Math 209: von Neumann Algebras

Brent Nelson

April 26, 2017

Contents

Introduction

This set of notes assumes a familiarity with C^*-algebras (in particular either *Math 206: Banach Algebras and Spectral Theory* or *Math 208: C*-algebras* at the University of California, Berkeley). In particular, one should be familiar with the continuous functional calculus.

With this in mind we make the following foreshadowing analogy: C^*-algebras are to continuous functions as von Neumann algebras are to essentially bounded measurable functions. This will be made precise later on, but for now take it as an indication that the intuition will shift from topological spaces to measure spaces. However, von Neumann algebras also offer a non-commutative context to study many other mathematical objects: groups, dynamical systems, equivalence relations, graphs, and random variables to name a few. It is an incredibly rich theory lying at an intersection of algebra and analysis (*cf.* Theorem 2.2.4), and though at times technically demanding it is well worth the effort to learn.

Though not as extensive as Takesaki's *Theory of Operator Algebras I*, these notes will follow roughly the same course. The ambitious reader is encouraged to check there for additional topics, exercises, and bragging rights. The intention is for these notes to be fully self-contained, but the occasional proof may be relegated to an exercise.

Chapter 1

Preliminaries

We recall some basic facts about Hilbert spaces and bounded operators, which we present—despite being familiar to anyone who has studied C^*-algebras—in order to establish some notation.

1.1 Hilbert Spaces

Let V be a vector space over $\mathbb{C}$.

Definition 1.1.1. An **inner product on** V is map $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{C}$ satisfying for all $x, y, z \in V$ and $a \in \mathbb{C}$

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (**Conjugate symmetry**)
2. $\langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle$ (**Linearity**)
3. $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$ (**Positive-definiteness**)

Observe that properties 1 and 2 imply the inner product is conjugate linear in the second coordinate:

$$\langle x, ay \rangle = \bar{a} \langle x, y \rangle \qquad x, y \in V \;\; a \in \mathbb{C}.$$

The inner product also naturally gives a norm on V given by

$$\|x\| := \sqrt{\langle x, x \rangle} \qquad x \in V.$$

The *Cauchy–Schwarz inequality* is automatically satisfied:

$$| \langle x, y \rangle | \leq \|x\| \|y\| \qquad x, y \in V.$$

Definition 1.1.2. A vector space $\mathcal{H}$ over $\mathbb{C}$ is called a **Hilbert space** if it is equipped with an inner product and is complete in the topology induced by the norm associated with the inner product. We say a Hilbert space is **separable** if its dimension is countable.

Remark 1.1.3. We can easily adjust the above definitions to give Hilbert space over $\mathbb{R}$. In the following, all Hilbert space will be complex unless otherwise specified.

1.2 Bounded Operators

Let $\mathcal{H}$ be a Hilbert space.

Definition 1.2.1. A **bounded operator on** $\mathcal{H}$ is a $\mathbb{C}$-linear map $x \colon \mathcal{H} \to \mathcal{H}$ such that

$$\sup_{\xi \in \mathcal{H}} \frac{\|x\xi\|}{\|\xi\|} < \infty.$$

This supremum is called the **operator norm of** x and is denoted $\|x\|$.

It is easily checked that x is bounded if and only if it is (Lipschitz) continuous on $\mathcal{H}$ (in the topology induced by the vector norm on $\mathcal{H}$).

If x is a bounded operator, its *adjoint*, denoted x^*, is the bounded operator uniquely determined by

$$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle \qquad \xi, \eta \in \mathcal{H}.$$

It follows that

$$\|x^*x\| = \|x\|^2 = \|x^*\|^2.$$

This equality is known as the *C^*-identity*.

We shall let $\mathcal{B}(\mathcal{H})$ denote the collection of bounded operators on $\mathcal{H}$. We then have that $\mathcal{B}(\mathcal{H})$ is a $*$-algebra with multiplication given by composition. It is easily seen that $\mathcal{B}(\mathcal{H})$ is closed under the operator norm, and since this satisfies the C^*-identity, $\mathcal{B}(\mathcal{H})$ is in fact a C^*-algebra.

1.3 Projections

Definition 1.3.1. Let $\mathcal{H}$ be a Hilbert space. We say $p \in \mathcal{B}(\mathcal{H})$ is a **projection** if $p = p^* = p^2$. Two projections $p, q \in \mathcal{B}(\mathcal{H})$ are said to be **orthogonal** if $pq = 0$. For projections $p, q \in \mathcal{B}(\mathcal{H})$, we write $p \leq q$ if $pq = p$. Equivalently, $p(1-q) = 0$.

Technically, the above definition is for *orthogonal* or *self-adjoint* projections, but since these are the only kind we shall consider we simply refer to them as *projections*. Note that for a projection p and $\xi \in \mathcal{H}$ we always have

$$\|p\xi\|^2 = \langle p\xi, p\xi\rangle = \langle p\xi, \xi\rangle \leq \|p\xi\|\|\xi\|,$$

or $\|p\xi\| \leq \|\xi\|$. Hence $\|p\| \leq 1$. Since $\|p\| = \|p^2\| = \|p\|^2$, we have $\|p\| = 1$ unless $p = 0$.

For any closed subspace $\mathcal{K} \subset \mathcal{H}$, there exists a unique projection p from $\mathcal{H}$ to $\mathcal{K}$ so that under the identification $\mathcal{H} \cong \mathcal{K} \oplus \mathcal{K}^\perp$, p acts as the identity on $\mathcal{K}$ and as the zero operator on $\mathcal{K}^\perp$. We may denote p by $[\mathcal{K}]$. More generally, for a subset $S \subset \mathcal{H}$, $[S]$ denotes the projection onto $\overline{\text{span}}(S)$.

1.4 Partial Isometries

Definition 1.4.1. Let $\mathcal{H}$ be a Hilbert space. $v \in \mathcal{B}(\mathcal{H})$ is an **isometry** if $\|v\xi\| = \|\xi\|$ for all $\xi \in \mathcal{H}$; equivalently, $v^*v = 1$. $v \in \mathcal{B}(\mathcal{H})$ is a **partial isometry** if there exists a closed subspace $\mathcal{K} \subset \mathcal{H}$ so that $v|_{\mathcal{K}}$ is an isometry and $v|_{\mathcal{K}^\perp} = 0$. $\mathcal{K}$ is called the **initial subspace of** v and is denoted $I(v)$. The range of $v\mathcal{K}$ (which is also easily seen to be a closed subspace), is called the **final subspace** and is denoted $F(v)$.

Proposition 1.4.2. *For $v \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

(i) v is a partial isometry;

(ii) v^*v is a projection;

(iii) $vv^*v = v$.

Moreover, if v is a partial isometry, then so is v^* and $I(v^*) = F(v)$ and $F(v^*) = I(v)$.

