.

(3) **Complex Roots** ($b^2 - 4ac < 0$): In this case r_1 and r_2 are two distinct complex numbers.[2]

Let's handle each of these cases separately.

2.1. **Two Distinct Real Roots.** Suppose that r_1 and r_2 are two distinct real roots of the auxiliary equation (2.1). Then we write down the general solution as:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where $\{e^{r_1 t}, e^{r_2 t}\}$ is the basis for our solution space and c_1 and c_2 are any real numbers.

Example 2.1. Consider the differential equation

$$y'' + 3y' + 2y = 0.$$

The auxiliary equation is

$$r^2 + 3r + 2 = 0,$$

which factors into

$$(r + 2)(r + 1) = 0.$$

Our two roots are thus $r_1 = -2$ and $r_2 = 1$. The general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{-t}.$$

2.2. **One Real Root.** Suppose that r is the unique real root that solves the auxiliary equation (2.1). Then we write down the general solution as

$$y(t) = c_1 e^{rt} + c_2 t e^{rt},$$

where $\{e^{rt}, te^{rt}\}$ is the basis for our solution space and c_1 and c_2 are any real numbers. Notice the factor of t in the second basis function; this is a theme that will show in later when we study the method of undetermined coefficients for non-homogeneous differential equations.

Example 2.2. Consider the differential equation

$$y'' + 2y' + 1y = 0.$$

The auxiliary equation is

$$r^2 + 2r + 1 = 0,$$

which factors into

$$(r + 1)^2 = 0.$$

Our double root is $r = -1$. The general solution is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

[2]In fact, they are conjugate to each other.

2.3. **Complex Roots.** Suppose that r_1 and r_2 are the complex roots that solves the auxiliary equation (2.1). The two resulting roots are conjugate to each other; they will have the form $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ for $\alpha, \beta \in \mathbf{R}$. Then we write down the general solution as

$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t),$$

where $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ is the basis for our solution space and c_1 and c_2 are any real numbers.

Example 2.3. Consider the differential equation

$$y'' - 4y' + 7y = 0.$$

The auxiliary equation is

$$r^2 - 4r + 7 = 0.$$

Solving using the quadratic equation, we obtain $r_1 = 2 + \sqrt{3}i$ and $r_2 = 2 - \sqrt{3}i$. The general solution is

$$y(t) = c_1 e^{2t}\cos(\sqrt{3}t) + c_2 e^{2t}\sin(\sqrt{3}t).$$

3. Initial Value Problems

3.1. **Background.** In the preceding discussion we observed that all general solutions of equation (1.1) are of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for two linearly independent solutions $y_1(t)$ and $y_2(t)$. A ***initial value problem (IVP)*** consists of an equation of the form (1.1) as well as two pieces of information that *uniquely* pin down the constants c_1 and c_2. This information takes the form of specifying $y(t_0)$ and $y'(t_0)$ for some $t_0 \in \mathbf{R}$.

To gain some intuition for these problems, let's consider a simpler case. Suppose we had the differential equation

$$y' + 3y = 0. \tag{3.1}$$

We can think of y as position and y' as velocity. For any time $t \in \mathbf{R}$, this equation tells us precisely what the velocity $y'(t)$ of our particle is if we know its position $y(t)$. Suppose, for example, that we know $y(0) = 3$. Then $y'(0) = -9$, and we know that $y(0.001)$ at time $t = 0.001$ going to be somewhere around $3 + (-9)(.001) = 2.991$. This is only an approximation, but we can continue this process to have a good idea of where the particle is going to be at time t, when all we knew was the governing equation (3.1) and $y(0) = 3$. But in fact calculus lets us make this approximation a reality: we can convince ourselves that if we infinitesimally small time steps forward, we can in fact know the location of the particle perfectly at any future (and past) time given only (3.1) and $y(0) = 3$. The evolution the particle are entirely governed by the above differential equation; the initial value $y(0) = 3$ simply gives us the starting state.

That was first-order case; we needed one piece of information to specify the particle's trajectory. In the second order case that we are considering, we have a formula for the acceleration of our particle in terms of its velocity and position. In a totally analogous manner, knowing $y(0) = 3$ and $y'(0) = 4$, for example, sets the starting state; the rest of the dynamics are totally determined by the second order differential equation (1.1).

This discussion leads us to the following theorem.

Theorem 3.1. *For any real numbers $a \neq 0, b, c, t_0, Y_0, Y_1$, there is a* ***unique*** *solution to the initial value problem (IVP)*

$$ay'' + by' + cy = 0, \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1. \tag{3.2}$$

The solution is valid for all time $t \in \mathbf{R}$.

In the next subsection, we will give the algorithm for solving an IVP, which builds on what we've done for the general solution.

3.2. **Solving an IVP.** Here we give the algorithm for solving an initial value problem (3.2). Here it is:

(1) Find the general solution for $ay'' + by' + cy = 0$. Say it is of the form $y(t) = c_1y_1(t) + c_2y_2(t)$.
(2) Plug in the initial value conditions (one must compute $y'(t)$ to apply $y'(t_0) = Y_1$) and obtain a system of two equations in c_1 and c_2. Solve this system for c_1 and c_2.[3]

Perhaps this is best illustrated in an example.

Example 3.2. Consider the IVP

$$y'' + 2y' + y = 0, \quad y(0) = 3, \quad y'(0) = 4.$$

We first find the general solution to $y'' + 2y' + y = 0$. But this is the same equation in Example 2.2, so we know that we have the general solution

$$y(t) = c_1e^{-t} + c_2te^{-t}.$$

Let's first calculate the derivative:

$$y'(t) = -c_1e^{-t} + c_2(e^{-t} - te^{-t}).$$

Plugging in $y(0) = 3$ gives us

$$3 = y(0) = c_1.$$

Plugging in $y'(0) = 4$ gives us

$$4 = y'(0) = -c_1 + c_2.$$

We thus see that $c_1 = 3$ and $c_2 = 7$, and so our solution is

$$y(t) = 3e^{-t} + 7te^{-t}.$$

[3]We may end up with some very ugly looking things like e^{-3} and $\cos(3)$ in our system. This is annoying, but remember that these will always simply be numbers and can be manipulated as such.

4. The Wronskian

Suppose we have two solutions $y_1(t)$ and $y_2(t)$ of the standard homogeneous equation (1.1). How do we know that they form a basis for the two-dimensional solution space? We must have a method of testing whether or not they are linearly independent. Thankfully, there is a way, using the Wronskian.

