MONOTONE SEQUENCE OF CONTINUOUS FUNCTIONS

We give a sequential proof to Exercise 41 in Chapter IV of Rosenlicht.

Let \((E, d)\) be a compact metric space and let \(f_n : E \rightarrow \mathbb{R}\) be a continuous function for each \(n \in \mathbb{N}\). Suppose \((f_n)_{n\in\mathbb{N}}\) converges pointwise to some continuous function \(f : E \rightarrow \mathbb{R}\) and that

\[
f_1(x) \leq f_2(x) \leq \cdots \quad \forall x \in E.
\]

We will show that, in fact, \((f_n)_{n\in\mathbb{N}}\) converges uniformly to \(f\).

We first note that it suffices to show that \((f - f_n)_{n\in\mathbb{N}}\) converges uniformly to the zero function. Denote \(h_n := f - f_n\) for each \(n \in \mathbb{N}\). From the properties of the \(f_n\), we know that \(h_n(x) \geq 0\) for all \(x \in E\), \((h_n)_{n\in\mathbb{N}}\) converges pointwise to zero, and that

\[
h_1(x) \geq h_2(x) \geq \cdots \quad \forall x \in E.
\]

Since \(h_n\) is a continuous function on a compact metric space, it attains its maximum value. Let \(x_n \in E\) be such that

\[
h_n(x_n) = \max_{x \in E} h_n(x).
\]

Note that for each \(x \in E\),

\[
|h_n(x) - 0| = h_n(x) \leq h_n(x_n).
\]

Thus if we can show \(\lim_{n \to \infty} h_n(x_n) = 0\), then we have shown \((h_n)_{n\in\mathbb{N}}\) converges uniformly to zero. Observe that

\[
h_{n+1}(x_{n+1}) \leq h_n(x_{n+1}) \leq h_n(x_n),
\]

so that the sequence \((h_n(x_n))_{n\in\mathbb{N}}\) is a monotone decreasing sequence of real numbers that is bounded below by zero. Consequently it converges, say to

\[
y := \lim_{n \to \infty} h_n(x_n),
\]

and \(h_n(x_n) \geq y\) for all \(n \in \mathbb{N}\). Suppose, towards a contradiction, that \(y > 0\). Since \((x_n)_{n\in\mathbb{N}}\) is a sequence in a compact metric space, it has a convergent subsequence \((x_{n_k})_{k\in\mathbb{N}}\), say with limit \(x \in E\). Note that we still have

\[
\lim_{k \to \infty} h_{n_k}(x_{n_k}) = y.
\]

Now, since \((h_n)_{n\in\mathbb{N}}\) converges pointwise to zero, \((h_{n_k})_{k\in\mathbb{N}}\) converges pointwise to zero and in particular

\[
\lim_{k \to \infty} h_{n_k}(x) = 0.
\]

Let \(K \in \mathbb{N}\) be such that for all \(k \geq K\) we have

\[
|h_{n_k}(x) - 0| < \frac{y}{2}.
\]

Equivalently, \(h_{n_k}(x) < \frac{y}{2}\). Now, \(h_{n_k}\) is continuous at \(x\), so there exists \(\delta > 0\) such that if \(x' \in E\) satisfies \(d(x', x) < \delta\) then \(|h_{n_k}(x') - h_{n_k}(x)| < \frac{y}{2}\). Consequently,

\[
h_{n_k}(x') \leq h_{n_k}(x') - h_{n_k}(x) + h_{n_k}(x) < \frac{y}{2} + \frac{y}{2} = y.
\]

So by the monotonicity condition on the \(h_n\), whenever \(k \geq K\) we have

\[
h_{n_k}(x') < y
\]

so long as \(d(x', x) < \delta\). We know that for sufficiently large \(k\), \(x_{n_k}\) satisfies this. Taking \(k \geq K\) as well we have

\[
h_{n_k}(x_{n_k}) < y.
\]

But since \(y \leq h_{n_k}(x_{n_k})\), this implies \(y < y\), a contradiction.

Thus it must be that \(y = 0\), which, as noted above, implies \((h_n)_{n\in\mathbb{N}}\) converges uniformly to zero. \(\square\)