Proof. [(i)$\Longrightarrow$(ii)]: Suppose v is a partial isometry with $I(v) = \mathcal{K}$. Let p be the projection from $\mathcal{H}$ to $\mathcal{K}$. Since $p - v^*v$ is self-adjoint, we have

$$\|p - v^*v\| = \sup_{\|\xi\|=1} |\langle (p - v^*v)\xi, \xi\rangle|.$$

For $\xi \in \mathcal{K}$ we have

$$\langle (p - v^*v)\xi, \xi\rangle = \langle \xi, \xi\rangle - \langle v\xi, v\xi\rangle = \|\xi\|^2 - \|v\xi\|^2 = 0.$$

For $\xi \in \mathcal{K}^\perp$ we have $(p - v^*v)\xi = 0$. Thus $\|p - v^*v\| = 0$, or $v^*v = p$.

[(ii)$\Longrightarrow$(iii)]: Assume v^*v is a projection. Then

$$\|(vv^*v - v)\xi\|^2 = \langle (v^*v-1)\xi, v^*(vv^*v - v)\xi\rangle = \langle (v^*v-1)\xi, (v^*v - v^*v)\xi\rangle = 0.$$

Thus $vv^*v = v$.

[(iii)$\Longrightarrow$(i)]: The assumed equality implies $v^*vv^*v = v^*v$. Since v^*v is self-adjoint, it is therefore a projection. Let $\mathcal{K} = v^*v\mathcal{H}$. Then $v\mid_{\mathcal{K}^\perp} = 0$ since $\ker(v) = \ker(v^*v)$. For $\xi \in \mathcal{K}$ we have

$$\|v\xi\|^2 = \langle v^*v\xi, \xi\rangle = \langle \xi, \xi\rangle = \|\xi\|^2.$$

Thus v is a partial isometry.

To see that v^* is also a partial isometry, note that $vv^*v = v$ implies $v^*vv^* = v^*$. Let $\eta = v\xi \in F(v)$. Then

$$\|v^*\eta\| = \|v^*v\xi\| = \|\xi\|.$$

For $\eta \in F(v)^\perp$ and any $\xi \in \mathcal{H}$ we have

$$\langle v^*\eta, \xi\rangle = \langle \eta, v\xi\rangle = 0.$$

Thus $v^*\mid_{F(v)^\perp} = 0$. Finally

$$F(v^*) = v^*I(v^*) = v^*F(v) = v^*vI(v) = I(v).$$

since v^*v is the projection onto $I(v)$. □

Remark 1.4.3. Note that we showed in the above proof that v^*v is the projection onto $I(v)$ and vv^* is the projection onto $F(v)$.

Example 1.4.4. Fix $n \in \mathbb{N}$. For each pair $i, j \in \{1, \ldots, n\}$, let $E_{ij} \in M_n(\mathbb{C})$ be the matrix with a one in the (i, j)th entry, and zeros elsewhere. Then E_{ij} is a partial isometry with $I(E_{ij}) = \mathbb{C}e_j$ and $F(E_{ij}) = \mathbb{C}e_i$, where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis for $\mathbb{C}^n$.

Example 1.4.5. Let (X, μ) be a measure space. Suppose $S, T \subset X$ are measurable subsets with $\mu(S) = \mu(T) < \infty$ and $f \colon S \to T$ is a measurable bijection. Then

$$L^2(X, \mu) \ni g \mapsto \chi_S g \circ f$$

defines a partial isometry $v \in \mathcal{B}(L^2(X, \mu))$ with $v^*v = \chi_T$ and $vv^* = \chi_S$.

1.5 Continuous Linear Functionals

Let A be a C^*-algebra. Recall that A^* is the set of continuous linear functionals $\varphi \colon A \to \mathbb{C}$, and

$$\|\varphi\| := \sup_{a\in A} \frac{|\varphi(a)|}{\|a\|} < \infty.$$

Definition 1.5.1. Let $\varphi \in A^*$. We say that φ is

- **hermitian** if $\varphi(a) = \overline{\varphi(a^*)}$ for all $a \in A$;
- **positive** if $\varphi(a^*a) \geq 0$ for all $a \in A$;
- **faithful** if $\varphi(a^*a) = 0$ if and only if $a = 0$ for all $a \in A$;
- a **state** if φ is positive and $\|\varphi\| = 1$;
- **tracial** if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.

Recall that any $\varphi \in A^*$ can be decomposed as a linear combination of positive elements. In particular,

$$\text{Re }(\varphi)(a) := \frac{1}{2}(\varphi(a) + \overline{\varphi(a^*)})$$
$$\text{Im }(\varphi)(a) := \frac{i}{2}(\varphi(a) - \overline{\varphi(a^*)})$$

define bounded hermitian linear functionals such that $\varphi = \text{Re }(\phi) + i\text{Im }(\varphi)$. By [1, Theorem 7.12], each these can be written uniquely as a the difference of two positive linear functionals.

We note that if A is a unital C^*-algebra, then for $\varphi \in A^*$ positive, $\|\varphi\| = \varphi(1)$. In particular, $\varphi \in A^*$ positive is a state if and only if $\varphi(1) = 1$.

We make A^* into an A-A bimodule via

$$(a \cdot \varphi \cdot b)(x) := \varphi(bxa) \qquad a, b, x \in A.$$

Note that $\|a \cdot \varphi \cdot b\| \leq \|a\|\|b\|\|\varphi\|$.

Chapter 2

Von Neumann Algebras

2.1 Strong and Weak Topologies

Let $\mathcal{H}$ be a Hilbert space. There is a natural (metrizable) topology on $\mathcal{B}(\mathcal{H})$ given by the operator norm. Studying this topology amounts to studying C^*-algebras. To study von Neumann algebras, we will need to consider two new topologies on $\mathcal{B}(\mathcal{H})$. There will be several others later on that are also important, but these first two will suffice to define a von Neumann algebra.

Definition 2.1.1. Let $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ be a net of bounded operators, and let $x \in \mathcal{B}(\mathcal{H})$. We say that (x_α) **converges strongly to** x if

$$\lim_\alpha \|(x_\alpha - x)\xi\| = 0 \qquad \forall \xi \in \mathcal{H}.$$

The topology induced by this convergence is called the **strong operator topology** (or **SOT**).

Viewing $\mathcal{H}$ as a metric space under its norm, strong convergence can be thought of as "pointwise convergence." Compare this to convergence under the operator norm, which should be thought of as "uniform convergence."

Definition 2.1.2. Let $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ be a net of bounded operators, and let $x \in \mathcal{B}(\mathcal{H})$. We say that (x_α) **converges weakly to** x if

$$\lim_\alpha \langle (x_\alpha - x)\xi, \eta\rangle = 0 \qquad \forall \xi, \eta \in \mathcal{H}.$$

The topology induced by this convergence is called the **weak operator topology** (or **WOT**).