Definition 4.1. Let $y_1(t)$ and $y_2(t)$ be two functions. The ***Wronskian*** is the following determinant:

$$W(y_1, y_2) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

Notice that **the Wronskian is a function of t, not a number**. The key property of the Wronskian is the following result.

Proposition 4.2. *Suppose that $y_1(t)$ and $y_2(t)$ solve the exact same differential equation of the form (1.1). Then $y_1(t)$ and $y_2(t)$ are linearly dependent if and only if the Wronskian is the zero function, or in other words, the Wronskian is 0 for all $t \in \mathbf{R}$. Phrased another way, $y_1(t)$ and $y_2(t)$ are linearly independent if and only if the Wronskian is not 0 for all $t \in \mathbf{R}$, i.e., there is some $t_0 \in \mathbf{R}$ where the Wronskian does not vanish.*

This result should not seem too far out of left field: in linear algebra the determinant vanishing is precisely the condition for when the column vectors of a square matrix are linearly dependent.

Example 4.3. Suppose that $y_1(t) = e^{3t}$ and $y_2(t) = te^{3t}$.[4] We then have

$$\begin{aligned} W(y_1, y_2) &= y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ &= (e^{3t})(e^{3t} + 3te^{3t}) - (te^{3t})(3e^{3t}) \\ &= e^{6t}(1 + 3t) - e^{6t}(3t) \\ &= e^{6t}. \end{aligned}$$

But e^{6t} is not the zero function, so y_1 and y_2 are linearly independent.

The hypothesis that y_1 and y_2 are solutions to the exact same differential equation is absolutely critical. If $f(t) = t^2$ and $g(t) = t \cdot |t|$, one can check that $W(f, g) = 0$ on all of $\mathbf{R}$, but f and g are *not* linearly dependent, contrary to what we saw above. Again, this is due to the fact that f and g do not solve the same homogeneous differential equation. The best we can do in the general case if the following.

Proposition 4.4. *Suppose f and g are any two differentiable functions on $\mathbf{R}$. If f and g are linearly dependent on $\mathbf{R}$, then $W(f, g)$ is the zero function. Equivalently, if $W(f, g)$ is not the zero function, i.e., there is some t where $W(f, g)(t) \neq 0$, then f and g are linearly independent.*[5]

[4] This example could arise in the double root case.

[5] This is the *contrapositive* of the previous statement. Try to figure out why they're equivalent if it is not immediately clear.

Here we are letting f and g be literally any random differentiable functions; they do not have to solve the same second order differential equation. At the risk of being unbearably repetitive, we emphasize that one **cannot** conclude that f and g are linearly dependent if their Wronskian is the zero function for any old differentiable functions f and g.

We state an interesting result. If there's time, try to prove it before reading the solution.

Exercise 4.5. Suppose that $f(t)$ and $g(t)$ are real functions such that their Wronskian $W(f,g)$ is the zero function, i.e., $W(f,g)(t) = 0$ for all $t \in \mathbf{R}$. If $g(t) \neq 0$ for all t, show that f and g are linearly dependent.

Proof. The functions $f(t)$ and $g(t)$ being linearly independent means there is a constant c such that $f(t) = cg(t)$ for all $t \in \mathbf{R}$. Since $g(t)$ is never zero, this is equivalent to having $f(t)/g(t) = c$ for all $t \in \mathbf{R}$. From calculus, this statement is the same as saying that $(f/g)'(t) = 0$ for all $t \in \mathbf{R}$. So we must show that the derivative of f/g vanishes for all t. Compute using the quotient rule:

$$\left(\frac{f(t)}{g(t)}\right)' = \frac{f'(t)g(t) - g'(t)f(t)}{g(t)^2} = \frac{-W(f,g)(t)}{g(t)^2} = \frac{0}{g(t)^2} = 0$$

for all t as $W(f,g)$ is the zero function by hypothesis. □

5. Non-Homogeneous Differential Equations

5.1. **Background.** Here we begin our analysis of non-homogeneous second order constant coefficient linear differential equations. These take the form

$$ay''(t) + by'(t) + cy(t) = f(t), \quad a, b, c \in \mathbf{R},\ a \neq 0. \tag{5.1}$$

The only difference between this equation and (1.1) is the presence of $f(t)$. Before diving into methods of solving this problem, we first make some general observations.

First, let's recall something from linear algebra. Let A be an $m \times n$ matrix, $\vec{x} \in \mathbf{R}^n$, and $\vec{b} \in \mathbf{R}^m$. Solving the inhomogeneous matrix-vector equation $A\vec{x} = \vec{b}$ can be split into two steps:

(1) First solve the homogeneous, or *complementary* equation $A\vec{x} = \vec{0}$. Say the solution set is of the form $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ for some linearly independent vectors $\{\vec{1}, \ldots, \vec{k}\}$, and for any real numbers $c_1, \ldots, c_k$.
(2) Find *any* solution to $A\vec{x} = \vec{b}$. Let's call the solution $\vec{x}^*$.
(3) The full solution to $A\vec{x} = \vec{b}$ is then given by

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + \vec{x}^*.$$

The reason why this works is because matrix multiplication by A is a *linear* operation. But in our case we also have a *linear* differential equation. Hence we can apply the exact same strategy, basically verbatim, to solve the inhomogeneous differential equation:

(1) First solve the homogeneous, or *complementary* equation

$$ay'' + by' + cy = 0.$$

Say the general solution is of the form $y_c(t) = c_1y_1(t) + c_2y_2(t)$.
(2) Find *any* solution to $ay'' + by' + cy = f$. Call this ***particular solution*** $y_p(t)$.

(3) The ***general solution*** to the differential equation is then given by

$$y(t) = y_c(t) + y_p(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

In practice, it's finding the particular solution $y_p(t)$ that is somewhat delicate. We start exploring these methods in the upcoming subsections.

5.2. **Method of Undetermined Coefficients.** Here we discuss a method of finding the particular solution $y_p(t)$ for (5.1) that works for particular forms of $f(t)$. The idea is as follows: there are certain functions whose first and second derivatives are "essentially" of the same form. In particular, the derivative of a polynomial is a polynomial, the derivative of an exponential is an exponential, and the derivative of $\sin t$ and $\cos t$ are $\cos t$ and $-\sin t$, respectively. We can use this fact to make an educated guess for $y_p(t)$.