It is clear that operator norm convergence implies strong convergence implies weak convergence, but the converses are not true. Here are some simple counter-examples:

Example 2.1.3. Let m be the Lebesgue measure on $\mathbb{R}$. For a measurable subset $E \subset \mathbb{R}$, let the characteristic function χ_E act on $\mathcal{B}(L^2(\mathbb{R}, m))$ by multiplication (so χ_E is a projection). Then $(\chi_{[-n,n]})_{n\in\mathbb{N}}$ converges strongly to the identity, but not in operator norm. For any $f \in L^2(\mathbb{R}, m)$ and any ϵ, there exists $N \in \mathbb{N}$ so that

$$\left(\int_{\mathbb{R}\setminus[-N,N]} |f|^2 \, dm\right)^{1/2} < \epsilon.$$

Thus, for any $n \geq N$ we have

$$\left\|(\chi_{[-n,n]} - 1)f\right\|_2 = \left(\int_{\mathbb{R}\setminus[-n,n]} |f|^2 \, dm\right)^{1/2}.$$

Thus this sequence of operators converges strongly to 1. However, $\|\chi_{[-n,n]} - 1\| = 1$ for all n, so the sequence does not converge in operator norm.

Example 2.1.4. Consider the following unitary operator on $\ell^2(\mathbb{Z})$:

$$(U\xi)(n) := \xi(n+1) \qquad \xi \in \ell^2.$$

For $n \in \mathbb{N}$, let $x_n := U^n$. Then we claim that $(x_n)_{n\in\mathbb{N}}$ converges weakly to the zero operator but not strongly. Indeed, fix $\xi, \eta \in \ell^2(\mathbb{Z})$. Let $\epsilon > 0$, then there exists $N \in \mathbb{N}$ sufficiently large so that

$$\left(\sum_{n\geq N} |\xi(n)|^2\right)^{1/2} < \epsilon$$

$$\left(\sum_{n<-N} |\eta(n)|^2\right)^{1/2} < \epsilon$$

Then for $m \geq 2N$ we have

$$\begin{aligned} |\langle x_m\xi, \eta\rangle| &\leq \sum_{n\in\mathbb{Z}} |\xi(n+m)||\eta(n)| \\ &= \sum_{n<-N} |\xi(n+m)||\eta(n)| + \sum_{n\geq m-N} |\xi(n)||\eta(n-m)| \\ &\leq \|\xi\|\epsilon + \epsilon\|\eta\|. \end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero. However, since U is a unitary,

$$\|x_n\xi\| = \|U^n\xi\| = \|\xi\| \qquad \forall\xi \in \ell^2(\mathbb{Z}),$$

thus $(x_n)_{n\in\mathbb{N}}$ does not converge to zero strongly.

2.2 Bicommutant Theorem

Definition 2.2.1. Let $\mathcal{H}$ be a Hilbert space and $X \subset \mathcal{B}(\mathcal{H})$ a set. The **commutant of** X, denoted X', is the set

$$X' := \{y \in \mathcal{B}(\mathcal{H}) \colon yx = xy \ \forall x \in X\}.$$

The **double commutant of** X is the set

$$X'' := (X')'$$

If $X \subset Y \subset \mathcal{B}(\mathcal{H})$ is an intermediate subset, we call $X' \cap Y$ a **relative commutant of** S.

Observe that, regardless of the structure of X, X' is always a unital algebra. If X is closed under taking adjoints, then X' is a $*$-algebra. It also easily checked (algebraically) that:

$$\begin{aligned} X \subset X'' &= (X'')'' = \cdots \\ X' &= (X')'' = \cdots \end{aligned}$$

Note that inclusions are reversed under the commutant: $X \subset Y$ implies $Y' \subset X'$.

Remarkably, the purely algebraic definition of the commutant has analytic implications. This culminates in The Bicommutant Theorem (Theorem 2.2.4), but is explained by the Lemma 2.2.3. We first require an additional definition.

Definition 2.2.2. Let $\mathcal{K} \subset \mathcal{H}$ be a subspace. For $x \in \mathcal{B}(\mathcal{H})$, we say $\mathcal{K}$ is **invariant for** x if $x\mathcal{K} \subset \mathcal{K}$. We say $\mathcal{K}$ is **reducing for** x if it is invariant for x and x^*. For $M \subset \mathcal{B}(\mathcal{H})$ a $*$-subalgebra, we say $\mathcal{K}$ is **reducing for** M if it is reducing for all elements in M. Equivalently, $M\mathcal{K} \subset \mathcal{K}$.

Lemma 2.2.3. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. Let $\mathcal{K} \subset \mathcal{H}$ be a closed subspace with p the projection from $\mathcal{H}$ to $\mathcal{K}$. Then $\mathcal{K}$ is reducing for M if and only if $p \in M'$.*

Proof. Assume $\mathcal{K}$ is reducing M. Let $x \in M$ and $\xi \in \mathcal{K}$. Then $x\xi \in \mathcal{K}$ so that

$$xp\xi = x\xi = px\xi.$$

If $\eta \in \mathcal{K}^\perp$, we have

$$\langle x\eta, \xi \rangle = \langle \eta, x^*\xi \rangle = 0,$$

since $x^*\xi \in \mathcal{K}$. Thus $x\eta \in \mathcal{K}^\perp$ and so $xp\eta = 0 = p\xi\eta$. It follows that $xp = px$ so that $p \in M'$.

Now suppose $p \in M'$. Let $x \in M$ and $\xi \in \mathcal{K}$. Then for $\eta \in \mathcal{K}^\perp$ we have

$$0 = \langle x\xi, p\eta \rangle = \langle px\xi, \eta \rangle = \langle xp\xi, \eta \rangle = \langle x\xi, \eta \rangle.$$

Thus $x\xi \in (\mathcal{K}^\perp)^\perp = \mathcal{K}$. Hence $M\mathcal{K} \subset \mathcal{K}$ so that $\mathcal{K}$ is reducing for M. □

We have the following theorem due to von Neumann from 1929. 1/18/2017

Theorem 2.2.4 (The Bicommutant Theorem)**.** *Let $M \subset B(\mathcal{H})$ be a $*$-algebra such that $1 \in M$. Then*

$$\overline{M}^{SOT} = \overline{M}^{WOT} = M''$$

Proof. We will show the following series of inclusions:

$$\overline{M}^{SOT} \subset \overline{M}^{WOT} \subset M'' \subset \overline{M}^{SOT}.$$

The first inclusion follows the fact that strong convergence implies weak convergence.

Now, suppose $x \in \overline{M}^{WOT}$, say with a net $(x_\alpha) \subset M$ converging weakly to it. Let $y \in M'$, then for any $\xi, \eta \in \mathcal{H}$ we have

$$\langle xy\xi, \eta \rangle = \lim_\alpha \langle x_\alpha y\xi, \eta \rangle = \lim_\alpha \langle yx_\alpha \xi, \eta \rangle = \langle yx\xi, \eta \rangle.$$

Since $\xi, \eta \in \mathcal{H}$ were arbitrary, we have $xy = yx$ and thus $x \in M''$.