Example 5.1. Suppose we wish to find a particular solution for the non-homogeneous differential equation

$$y'' + 3y' + 2y = e^{4t}.$$

Given the form on the right hand side, we can make an educated guess $y_p(t) = Ae^{4t}$ for some constant A. Plugging this guess into the equation gives us

$$16Ae^{4t} + 12Ae^{4t} + 2Ae^{4t} = e^{4t}.$$

Dividing by e^{4t}, which is never zero, we end up with $30A = 1$, so $A = \frac{1}{30}$. Hence $y_p(t) = \frac{1}{30}e^{4t}$.

Let's now consider a modified version of this problem where we solve

$$y'' + 3y' + 2y = e^{-2t}.$$

Following the above procedure, we take an educated guess Ae^{-2t}. Plugging in on both sides gives us

$$4Ae^{-2t} - 6Ae^{-2t} + 2Ae^{-2t} = e^{-2t}.$$

Dividing by e^{-2t}, we get $4A - 6A + 2A = 0$ on the left and side and 1 on the right hand side. What went wrong was that e^{-2t}, as we saw in Example 2.1, is in fact a solution to the (homogeneous) complementary equation

$$y'' + 3y' + 2y = 0.$$

Perhaps a better way to say the same thing: the number -2 is a solution to the auxiliary equation $r^2 + 3r + 2 = 0$.

To fix this, we will throw in an extra factor of t into our guess, not unlike when the auxiliary equation for the homogeneous equation has a double root (Example 2.2). Our new guess is $Ate^{=2t}$. Plugging this in and simplifying in fact gives

$$-Ae^{-2t} = e^{-2t},$$

so we can take $A = -1$. Hence our particular solution is $y_p(t) = -te^{-2t}$. For completeness the full solution to the problem is then

$$y(t) = y_c(t) + y_p(t) = c_1 e^{-2t} + c_2 e^{-t} - te^{-2t}.$$

In the previous example we saw that there is some extra work to do when our guess overlaps with the solutions to the complementary equation. Here's another example:

Example 5.2. Consider the differential equation

$$y'' + 2y' = 9.$$

To find a particular solution, we rewrite the right hand side to get

$$y'' + 2y' = 9e^{0t}.$$

This might seem silly, but it lets us relate the structure on the left hand side to the solution to the complementary equation. Indeed, perhaps our naive guess for $y_p(t)$ would be $Ae^{0t} = A$, some constant.

But the auxiliary equation associated to the homogeneous equation is $r^2 + 2r = 0$. As 0 is a simple (degree one) root and we are dealing with Ae^{0t}, we have one degree of "overlap" so we should multiply our guess by t. Try plugging in Ae^{0t}; we would obtain $0 = 9$, a contradiction. Yes, we are just dealing with constants here. But writing things in terms of e^{0t} lets us relate this example to the previous one.

If we guess $Ate^{0t} = At$ as the particular solution, we can plug this into the equation to obtain $2A = 9$ and $A = \frac{9}{2}$. Hence the particular solution is $y_p(t) = \frac{9}{2}t$.

5.3. **An Algorithm for Guessing Particular Solutions.** With these examples out of the way, we can list a general algorithm for solving these differential equations. **What will truly matter are the roots of the auxiliary equation, and whether we have a single or double root.** If the roots "match" with $f(t)$ on the right hand side, then we must multiply by t or t^2 to correct things.

Let's cover the two basic cases (lifted directly from the book):

5.3.1. *Case 1.* Suppose first we have a differential equation of the form

$$ay'' + by' + cy = Cp_m(t)e^{rt},$$

where $p_m(t)$ is a degree m polynomial in t and r is a real number. Notice that this includes the case $r = 0$, where the right hand side would only be a polynomial. Then our guess will be the following:

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}.$$

Here $\{A_0, \ldots, A_m\}$ are real numbers. Notice that our guess is a polynomial of degree m multiplied by the exponential e^{rt}, but we also have the correction term t^s in case of "overlap" with solutions of the homogeneous equation.

The value of s is determined solely by whether what degree root r is in the associated auxiliary equation $ar^2 + br + c = 0$. Indeed, we have

(i) $s = 0$ if r is not a root of the associated auxiliary equation;
(ii) $s = 1$ if r is a simple root of the associated auxiliary equation; and
(iii) $s = 2$ if r is a double root of the associated auxiliary equation.

Note: **nothing about s depends on the polynomial term** $p_m(t)$.

5.3.2. *Case 2.* Suppose now we have a differential equation of the form

$$ay'' + by' + cy = \begin{cases} Cp_m(t)e^{\alpha t}\cos(\beta t) \\ Cp_m(t)e^{\alpha t}\sin(\beta t) \end{cases},$$

where once again $p_m(t)$ is a polynomial of degree m. Then our guess will be the following:

$$\begin{aligned} y_p(t) =& t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) \\ &+ t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t) \end{aligned}$$

Here $\{A_0, \ldots, A_m\}$ and $\{B_0, \ldots, B_m\}$ are real numbers. Notice that even if we have only one of $\cos(\beta t)$ or $\sin(\beta t)$, our guess requires both of them. This is because the derivative sin is cos, which may appear when we plug our guess in.

Once again there is a correction term t^s. Again, the value of s depends on the roots of the associated auxiliary equation $ar^2 + br + c = 0$. Here is the rule:

(i) $s = 0$ if $r = \alpha + i\beta$ is not a root of the associated auxiliary equation; and
(ii) $s = 1$ if $r = \alpha + i\beta$ is a root of the associated auxiliary equation.

Again, the exponent s on the correction term does not depend on the polynomial term at all. The reason why we do not have an $s = 2$ case is because complex roots of the auxiliary equation come in conjugate pairs.

In general, the form of $f(t)$ might be the sum of multiple terms of the form we studied above. In that situation our guess is simply the sum of our guesses for each term.

Example 5.3. Consider the non-homogeneous equation

$$y'' - y' - 2y = 6e^{2t}.$$

The associated auxiliary equation is $r^2 - r - 2y = 0$, which factors into $(r-2)(r+1) = 0$. The two roots are $r_1 = 2$ and $r_1 = -1$. But on the right hand side, we have $6e^{2t}$, so we have one degree of "overlap." Looking at the form in Case 1 (5.3.1), we see that our guess should be

$$y_p(t) = Ate^{2t}.$$

If we plug this in and calculate $(y_p(t))'' - (y_p(t))' - 2y_p(t) = 6e^{2t}$, we end up with $A = 2$. Hence the particular solution is $y_p(t) = 2te^{2t}$.

From above, we see that the complementary solution is $y_c(t) = c_1 e^{2t} + c_2 e^{-t}$. Then the general solution is:

$$y(t) = y_c(t) + y_p(t) = c_1 e^{2t} + c_2 e^{-t} + 2te^{2t}.$$

Example 5.4. Consider the non-homogeneous equation

$$x''(t) - 4x'(t) + 4x(t) = te^{2t}. \tag{5.2}$$

We first find the roots of the associated auxiliary equation and solve for the complementary solution. We reduce to $r^2 - 4x + 4 = 0$, which factors as $(r-2)^2 = 0$, a double root. We get $x_c(t) = c_1 e^{2t} + c_2 te^{2t}$.

Let's now make a guess for $y_p(t)$. We are in Case 1 (5.3.1). Using that notation, we see that te^{2t} on the right hand side implies $r = 2$, and so $s = 2$. Our final guess is thus

$$x_p(t) = t^2(A_1 t + A_0)e^{2t}.$$

Now we can plug in our guess and solve for the constants A_1 and A_0:

$$\begin{aligned} (x_p(t))'' &= 4A_1 t^3 e^{2t} + (4A_0 + 12A_1)t^2 e^{2t} + (8A_0 + 6A_1)te^{2t} + 2A_0 e^{2t}; \\ -4(x_p(t))' &= -8A_1 t^3 e^{2t} + (-8A_0 + 3A_1)t^2 e^{2t} + 2A_0 te^{2t}; \end{aligned}$$

$$4x_p(t) = 4A_1t^3e^{2t} + 4A_0t^2e^{2t}.$$

Plugging back into (5.2) we obtain, the following:

$$6A_1te^{2t} + 2A_0e^{2t} = te^{2t}.$$

Now there is no e^{2t} term on the right, so for the above to hold for *all* $t \in \mathbf{R}$, we must have $A_0 = 0$. We also see that $A_1 = \frac{1}{6}$. Hence the particular solution is $x_p(t) = \frac{1}{6}t^3e^{2t}$ and we have as the general solution:

$$x(t) = x_c(t) + x_p(t) = c_1e^{2t} + c_2te^{2t} + \frac{1}{6}t^3e^{2t}.$$

Example 5.5. Consider the non-homogeneous equation

$$y'' - 4y' + 3y = 3te^{2t}\cos(t) + (t^2+4)\sin(2t).$$

The auxiliary equation is $r^2 - 4r + 3 = 0$ which factors into $(r-3)(r-1) = 0$. The two roots are thus $r_1 = 3$ and $r_2 = 1$.

Let's split the right hand side up into two distinct terms:

$$f_1(t) = 3te^{2t}\cos(t), \quad f_2(t) = (t^2+4)\sin(2t).$$

The guess for $f_1(t)$ is Case 2 (5.3.2). In this case $\alpha = 2$ and $\beta = 1$. As $\alpha + i\beta = 2 + i$ is not a root of the auxiliary equation, we guess

$$y_1(t) = (A_1t + A_0)e^{2t}\cos(t) + (B_1t + B_0)e^{2t}\sin(t).$$

The guess for $f_2(t)$ is also case (2) from above. In this case $\alpha = 0$ and $\beta = 2$. As $\alpha + i\beta = 2i$ is not a root of the auxiliary equation, we guess

$$y_2(t) = (C_2t^2 + C_1t + C_0)\sin(2t) + (D_2t^2 + D_2t + D_0)\cos(2t).$$

Our final guess is $y_p(t) = y_1(t) + y_2(t)$. It will have 10 unknowns, and solving for them would be a ludicrous undertaking.

Example 5.6. Consider the following differential equation:

$$y'' - 2y' + 2y = e^t\cos t + 3e^t\sin t.$$

The associated auxiliary equation is $r^2 - 2r + 2 = 0$. The roots are $r_1 = 1+i$ and $r_2 = 1-i$. We are in Case 2 (5.3.2). We see that $\alpha = 1$ and $\beta = 1$, and in fact $\alpha + i\beta = 1 + i$. Then $s = 1$ in that notation. We then follow the algorithm and guess

$$y_p(t) = C_1te^t\cos t + C_2te^t\cos t.$$

Notice that even though there are *two* terms on the right hand side of the differential equation, they both correspond to the same $1+i$ root, so we do not need to write

$$y_p(t) = C_1te^t\cos t + C_2te^t\sin t + D_1te^t\cos t + D_2te^t\sin t.$$

Indeed, we can combine like terms to get

$$y_p(t) = (C_1 + D_1)te^t\cos t + (C_2 + D_2)te^t\sin t,$$

which is the same thing as our original guess.

In this previous example, we made some ad-hoc simplification based on the structure of the right hand side to reduce the number of constants we had to solve for, in some capacity. In more complicated examples, this type of maneuver can also appear, as will be seen in the table below.

We now list a table with non-homogeneous terms with their associated guesses. There will be no proof, but these examples should be able to highlight what's going on in the cases above. In the following table, remember that the guess depends on the solutions to the complementary equation.

Differential Equation	r_1, r_2	Guess For Particular Solution
$y'' - 6y' + 9y = 3e^t$	$3, 3$	Ae^t
$y'' - 6y' + 9y = 2t^2e^t$	$3, 3$	$(A_2t^2 + A_1t + A_0)e^t$
$y'' - 6y' + 9y = 4e^{3t}$	$3, 3$	$t^2(Ae^{3t})$
$y'' - 6y' + 9y = -t^2e^{3t}$	$3, 3$	$t^2(A_2t^2 + A_1t + A_0)e^{3t}$
$y'' - 6y' + 5y = -2e^{5t}$	$1, 5$	$t(Ae^{5t})$
$y'' - 6y' + 5y = -2t^3e^{5t}$	$1, 5$	$t(A_3t^3 + A_2t^2 + A_1t + A_0)e^{5t}$
$y'' - 6y' + 5y = -2e^{2t} + 3e^{4t}$	$1, 5$	$Ae^{2t} + Be^{4t}$
$y'' - 6y' + 5y = -2e^t + 3e^{2t}$	$1, 5$	$t(Ae^t) + Be^{2t}$
$y'' - 6y' + 5y = -2te^t + 3e^{2t}$	$1, 5$	$t(A_1t + A_0)e^t + Be^{2t}$
$y'' - 6y' + 5y = 2e^{3t}\cos(2t)$	$1, 5$	$Ae^{3t}\cos(2t) + Be^{3t}\sin(2t)$
$y'' - 6y' + 5y = 2e^{3t}\sin(2t)$	$1, 5$	$Ae^{3t}\cos(2t) + Be^{3t}\sin(2t)$
$y'' - 6y' + 5y = 2e^{3t}\sin(2t) - e^{3t}\cos(2t)$	$1, 5$	$Ae^{3t}\cos(2t) + Be^{3t}\sin(2t)$
$y'' - 2y' + 5y = t^2 + 1 + 2e^{2t}$	$1 \pm 2i$	$(A_2t^2 + A_1t + A_0) + (Be^{2t})$
$y'' - 2y' + 5y = -e^t\cos(2t)$	$1 \pm 2i$	$t(Ae^t\cos(2t)) + t(Be^t\sin(2t))$
$y'' - 2y' + 5y = -te^t\sin(2t)$	$1 \pm 2i$	$t(A_1t + A_0)e^t\cos(2t) + t(B_1t + B_0)e^t\sin(2t)$
$y'' - 2y' + 5y = -e^t\cos(2t) + 2e^t\sin(2t)$	$1 \pm 2i$	$t(Ae^t\cos(2t)) + t(Be^t\sin(2t))$
$y'' - 2y' + 5y = t^2e^{3t}$	$1 \pm 2i$	$(A_2t^2 + A_1t + A_0)e^{3t}$
$y'' - 2y' + 5y = 2t^3 - 3t^2 + 4$	$1 \pm 2i$	$A_3t^3 + A_2t^2 + A_1t + A_0$