Finally, suppose $x \in M''$. Note that to show $x \in \overline{M}^{SOT}$, it suffices to show for all $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$ that there exists $x_0 \in M$ with

$$\|(x - x_0)\xi_j\| < \epsilon \qquad j = 1, \ldots, n.$$

Fix $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$. Write $\Xi := (\xi_1, \ldots, \xi_n) \in \mathcal{H}^{\oplus n}$, let $\mathcal{S}$ denote the closure of the subspace

$$\{(x_0\xi_1, \ldots, x_0\xi_n) \colon x_0 \in M\} = \{(x_0 \otimes I_n)\Xi \colon x_0 \in M\}.$$

$\mathcal{S}$ is clearly reducing for $M \otimes I_n$. Thus if p is the projection of $\mathcal{H}^{\oplus n}$ onto $\mathcal{S}$, then Lemma 2.2.3 implies

$$p \in (M \otimes I_n)' = M' \otimes M_n(\mathbb{C}),$$

where the equality follows from an easy computation. It is also easily checked that $x \otimes I_n \in (M' \otimes M_n(\mathbb{C}))'$, so

$$p(x \otimes I_n) = (x \otimes I_n)p.$$

In particular, $(x \otimes I_n)\Xi \in \mathcal{S}$ (note that here we are using $1 \in M$ to assert $\Xi = (1 \otimes I_n)\Xi \in \mathcal{S}$). Thus, there exists $x_0 \in M$ so that $\|\left((x \otimes I_n) - (x_0 \otimes I_n)\right)\Xi\| < \epsilon$. This gives the desired condition. □

2.3 Definition of a von Neumann algebra

Definition 2.3.1. We say a unital $*$-subalgebra $1 \in M \subset \mathcal{B}(\mathcal{H})$ is a **von Neumann algebra** if it is closed in any of the equivalent ways of Theorem 2.2.4.

From this definition we immediately have two examples of von Neumann algebras in $\mathcal{B}(\mathcal{H})$: $\mathcal{B}(\mathcal{H})$ and $\mathbb{C}1$. We will explore further examples in Section 2.4, but first must define a few related concepts.

From the observation following Definition 2.2.1, we see that for M a von Neumann algebra, M' is also a von Neumann algebra. Consequently, so is $M \cap M'$ we which we give a name to here:

Definition 2.3.2. For M a von Neumann algebra, the **center of** M, denoted $\mathcal{Z}(M)$, is the von Neumann subalgebra $M \cap M'$. If $\mathcal{Z}(M) = \mathbb{C}1$, we say M is a **factor**. If $\mathcal{Z}(M) = M$, we say M is **abelian**.

2.4 Basic Examples

2.4.1 $\mathcal{B}(\mathcal{H})$ and Matrix Algebras

As previously mentioned, the $*$-algebra of bounded operators $\mathcal{B}(\mathcal{H})$ forms a von Neumann algebra. Indeed, the Principle of Uniform Boundedness (or Banach–Steinhaus Theorem) implies that it is SOT-closed. Furthermore, $\mathcal{B}(\mathcal{H})$ is always a factor (fixing an orthonormal basis, check the commutation relationship against projections onto single basis vectors and partial isometries permuting pairs of basis vectors). In particular, if $\mathcal{H} = \mathbb{C}^d$ is finite dimensional, then $\mathcal{B}(\mathcal{H})$ is simply the $d \times d$ matrices over $\mathbb{C}$, which we will denote by $M_d(\mathbb{C})$. Though an elementary example, $M_d(\mathbb{C})$ will eventually inform a great deal of our intuition about von Neumann algebras. Recall that the *unnormalized trace* on $M_d(\mathbb{C})$ is a linear functional $\mathrm{Tr}\colon M_d(\mathbb{C}) \to \mathbb{C}$ defined as

$$\mathrm{Tr}(A) = \sum_{i=1}^{d} [A]_{ii}.$$

Note that if $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{C}^d$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{d} \langle Ae_i, e_i \rangle .$$

In fact, if $\{f_1, \ldots, f_d\}$ is *any* orthonormal basis for $\mathbb{C}^d$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{d} \langle Af_i, f_i \rangle .$$

This is because if U is the unitary matrix whose columns are $f_1, \ldots, f_d$, then $Ue_i = f_i$ for each $i = 1, \ldots, d$. Consequently

$$\sum_{i=1}^{d} \langle Af_i, f_i \rangle = \sum_{i=1}^{d} \langle AUe_i, Ue_i \rangle = \sum_{i=1}^{d} \langle U^*AUe_i, e_i \rangle = \mathrm{Tr}(U^*AU) = \mathrm{Tr}(AUU^*) = \mathrm{Tr}(A).$$

We will use this later to define a trace for infinite dimensional Hilbert spaces. The domain of this trace will be the so-called *trace class operators* in $\mathcal{B}(\mathcal{H})$, which we will later see play an important role in the study of von Neumann algebras.

2.4.2 Measure Spaces

Let (X, Ω, μ) be a σ-finite measure space. With $\mathcal{H} = L^2(X, \mu)$, for each $f \in L^\infty(X, \mu)$ identify it with the following bounded operator on $\mathcal{H}$:

$$m_f \xi := f\xi \qquad \xi \in \mathcal{H}.$$

We claim that $M := L^\infty(X, \mu)$ is an abelian von Neumann algebra. First note that M is indeed a unital (abelian) $*$-algebra. By the Bicommutant Theorem, it suffices to show $M'' \subset M$.

Let $x \in M''$. As the measure space is σ-finite, we can find an ascending sequence of finite measure subsets whose union is X: $E_1 \subset E_2 \subset \cdots$. With χ_{E_n} the characteristic function on E_n, let $f\colon X \to \mathbb{C}$ be the function uniquely determined by

$$f\chi_{E_n} = x\chi_{E_n} \qquad \forall n \in \mathbb{N}.$$

For some $n \in \mathbb{N}$, let $A \subset E_n$ be measurable (and consequently of finite measure). Since $\chi_A \in L^\infty(X, \mu) \cap L^2(X, \mu)$, we have $\chi_A \in M$. Since M is abelian, $M \subset M'$ and hence $x \in M''$ commutes with χ_A. Thus for each $n \in \mathbb{N}$, we have

$$x\chi_A = x\chi_A\chi_{E_n} = \chi_A x\chi_{E_n} = \chi_A f\chi_{E_n} = f\chi_A.$$

Hence

$$\frac{1}{\mu(A)} \int_X |f\chi_A|^2 \, d\mu = \frac{1}{\mu(A)} \|x\chi_A\|_2^2 \leq \|x\|^2.$$

This implies $f \in L^\infty(X,\mu)$ with $\|f\|_\infty \leq \|x\|$. Now, for an arbitrary finite measure subset $A \subset X$ we have $\|\chi_{A\cap E_n} - \chi_A\|_2 \to 0$. Since x and m_f are both continuous, we have by the previous computation

$$\|(x - m_f)\chi_A\|_2 = \lim_{n\to\infty} \|(x - m_f)\chi_{A\cap E_n}\|_2 = 0.$$

Thus $x\chi_A = m_f\chi_A$. This implies x and m_f agree on the dense subspace of simple functions and therefore must be equal. The equality $\|f\|_\infty = \|x\|$ follows.

Observe that we really only used $x \in M'$. Thus we have shown that $M' = M$. A von Neumann algebra satisfying this condition is known as a *maximal abelian subalgebra* (or *MASA*), in this case we are viewing M as a subalgebra of $\mathcal{B}(\mathcal{H})$. The name comes from the fact that if A is an abelian von Neumann algebra such that $M \subset A \subset \mathcal{B}(\mathcal{H})$, then $M = A$.

The following proposition provides a more concrete example of the WOT. We leave its proof to the reader.

Proposition 2.4.1. *Let (X,Ω,μ) be a σ-finite measure space. Thinking of $L^\infty(X,\mu)$ as the dual space to $L^1(X,\mu)$, a net $(f_\alpha) \subset L^\infty(X,\mu)$ converges in the weak*-topology if and only if $(m_{f_\alpha}) \subset \mathcal{B}(L^2(X,\mu))$ converges in the WOT.*

As we shall later see, all abelian von Neumann algebras are of this form. It is for this reason that the theory of von Neumann algebras is often considered non-commutative measure theory. In fact, many of the celebrated results in measure theory have non-commutative analogs (e.g. Egorov's theorem, Lusin's theorem, etc.). When considering probability measures, the non-commutative analog is precisely Voiculescu's *free probability theory*. As with matrix algebras, measure theory will inform a lot of our intuition for von Neumann algebras. 1/20/2017

2.4.3 Group von Neumann Algebras

Let Γ be a discrete group. With $\mathcal{H} = \ell^2(\Gamma)$, we consider the left regular representation $\lambda\colon \Gamma \to \mathcal{B}(\mathcal{H})$:

$$[\lambda(g)\xi](h) = \xi(g^{-1}h) \qquad \xi \in \ell^2(\Gamma),\ h \in \Gamma$$

Equivalently, if for $g \in \Gamma$ we let $\delta_g \in \ell^2(\Gamma)$ be the function $\delta_g(h) = \delta_{g=h}$, then $\lambda_g\delta_h = \delta_{gh}$ for all $h \in \Gamma$.

The *group von Neumann algebra for* Γ is $L(\Gamma) := \mathbb{C}[\lambda(\Gamma)]''$, where $\mathbb{C}[\lambda(\Gamma)]$ is the $*$-algebra generated by $\lambda(G)$.

We consider the linear functional $\tau\colon L(\Gamma) \to \mathbb{C}$ defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$, which we note is clearly WOT-continuous and a state. Since $\tau(\lambda(g)) = \delta_{g=e}$, τ encodes the group relations; that is, $g_1g_2\cdots g_n = e$ for $g_1,\ldots,g_n \in \Gamma$ if and only if $\tau(\lambda(g_1)\cdots\lambda(g_n)) = 1$. Since $gh = e$ if and only if $hg = e$ for $g,h \in \Gamma$, it follows that $\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g))$. The WOT-continuity and linearity of τ then implies $\tau(xy) = \tau(yx)$ for all $x, y \in M$. In this case we say τ is a *tracial* state.

When Γ is abelian, it follows that $L(\Gamma)$ is abelian. Let us demonstrate a condition on Γ that guarantees $L(\Gamma)$ is a factor. We say Γ has the *infinite conjugacy class property* (or is an *i.c.c. group*) if $\{f^{-1}gf\colon f \in \Gamma\}$ is infinite for all $g \in \Gamma \setminus \{e\}$. Suppose Γ is an i.c.c. group.

First note that for a *finite* linear combination $x_0 = \sum c_g\lambda(g)$,

$$c_g = \left\langle x_0\delta_{g^{-1}h}, \delta_h\right\rangle \qquad \forall h \in \Gamma$$

Since $x \in L(\Gamma)$ is the WOT-limit of such finite linear combinations, it follows that

$$\Gamma \ni h \mapsto \left\langle x\delta_{g^{-1}h}, \delta_h\right\rangle$$

is a constant map. We denote its value by $c_h(x)$. Now, if $x \in Z(L(\Gamma))$ then x commutes with $\lambda(f)$ for all $f \in \Gamma$. In particular, $\lambda(f)x\lambda(f^{-1}) = x$ for all $f \in \Gamma$. Hence for any $h \in \Gamma$ we have

$$c_g(x) = \left\langle x\delta_{g^{-1}h}, \delta_h\right\rangle = \left\langle \lambda(f)x\lambda(f^{-1})\delta_{g^{-1}h}, \delta_h\right\rangle = \left\langle x\delta_{f^{-1}g^{-1}h}, \delta_{f^{-1}h}\right\rangle = \left\langle x\delta_{f^{-1}g^{-1}fh'}, \delta_{h'}\right\rangle = c_{f^{-1}gf}(x),$$

where $h' = f^{-1}h$ in the second-to-last equality. Thus for $x \in Z(L(\Gamma))$, $g \mapsto c_g(x)$ is constant on the conjugacy classes of Γ. However, if we let $\xi = x\delta_e \in \ell^2(\Gamma)$, then one easily checks that $\xi = \sum c_g(x)\delta_g$. Hence

$(c_g(x))_{g\in\Gamma}$ is a square-summable sequence. This is only possible if $c_g(x) = 0$ for all $g \in \Gamma\backslash\{e\}$. Consequently, $x\delta_g = c_e(x)\delta_g$ for all $g \in \Gamma$, which means $x = c_e(x)1$. Thus $L(\Gamma)$ is a factor.

In constructing the group von Neumann algebra, one could instead use the right regular representation:

$$[\rho(g)\xi](h) = \xi(hg) \qquad \xi \in \ell^2(\Gamma),\ h \in \Gamma,$$

in which case one denotes by $R(\Gamma) = \mathbb{C}[\rho(\Gamma)]''$. It turns out that $R(\Gamma) = L(\Gamma)'$.

We conclude this section by noting that group von Neumann algebras remain far from fully understood. On the one hand, by deep results of von Neumann and Connes all amenable i.c.c. groups yield the same von Neumann algebra (the *hyperfinite* II_1 *factor* $\mathcal{R}$, see the next example). On the other hand, it is still an open problem whether or not $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$, where $\mathbb{F}_k$ is the free group with k generators. A very active area of research in von Neumann algebras is focused on how much of Γ is "rememebered" by $L(\Gamma)$. The most powerful results to date have relied on a powerful collection of techniques known as Popa's deformation/ridigity theory. 1/23/2017

2.4.4 The Hyperfinite II_1 Factor

Fix $d \in \mathbb{N}$, $d \geq 2$. Observe that we can embed $M_d(\mathbb{C})$ into $M_d(\mathbb{C})^{\otimes 2}$ ($\cong M_{d^2}(\mathbb{C})$) via

$$M_d(\mathbb{C}) \ni x \mapsto \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} = A \otimes I_d.$$

In fact, the definition gives an embedding of $M_d(\mathbb{C})^{\otimes n}$ into $M_d(\mathbb{C})^{\otimes(n+1)}$ for all $n \in \mathbb{N}$. If we let tr denote the normalized trace, observe that this embedding preserves tr.