5.4. **Initial Value Problems in the Non-Homogeneous Case.** We can also solve initial value problems in the non-homgeneous case. The idea is exactly the same as in the homogeneous case: we first find the general solution, then determine the coefficients by plugging in the initial value conditions. There is really nothing new going on here; it's perhaps best seen in an example.

Example 5.7. Consider the following IVP:

$$z''(x) + z'(x) = 2e^{-x}, \quad z(0) = 0,\ z'(0) = 0.$$

The associated auxiliary equation is $r^2 + r = 0$, which factors into $r(r + 1) = 0$. The two roots are then $r_1 = 0$ and $r_2 = -1$. In particular, the complementary solution is $z_c(x) = c_1 + c_2e^{-x}$.

We now make our guess for the particular solution. Since the right hand side is $2e^{-1\cdot x}$ and $r_2 = -1$ is a root, following the algorithm tells us that we should guess $z_p(x) = Axe^{-x}$, where the factor of x comes from the "overlap." Plugging this in to extract the value of A gives us:

$$(Axe^{-x})'' + Axe^{-x} = 2e^{-x},$$

which one can solve into $A = -2$. Hence our general solution is

$$z(x) = z_c(x) + z_p(x) = c_1 + c_2 e^{-x} - 2xe^{-x}.$$

Let's now use the initial conditoins. We first compute the derivative

$$z'(x) = -c_2 e^{-x} - 2e^{-x} + 2xe^{-x} = -(c_2 + 2)e^{-x} + 2xe^{-x}.$$

Then we have

$$\begin{aligned} 0 &= z(0) = c_1 + c_2 \\ 0 &= z'(0) = -c_2 - 2. \end{aligned}$$

Then we have $c_2 = -2$ and $c_1 = 2$. The unique solution to our IVP is thus

$$z(x) = 2 - 2e^{-x} - 2xe^{-x}.$$

The previous example encapsulates all of the techniques we have learned up to this point.

6. Variation of Parameters

Here we introduce another method for finding the particular solution for a non-homogeneous differential equation, the variation of parameters. It applies in more generality than the aforementioned method of undetermined coefficients, but may require, in some cases, a hairy integral or two. In the first subsection we give some motivation for the procedure from physics. Feel free to skip it. The second subsection describes how to use it in practice.

6.1. **Motivation.**

The method of variation of parameters can be encapsulated in the following mantra: "non-homogeneous differential equations can be recast as a series of initial value problems of the associated homogeneous differential equation."[6]

Let's begin with the following phsyical problem. Suppose we have a mass of 1 kilogram on a spring as follows, and let x be the coordinate that determines the position of the mass:

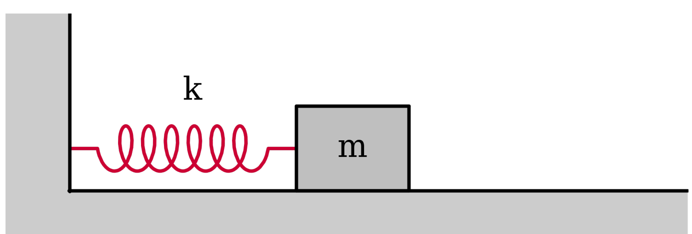

Figure 1. Mass on a spring.

Here x depends on time; we are going to analyze how this physical system should evolve. As the spring moves, the net force will consist of the spring force and a friction/drag term. Let's assume that the spring force $-kx$ depends only on the position of the mass and the drag force $-bx'$ depends only on the velocity of the mass. Then Newton's second law $F_{\text{net}} = ma = mx''$ implies that

$$mx'' = -kx - bx' \Rightarrow mx'' + bx' + kx = 0. \tag{6.1}$$

[6]In a more general form, this goes by Duhamel's principle.

This homogeneous differential equation governs the evolution of the spring-mass system. Phrased in terms of an initial value problem, we can say that given some initial conditions $x(t_0) = x_0$ and $x'(t_0) = v_0$, the equation (6.1) will completely determine the trajectory of the mass on the spring for all time.

Let's suppose now that the mass begins at rest, and we quickly hit it with a hammer. How does this change our differential equation? Well, the hammer is going to exert some force on the mass (albeit for a very short amount of time). A graph of the force over time $f(t)$ imparted by the hammer might look like the following:

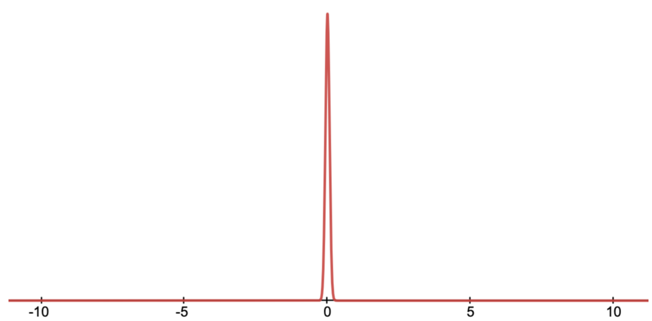

FIGURE 2. Force against time for a hammer blow.