We let

$$\bigotimes_{n=1}^{\infty} M_d(\mathbb{C}) = \varinjlim M_d(\mathbb{C})^{\otimes n},$$

and let $\tau\colon \bigotimes_{n=1}^{\infty} M_d(\mathbb{C}) \to \mathbb{C}$ be the linear functional induced by tr. Equivalently, a typical $x \in \bigotimes_{n=1}^{\infty} M_d(\mathbb{C})$ is of the form

$$x = x_1 \otimes \cdots \otimes x_n \otimes I_d \otimes \cdots,$$

in which case

$$\tau(x) = \mathrm{tr}(x_1)\cdots\mathrm{tr}(x_n).$$

Let $(\mathcal{H}, \pi)$ be the GNS representation of $M_0 := \bigotimes_{n=1}^{\infty} M_d(\mathbb{C})$ with respect to τ. Recall that M_0 forms a dense subspace of $\mathcal{H}$, and for $x \in M_0$ we will let $\hat{x} \in \mathcal{H}$ denote the associated vector. Hence for $x, y \in M_0$

$$\pi(x)\hat{y} = \widehat{xy}.$$

Consider the von Neumann algebra

$$\mathcal{R} := \pi(M_0)'' \subset \mathcal{B}(\mathcal{H}).$$

We extend τ to $\mathcal{R}$ via the vector state $\tau(x) = \langle x\hat{1}, \hat{1}\rangle$, $x \in \mathcal{R}$. Clearly τ is WOT-continuous and a state. Moreover, since τ satisfies the trace property on $\bigotimes_{n=1}^{\infty} M_d(\mathbb{C})$, the WOT-continuity implies τ is a tracial state on $\mathcal{R}$.

$\mathcal{R}$ is called the **hyperfinite II_1 factor**. The significance of terminology "II_1," which relates to the classification of von Neumann algebras, will appear later. Note that $\mathcal{R}$ does not depend on d.

Let us prove that $\mathcal{R}$ is indeed a factor. Let $z \in Z(\mathcal{R})$. Since $z^*z \in Z(\mathcal{R})$, we may assume that $z \geq 0$. By renormalizing, we can also assume $\tau(z) = 1$. Consider the linear functional $\varphi\colon \mathcal{R} \to \mathbb{C}$ defined by

$$\varphi(x) = \tau(xz) \qquad \forall x \in \mathcal{R}.$$

Observe that φ is a WOT-continuous tracial state. In particular, $\varphi \circ \pi$ restricted to $M_d(\mathbb{C})^{\otimes n}$ is a tracial state. Since $M_d(\mathbb{C})^{\otimes n} = M_{d^n}(\mathbb{C})$ has the unique tracial state tr (check this), we have

$$\varphi \circ \pi\,|_{M_d(\mathbb{C})^{\otimes n}} = \mathrm{tr}.$$

The WOT-continuity of φ and τ along with the WOT density of $\pi\left(\bigotimes_{n=1}^{\infty} M_d(\mathbb{C})\right)$ (via the Bicommutant Theorem) implies $\varphi = \tau$. That is,

$$\tau(x) = \tau(xz) \qquad \forall x \in \mathcal{R}.$$

Equivalently, $\tau((x(1-z)) = 0$ for all $x \in \mathcal{R}$, and in particular

$$0 = \tau((1-z)^*(1-z)) = \|(1-z)\hat{1}\|$$

Thus $(1-z)\hat{1} = 0$. If we can show that $\hat{1}$ is separating for $\mathcal{R}$ (*cf.* Definition 3.2.1), then we will have $z = 1 \in \mathbb{C}1$. By Proposition 3.2.3, it suffices to show that $\hat{1}$ is cyclic for $\mathcal{R}'$. For $x, y \in M_0$, define

$$\rho(x)\hat{y} = \widehat{yx}.$$

Using that τ is a tracial state we have

$$\begin{aligned}\|\rho(x)\hat{y}\|^2 = \|\widehat{yx}\|^2 &= \left\langle \pi(x^*y^*yx)\hat{1}, \hat{1}\right\rangle = \tau(\pi(x^*y^*)\pi(yx)) = \tau(\pi(yx)\pi(x^*y^*))\\ &= \|\pi(x^*)\widehat{y^*}\|^2 \leq \|\pi(x^*)\|^2\tau(\pi(y)\pi(y^*)) = \|\pi(x)\|^2\tau(\pi(y^*)\pi(y)) = \|\pi(x)\|^2\|\hat{y}\|^2.\end{aligned}$$

Thus $\rho(x)$ extends to a bounded operator on $\mathcal{H}$ with $\|\rho(x)\| \leq \|\pi(x)\|$. It is easy to check that $\rho(x) \in \mathcal{R}'$. Clearly $\hat{1}$ is cyclic for $\rho(M_0)$, and hence it is cyclic for $\mathcal{R}'$. It follows that $\mathcal{R}$ is a factor.

2.5 Operations on von Neumann Algebras

In this section we discuss various operations on von Neumann algebras which yield new von Neumann algebras.

2.5.1 Direct Sums

Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces. For each $i = 1, \ldots, n$, define an isometric embedding $\pi_i \colon \mathcal{B}(\mathcal{H}_i) \to \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$ by

$$\pi_i(x)(\xi_1, \ldots, \xi_n) = (0, \cdots, 0, x\xi_i, 0, \ldots, 0) \qquad (\xi_1, \ldots, \xi_n) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n,$$

for $x \in \mathcal{B}(\mathcal{H}_i)$. We can also think of $\pi_i(x)$ as an $n \times n$ matrix with x in the (i,i)th-entry, and zeros elsewhere.

Definition 2.5.1. Let $M_i \subset \mathcal{B}(\mathcal{H}_i)$ be a von Neumann algebra for each $i = 1, \ldots, n$. The **direct sum of** $M_1, \ldots, M_n$ is

$$M_1 \oplus \cdots \oplus M_n := \operatorname{span}\{\pi_i(x) \colon i = 1, \ldots, n,\ x \in M_i\},$$

which is easily seen to be a von Neumann algebra. It may also be denoted $\displaystyle\bigoplus_{i=1}^{n} M_i$.

Observe that

$$\mathcal{Z}(M_1 \oplus \cdots \oplus M_n) = \mathcal{Z}(M_1) \oplus \cdots \oplus \mathcal{Z}(M_n).$$

In particular, $\pi_1(1), \ldots, \pi_n(1) \in \mathcal{Z}(M_1 \oplus \cdots \oplus M_n)$. Thus if $n \geq 2$, then these are non-trivial projections in the center of $M_1 \oplus \cdots \oplus M_n$, which is therefore not a factor.

Example 2.5.2. Suppose $X = \{t_1, \ldots, t_n\}$ is a finite set equipped with a finite measure μ. Let $p_j = \chi_{\{t_j\}}$ for $j = 1, \ldots, n$. Then

$$L^\infty(X, \mu) \cong \mathbb{C}p_1 \oplus \cdots \oplus \mathbb{C}p_n$$

In particular, integration against μ on the left-hand side induces a linear functional φ on the right-hand side that is determined by $\varphi(p_j) = \mu(\{t_j\})$, $j = 1, \ldots, n$.

The above is quite a trivial example, but it alludes to another operation on von Neumann algebra which we may visit later: *direct integrals*.

2.5.2 Tensor Products

Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces. We let the algebraic tensor product $\mathcal{B}(\mathcal{H}_1)\otimes\cdots\otimes\mathcal{B}(\mathcal{H}_n)$ act on $\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n$ by

$$x_1 \otimes \cdots \otimes x_n(\xi_1 \otimes \cdots \otimes \xi_n) = (x_1\xi_1) \otimes \cdots \otimes (x_n\xi_n) \qquad \xi_1 \in \mathcal{H}_1, \ldots, \xi_n \in \mathcal{H}_n,$$

for $x_1 \in \mathcal{B}(\mathcal{H}_1), \ldots, x_n \in \mathcal{B}(\mathcal{H}_n)$.

Definition 2.5.3. Let $M_i \subset \mathcal{B}(\mathcal{H}_i)$ be a von Neumann algebra for each $i = 1, \ldots, n$. The **the tensor product of** $M_1, \ldots, M_n$ is

$$M_1\bar{\otimes}\cdots\bar{\otimes}M_n := (M_1 \otimes \cdots \otimes M_n)'' \cap \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n).$$

It may also be denoted $\overline{\bigotimes}_{i=1}^n M_i$.

Lemma 2.5.4. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces so that at most one is infinite dimensional. If $M_i \subset \mathcal{B}(\mathcal{H}_i)$ is a von Neumann algebra for each $i = 1, \ldots, n$, then $M_1\bar{\otimes}\cdots\bar{\otimes}M_n = M_1 \otimes \cdots \otimes M_n$.*

Proof. By induction, it suffices to consider $n = 2$ with $\mathcal{H}_2$ finite dimensional, say with $\dim(\mathcal{H}_2) = d$. Then $\mathcal{B}(\mathcal{H}_2) \cong M_d(\mathbb{C})$, and if $\{e_1, \ldots, e_d\}$ is an orthonormal basis for $\mathcal{H}_2$ then

$$\langle x, y\rangle_2 := \operatorname{Tr}(y^*x) = \sum_{i=1}^d \langle y^*xe_i, e_i\rangle \qquad x, y \in M_d(\mathbb{C})$$

makes $M_d(\mathbb{C})$ into a Hilbert space. M_2 is a closed subspace of this Hilbert space since it is SOT closed. Thus there exists $\{y_1, \ldots, y_n\}$ an orthonormal basis for M_2 with respect to the above inner product. In particular, it is a basis and so every element $X \in M_1 \otimes M_2$ is of the form

$$X = \sum_{i=1}^n x_i \otimes y_i$$

for $x_1, \ldots, x_n \in M_1$. We denote $X^{(i)} := x_i$ for $i = 1, \ldots, n$. Observe that for X as above we have

$$\begin{aligned}\sum_{j=1}^d \|X(\xi \otimes e_j)\|^2 &= \sum_{j=1}^d \sum_{i,k=1}^n \langle (x_i \otimes y_i)(\xi \otimes e_j), (x_k \otimes y_k)(\xi \otimes e_j)\rangle \\ &= \sum_{i,k=1}^n \langle x_i\xi, x_k\xi\rangle \sum_{j=1}^d \langle y_ie_j, y_ke_j\rangle \\ &= \sum_{i,k=1}^n \langle x_i\xi, x_k\xi\rangle \langle y_i, y_k\rangle_2 = \sum_{i=1}^n \|x_i\xi\|^2\end{aligned}$$

Thus if $(X_\alpha) \subset M_1 \otimes M_2$ converges strongly then $(X_\alpha^{(i)}) \subset M_1$ converges strongly for each $i = 1, \ldots, n$ to some $X^{(i)}$. Since M_1 is SOT closed, $X^{(1)}, \ldots, X^{(n)} \in M_1$. We claim that (X_α) converges strongly to

$$X := \sum_{i=1}^n X^{(i)} \otimes y_i \in M_1 \otimes M_2.$$

Indeed, the above computation shows for each $j = 1, \ldots, d$

$$\|(X - X_\alpha)(\xi \otimes e_j)\|^2 \leq \sum_{j=1}^d \|(X - X_\alpha)(\xi \otimes e_j)\|^2 = \sum_{i=1}^n \|(X^{(i)} - X_\alpha^{(i)})\xi\|^2 \to 0.$$

Consequently, for $\xi_1, \ldots, \xi_d \in \mathcal{H}_1$ we have

$$\left\|(X - X_\alpha)\left(\sum_{j=1}^d \xi_j \otimes e_j\right)\right\| \leq \sum_{j=1}^d \|(X - X_\alpha)(\xi_j \otimes e_j)\| \to 0.$$

So X is the SOT limit of (X_α) and thus $M_1\bar{\otimes}M_2 = \overline{M_1 \otimes M_2}^{SOT} \subset M_1 \otimes M_2$. ☐

Lemma 2.5.5. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $(M\bar{\otimes}\mathcal{B}(\mathcal{H}))' = M' \otimes \mathbb{C}$.*

Proof. One inclusion is immediate:

$$M' \otimes \mathbb{C} \subset (M \otimes \mathcal{B}(\mathcal{H}))' = (M\bar{\otimes}\mathcal{B}(\mathcal{H}))'.$$

Let $T \in (M\bar{\otimes}\mathcal{B}(\mathcal{H}))'$. Fix $\eta, \zeta \in \mathcal{K}$ and observe that

$$\mathcal{H} \times \mathcal{H} \ni (\xi_1, \xi_2) \mapsto \langle T(\xi_1 \otimes \eta), \xi_2 \otimes \zeta \rangle$$

defines a bounded sesquilinear form. Thus there exists $x_{\eta,\zeta} \in \mathcal{B}(\mathcal{H})$ so that

$$\langle x_{\eta,\zeta}\xi_1, \xi_2 \rangle = \langle T(\xi_1 \otimes \eta), \xi_2 \otimes \zeta \rangle \qquad \xi_1, \xi_2 \in \mathcal{H}.$$