Using Newton's second law again, we must take into account this external force of the hammer blow that we now call $f(t)$. We now have

$$mx'' = -kx - bx' + f \Rightarrow mx'' + bx' + bx = f.$$

So this is almost the same as the equation (6.1) that we saw before, but with a non-homogeneous term f on the right hand side.

Now Figure 2 gives us force over time. We know from physics that $\int_0^t f(t)\,dt = mx'(t) - mx'(0)$, i.e., force over time is impulse, and applying an impulse yields change in momentum mx'. But take a look at the shape of the graph in Figure 2. It is a spike concentrated at the time $t = 0$. Physically, this means that the hammer blow is giving our mass some momentum basically instantaneously, which certainly agrees with our intuition. And we can write the amount of momentum imparted as $f(0)\,ds$, where ds should be interpreted as a very small interval of time. This makes sense as $f(0)\,ds$ is a good approximation of what the integral of the spike $f(t)$ should be.

Putting everything together, here's the takeaway: if we have the differential equation

$$mx'' + bx' + kx = f,$$

where f is a spike concentrated at 0, solving this is essentially equivalent to solving the initial value problem for the homogeneous equation given by:

$$mx'' + bx' + kx = 0, \quad x(0) = 0,\ x'(0) = \frac{f(0)}{m}\,ds. \tag{6.2}$$

We divide by m on the right hand side as x' is velocity, not momentum. The physical intuition is as follows: hitting the mass with a hammer applies a force to the mass for such a small amount of time that it's basically the same as giving the mass an initial velocity; the dynamics are then governed by the homogeneous equation (6.1).

Let's now consider a general form of $f(t)$, one that does not have to look like a spike:

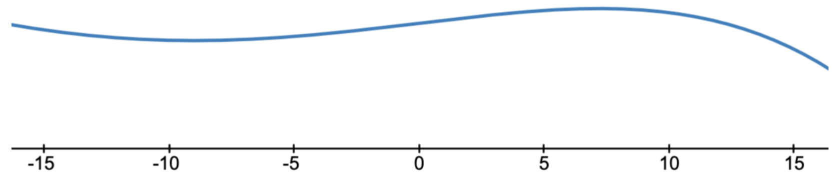

FIGURE 3. A generic function $f(t)$.

Physically, this could be interpreted as someone using their hand to apply pressure to the mass at different levels of force over time. We now reduce this to the previous case using an idea borrowed from calculus. We split $f(t)$ into rectangles:

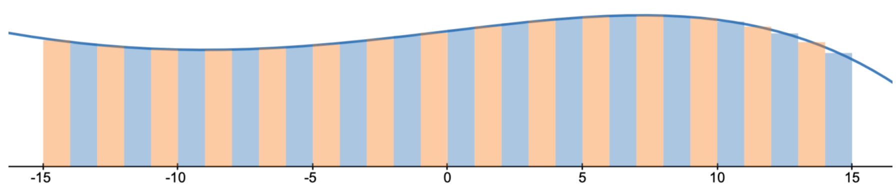

FIGURE 4. The function $f(t)$ is partitioned into rectangles. Each box represents an amount of impulse applied to the mass, which results in a change in momentum.

In Figure 4, we can interpret each rectangle as "duller" hammer blow that imparts momentum equivalent to the area of the triangle. Letting the widths of the rectangles approach zero, we can say that using one's hand to apply impulse to the mass results in a change in momentum that can be perfectly approximated by a very, *very* high number of rapid hammer blows to the mass of adequate force.

Under this interpretation, how does the system evolve? At the first instant, we have a hammer blow at time $t = 0$, which induces movement according to the solution $x_0(t)$ that solves

$$mx'' + bx' + kx = 0, \quad x(0) = 0,\ x'(0) = \frac{f(0)}{m}\, ds.$$

Once again we divide by m to convert momentum to velocity. This is precisely the equation (6.2) we saw earlier with one hammer blow. The next instant, we can imagine having another hammer blow at a time $\varepsilon > 0$, where ε is a very, *very* tiny number, for being barely past zero. This hammer blow induces movement according to the solution $x_\varepsilon(t)$

that solves

$$mx'' + bx' + kx = 0, \quad x(\varepsilon) = 0,\ x'(0) = \frac{f(\varepsilon)}{m}\, ds.$$

The initial conditions encode the fact that this "second" hammer blow is unaware of the previous one; it's imposing its own trajectory on the mass. In general, for each time s, we solve the initial value problem

$$mx'' + by' + kx = 0, \quad x(s) = 0,\ x'(s) = \frac{f(s)}{m}\, ds \tag{6.3}$$

and obtain a solution $x_s(t)$. Note: here the subscript s is indicating that we are solving the problem for the micro-hammer-blow at time s. The function $x_s(t)$ is then encodes the unique evolution of the mass over time as a result of that micro-hammer-blow. Also, as ds gets tiny and the number of rectangles we have increases, the impulse $f(s)\, ds$ also gets very tiny as $f(s)$ is fixed. Hence the induced dynamics and trajectories will also be very tiny, and indeed $x_s(t)$ will have the form $x_s(t) = [\text{functions in t}]\, ds$. For example, the initial value problem

$$x'' + x = 0 \quad x(s) = 0,\ x'(s) = f(s)\, ds$$

has solution

$$x_s(t) = f(s)\sin(t - s)\, ds.$$

In a bit we will be more explicit about how this works.

For now, here comes the crucial idea: as the differential equation is linear, we can simply sum all of these trajectories up via superposition. The location of the mass at some time t is going to be the continuous sum, or integral, of all of the trajectories imposed by the micro hammer blows coming before. This yields the following integral as a (particular) solution for the non-homogeneous equation

$$mx'' + bx' + kx = f$$

for a general function f:

$$x(t) = \int_0^t x_s(t).$$

The above integral is with respect to s. But we do not write a separate ds as the differential ds is already built into $x_s(t)$.[7] Notice that there are two parameters here: the time itself as well as family of IVP solutions. We are integrating over the latter. So far, we have been reasoning about things physically, and we came up with this integral. But now that we have an actual solution, we can simply check that it works, i.e., that $mx'' + bx' + kx = f$ actually holds.

Using the Leibniz rule to differentiate, we obtain the following:

$$x'(t) = x_t(t) + \int_0^t x_s'(t) = \int_0^t x_s'(t).$$

[7] This is a very technical point. Don't really worry about it; understanding the intuition behind this type of argument is far more important.

The first term vanishes by the initial condition. We also get

$$x''(t) = \frac{x_t'(t)}{ds} + \int_0^t x_s''(t) = \frac{f(t)}{m} + \int_0^t x_s''(t)\,ds$$

which also follows from the initial condition. Hence we obtain

$$mx''(t) + bx'(t) + kx(t) = f(t) + \int_0^t mx_s'' + bx_s' + kx_s = f(t)$$

as $x_s(t)$ solves the homogeneous equation for all s! **We have written down the solution to a non-homogeneous differential equation in terms of solutions of initial value problems of the associated homogeneous differential equation.**[8]

Let's now analyze the structure of $x_s(t)$. Say the general solution to the homogeneous equation $mx'' + bx' + kx = 0$ is $c_1x_1(t) + c_2x_2(t)$; we know from Section 2 that it must be of this form. As $x_s(t)$ solves equation (6.3), the same homogeneous equation with initial conditions $x(s) = 0$ and $x'(s) = \frac{f(s)}{m}\,ds$, we know that $x_s(t)$ will take on the following form:

$$x_s(t) = [c_1(s)x_1(t) + c_2(s)x_2(t)]\,ds.$$

where $c_1(s)x_1(t) + c_2(s)x_2(t)$ solves the initial value problem

$$mx'' + bx' + kx = 0, \quad x(s) = 0,\ x'(s) = \frac{f(s)}{m}.$$

This should make sense; it's the same IVP we've been considering without the ds.

Note two parameters we have flying around here: s and t. For each *fixed* s, we are solving the same homogeneous differential equation with a particular set of initial conditions: this solution to this IVP is found with constants $c_1(s)$ and $c_2(s)$. That is why the constants are in fact parameterized by s. We are, in fact, **varying the parameters**.

We can now apply all initial conditions at the same time. In particular, we have that for all s:

$$0 = x_s(s) = [c_1(s)x_1(s) + c_2(s)x_2(s)]\,ds$$
$$\frac{f(s)}{m}\,ds = x_s'(s) = [c_1(s)x_1'(s) + c_2(s)x_2'(s)]\,ds.$$

Note in the second equation that we only take derivatives on x_1 and x_2 as $x_s(t)$ is a function of t. We can divide away and ignore ds in the first equation, and the differentials ds cancel out in the second equation. We are left with solving the following system for $c_1(s)$ and $c_2(s)$:

$$0 = c_1(s)x_1(s) + c_2(s)x_2(s)$$
$$\frac{f(s)}{m} = c_1(s)x_1'(s) + c_2(s)x_2'(s).$$

The solution can be written down using Cramer's rule. We obtain the following:

$$c_1(s) = \frac{-x_2(s)f(s)}{mW(x_1, x_2)}, \quad c_2(s) = \frac{x_1(s)f(s)}{mW(x_1, x_2)},$$

where $W(x_1, x_2)$ is Wronksian (Section 4).

[8]It is the linearity of the differential equation that lets us combine them under the integral!

Now let's return back to our integral form solution of the nonhomogeneous problem. We now have the following:

$$x(t) = \int_0^t x_s(t) = \int_0^t [c_1(s)x_1(t) + c_2(s)x_2(t)]\, ds = \int_0^t c_1(s)\, ds \cdot x_1(t) + \int_0^t c_2(s)\, ds \cdot x_2(t).$$

Notice that we can replace the definite integral $\int_0^t c_1(s)\, ds$ with $\int c_1(t)$ where $\int c_1(t)$ is any antiderivative of $c_1(t)$. The reason is that $\int_0^t c_1(s)\, ds - \int c_1(t)$ is a constant, and as $x_1(t)$ solves the homogeneous equation, its coefficient in front differing by a constant simply won't matter in our search for a particular solution. The same reasoning lets us replace $\int_0^t c_2(s)\, ds$ with $\int c_2(t)$.

If we define $g(t) \coloneqq \int c_1(t)$ and $h(t) \coloneqq \int c_2(t)$, we obtain the following particular solution:

$$x_p(t) = g(t)x_1(t) + h(t)x_2(t)$$

where

$$g(t) = \int \frac{-x_2(t)f(t)}{mW(x_1, x_2)}, \quad h(t) = \int \frac{x_1(t)f(t)}{mW(x_1, x_2)}.$$

Hopefully this looks familiar.[9]

6.2. **Computing With Variation of Parameters.** As mentioned at the beginning of this section, variation of parameters will let us find a particular solution for a particular differential equation. And as is always the case, the particular solution will let us write down a general solution, which can then be used to solve an initial value problem.

Let's begin with finding the general solution to the constant coefficient differential equation

$$ay''(t) + by'(t) + cy(t) = f(t).$$

The steps to apply the variation of parameters are quite simple:

(1) Find the complementary solution
$$y_c(t) = c_1y_1(t) + c_2y_2(t)$$
for the associated homogeneous differential equation $ay'' + by' + cy = 0$ (Section 2). *Sometimes $y_1(t)$ and $y_2(t)$ might already be given; we will discuss this in a bit.*

(2) Set the particular solution to be $y_p(t) = g(t)y_1(t) + h(t)y_2(t)$.

(3) Compute $g(t)$ and $h(t)$ via the following formulas:
$$g(t) = \int \frac{-y_2(t)f(t)}{aW(y_1, y_2)}, \quad h(t) = \int \frac{y_1(t)f(t)}{aW(y_1, y_2)},$$
where $W(y_1, y_2)$ is the Wronskian (Section 4), and $\int$ denotes finding *any* antiderivative.

(4) Write down the general solution:
$$y(t) = y_c(t) + y_p(t) = c_1y_1(t) + c_2y_2(t) + g(t)y_1(t) + h(t)y_2(t).$$

Let's do a quick example.

[9] We can replace m, b, and k with functions of t; we just need to be careful to avoid regions where such functions are undefined.

Example 6.1. Consider the differential equation:

$$y'' - y' - 2y = 2e^{-t}.$$

The auxiliary equation factors as $(r-2)(r+1) = 0$, and so the complementary solutions are $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$. The particular solution has the form

$$y_p(t) = g(t)y_1(t) + h(t)y_2(t).$$

The Wronskian is:

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' = e^{2t} \cdot -e^{-t} - e^{-t} \cdot 2e^{2t} = -3e^t.$$

With $a = 1$ and $f(t) = 2e^{-t}$, we compute:

$$g(t) = \int \frac{-y_2(t)f(t)}{aW(y_1, y_2)}\, dt = \int \frac{-e^{-t} \cdot 2e^{-t}}{-3e^t}\, dt = \int \frac{2}{3}e^{-3t}\, dt = -\frac{2}{9}e^{-3t}.$$

We also get

$$h(t) = \int \frac{y_1(t)f(t)}{aW(y_1, y_2)}\, dt = \int \frac{e^{2t} \cdot 2e^{-t}}{-3e^t}\, dt = \int -\frac{2}{3}\, dt = -\frac{2}{3}t.$$

Plugging into the above, we have the particular solution is $y_p(t) = -\frac{2}{9}e^{-t} - \frac{2}{3}te^{-t}$, and so the general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{9}e^{-t} - \frac{2}{3}te^{-t} = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3}te^{-t},$$

where we absorbed one term into the homogeneous solution.

Notice that in this case we also could have applied the method of undetermined coefficients with guess $t(Ae^{-t})$. It probably would have been a lot faster. Usually, when the method of undetermined coefficients is available (when $f(t)$ is some combination of polynomials, exponentials, sines, and cosines), it probably yields a particular solution with less effort. But this is not a hard and fast rule.

Let's do a slightly more interesting example, one that cannot be solved using the method of undetermined coefficients:

Example 6.2. Consider the differential equation:

$$y'' + 3y' + 2y = \frac{1}{1+e^x}.$$

The auxiliary equation factors as $(r+2)(r+1) = 0$, and so the complementary solutions are $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$. The particular solution has the form

$$y_p(t) = g(t)y_1(t) + h(t)y_2(t).$$

The Wronskian is:

$$W(y_1, y_2)(x) = y_1 y_2' - y_2 y_1' = e^{-x} \cdot -2e^{-x} - e^{-2x} \cdot -e^{-x} = -e^{-3x}.$$

With $a = 1$ and $f(x) = \frac{1}{1+e^x}$, we compute:

$$g(x) = \int \frac{-y_2(x)f(x)}{aW(y_1, y_2)}\, dx = \int \frac{-e^{-2x} \cdot \frac{1}{1+e^x}}{-e^{-3x}}\, dx = \int \frac{e^x}{1+e^x}\, dx = \ln(1+e^x).$$

We don't need absolute value bars for the logarithm as $1+e^x > 0$ for all x. We also get

$$h(x) = \int \frac{y_1(t)f(t)}{aW(y_1,y_2)}\,dt = \int \frac{e^{-x}\cdot\frac{1}{1+e^x}}{-e^{-3x}}\,dx = \int -\frac{e^{2x}}{1+e^x}\,dx.$$

Alright, we have to do something about this integral. Let $u = e^x$, then $du = e^x\,dx$. Then we have

$$\int -\frac{e^{2x}}{1+e^x}\,dx = \int -\frac{u}{1+u}\,du = \int -\frac{(1+u)-1}{1+u}\,du = \int -1+\frac{1}{1+u}\,du = -u+\ln|1+u|.$$

Substituting back in yields

$$h(x) = -e^x + \ln(1+e^x).$$

Plugging into the above, we have the particular solution is

$$\begin{aligned} y_p(x) &= \ln(1+e^x)e^{-x} + (-e^x + \ln(1+e^x))e^{-2x} \\ &= \ln(1+e^x)e^{-x} - e^{-x} + \ln(1+e^x)e^{-2x}. \end{aligned}$$

The general solution is thus

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= c_1y_1(t) + c_2y_2(t) + y_p(t) \\ &= c_1e^{-x} + c_2e^{-2x} + \ln(1+e^x)e^{-x} - e^{-x} + \ln(1+e^x)e^{-2x} \\ &= c_1e^{-x} + c_2e^{-2x} + \ln(1+e^x)e^{-x} + \ln(1+e^x)e^{-2x}, \end{aligned}$$

where we absorbed one term into the homogeneous solution.

Finally, note that variation of parameters can even be applied when the coefficients are not constants. An example would be the differential equation

$$ty'' - (t+1)y' + y = t^2.$$

Let's think about what needs to change in our process. First, we simply have no method to find the two complementary solutions $y_1(t)$ and $y_2(t)$ that solve $ty'' - (t+1)y' + y = 0$. We could resort to guessing. But what is most likely going to happen is that $y_1(t)$ and $y_2(t)$ will be given. So let's now assume we're given

$$y_1(t) = e^t, \quad y_2(t) = t+1.$$

We can readily verify that $y_1(t)$ and $y_2(t)$ indeed solve the homogeneous equation.

Next, observe that in the constant coefficient case, the coefficients b and c in front of y' and y respectively never made an appearance in our calculations. Only the coefficient a did in the denominator for solving for $g(t)$ and $h(t)$. Well, we'll simply replace a with whatever function of t is in front of y''.

To recapitulate, if we are confronted with

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t)$$

and given complementary solutions $y_1(t)$ and $y_2(t)$ of the associated homogeneous equation, then our guess for the particular solution

$$y_p(t) = g(t)y_1(t) + h(t)y_2(t)$$

is solved by

$$g(t) = \int \frac{-y_2(t)f(t)}{a(t)W(y_1,y_2)}, \quad h(t) = \int \frac{y_1(t)f(t)}{a(t)W(y_1,y_2)}.$$

Let's finish the example we started.

Example 6.3. Take the differential equation:

$$ty'' - (t+1)y' + y = t^2,$$

and assume we are given the complementary solutions $y_1(t) = e^t$ and $y_2(t) = t+1$. The particular solution has the form

$$y_p(t) = g(t)y_1(t) + h(t)y_2(t).$$

The Wronskian is:

$$W(y_1,y_2)(t) = y_1y_2' - y_2y_1' = e^t \cdot 1 - (t+1)\cdot e^t = -te^t.$$

With $a(t) = t$ and $f(t) = t^2$, we compute:

$$g(t) = \int \frac{-y_2(t)f(t)}{a(t)W(y_1,y_2)}\,dt = \int \frac{-(t+1)\cdot t^2}{t\cdot -te^t}\,dt = \int (t+1)e^{-t}\,dt = -e^{-t}(t+2).$$

We also get

$$h(t) = \int \frac{y_1(t)f(t)}{a(t)W(y_1,y_2)}\,dt = \int \frac{e^t\cdot t^2}{t\cdot -te^t}\,dt = \int -1\,dt = -t.$$

Plugging into the above, we have the particular solution is

$$y_p(t) = -e^{-t}(t+2)e^t - t(t+1) = -t-2-t^2-t = -t^2-2t-2.$$

So the general solution is

$$y(t) = c_1e^t + c_2(t+1) - t^2 - 2t - 2 = c_1e^t + c_2(t+1) - t^2,$$

where we absorbed two terms into the homogeneous solution.

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