Since T commutes with $M \otimes 1$ we have for $a \in M$

$$\begin{aligned} \langle x_{\eta,\zeta} a\xi_1, \xi_2 \rangle &= \langle T((a\xi_1) \otimes \eta), \xi_2 \otimes \zeta \rangle = \langle (a \otimes 1)T(\xi_1 \otimes \eta), \xi_2 \otimes \eta \rangle \\ &= \langle T(\xi_1 \otimes \eta), (a^*\xi_2) \otimes \zeta \rangle = \langle x_{\eta,\zeta}\xi_1, a^*\xi_2 \rangle = \langle a x_{\eta,\zeta}\xi_1, \xi_2 \rangle . \end{aligned}$$

Thus $x_{\eta,\zeta} \in M'$. Fixing $\xi_1, \xi_2 \in \mathcal{H}$

$$\mathcal{K} \times \mathcal{K} \ni (\eta, \zeta) \mapsto \langle x_{\eta,\zeta}\xi_1, \xi_2 \rangle = \langle T(\xi_1 \otimes \eta), \xi_2 \otimes \zeta \rangle$$

also defines a bounded sesquilinear form. Using the same argument as above, there exists $y \in \mathcal{B}(\mathcal{K})' = \mathbb{C}$ (depending only on ξ_1, ξ_2) so that

$$\langle x_{\eta,\zeta}\xi_1, \xi_2 \rangle = \langle y\eta, \zeta \rangle = y \langle \eta, \zeta \rangle .$$

It follows that $x_{\eta,\zeta} = 0$ whenever $\eta \perp \zeta$, and $x_{\eta,\eta} = \|\eta\|^2 x$ some fixed $x \in M'$. We claim $T = x \otimes 1 \in M' \otimes \mathbb{C}$.

Fix orthonormal bases $\{\xi_i\}_{i \in I}$ and $\{\eta_j\}_{j \in J}$ for $\mathcal{H}$ and $\mathcal{K}$, respectively. For $i, i' \in I$ and $j, j' \in J$ we have

$$\langle T(\xi_i \otimes \eta_j), \xi_{i'} \otimes \eta_{j'} \rangle = \left\langle x_{\eta_j,\eta_{j'}}\xi_i, \xi_{i'} \right\rangle = \langle x\xi_i, \xi_{i'} \rangle = \langle (x \otimes 1)\xi_i \otimes \eta_j, \xi_{i'} \otimes \eta_{j'} \rangle ,$$

which yields the claimed equality. □

2.5.3 Compressions

Definition 2.5.6. For $M \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra and $p \in \mathcal{B}(\mathcal{H})$ a projection, pMp is called **compression of** M or a **corner of** M.

The terminology comes from the fact that under the identification $\mathcal{H} \cong p\mathcal{H} \oplus (1-p)\mathcal{H}$, pxp for $x \in M$ is identified with

$$\begin{pmatrix} pxp & 0 \\ 0 & 0 \end{pmatrix}.$$

If $p < 1$, then pMp is a $*$-algebra in $\mathcal{B}(\mathcal{H})$ but it is not unital. However, pMp can be identified with a $*$-algebra in $\mathcal{B}(p\mathcal{H})$, in which case $p \in pMp$ is the unit. Note that if $p \in M'$, then $pMp = Mp$.

Theorem 2.5.7. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in M$ a projection. Then pMp and $M'p$ are von Neumann algebras in $\mathcal{B}(p\mathcal{H})$.*

Proof. We first show the following equalities:

$$(M'p)' \cap \mathcal{B}(p\mathcal{H}) = pMp$$
$$(pMp)' \cap \mathcal{B}(p\mathcal{H}) = M'p.$$

The result easily follows from these and the bicommutant theorem.

The inclusion $pMp \subset (M'p)' \cap \mathcal{B}(p\mathcal{H})$ is immediate. Suppose $x \in (M'p)' \cap \mathcal{B}(p\mathcal{H})$. Define $\tilde{x} = xp = px \in \mathcal{B}(\mathcal{H})$. Then if $y \in M'$ we have

$$y\tilde{x} = ypx = x(yp) = xpy = \tilde{x}y.$$

Thus $\tilde{x} \in M'' = M$, and $x = p\tilde{x}p \in pMp$.

The inclusion $M'p \subset (pMp)' \cap \mathcal{B}(p\mathcal{H})$ is immediate. Suppose $y \in (pMp)' \cap \mathcal{B}(p\mathcal{H})$. Using the functional calculus to write y as a linear combination of four unitaries, we may assume $y = u$ is a unitary. We will extend u to an element of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{K} = \overline{Mp\mathcal{H}}$ and $q = [\mathcal{K}]$. Since $\mathcal{K}$ is clearly reducing for both M and M', we have $q \in \mathcal{Z}(M)$ by Lemma 2.2.3. We first extend u to $\mathcal{K}$. Define $\tilde{u}$ by

$$\tilde{u} \sum_i x_i p \xi_i = \sum_i x_i u p \xi_i,$$

for $x_i \in M$ and $\xi_i \in \mathcal{H}$. Observe that

$$\begin{aligned} \left\| \tilde{u} \sum_i x_i p \xi_i \right\|^2 &= \sum_{i,j} \langle x_i u p \xi_i, x_j u p \xi_j \rangle \\ &= \sum_{i,j} \left\langle p x_j^* x_i p u \xi_i, u \xi_j \right\rangle \\ &= \sum_{i,j} \left\langle u p x_j^* x_i p \xi_i, u \xi_j \right\rangle \\ &= \sum_{i,j} \left\langle p x_j^* x_i p \xi_i, \xi_j \right\rangle = \left\| \sum_i x_i p \xi_i \right\|^2. \end{aligned}$$

Thus $\tilde{u}$ is well-defined and an isometry, which we extend to $\mathcal{K}$. By definition, $\tilde{u}$ commutes with M on $Mp\mathcal{H}$, and consequently they commute on $\mathcal{K}$. It follows that for $x \in M$ and $\xi \in \mathcal{H}$ we have

$$x\tilde{u}q\xi = \tilde{u}xq\xi = \tilde{u}qx\xi.$$

That is, $\tilde{u}q = M' \cap \mathcal{B}(\mathcal{H})$. Since

$$\tilde{u}qp\xi = \tilde{u}q1p\xi = \tilde{u}1p\xi = 1up\xi,$$

we have $u = \tilde{u}qp$ in $\mathcal{B}(p\mathcal{H})$. Thus $u \in M'p$. □

Corollary 2.5.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in M$ a projection. If M is a factor then pMp and $M'p$ are factors.*

Proof. Let q be the projection as in the proof of Theorem 2.5.7. Since $q \in \mathcal{Z}(M) = \mathbb{C}$, we must have $q = 1$. We claim that this implies $M' \ni y \mapsto yp \in M'p$ is a $*$-algebra isomorphism. Indeed, if $yp = 0$ for $y \in M'$, then for all $x \in M$ and $\xi \in \mathcal{H}$ we have

$$yxp\xi = xyp\xi = 0.$$

Since $q = 1$, $Mp\mathcal{H}$ is a dense subset of $\mathcal{H}$ and thus $y = 0$. Since the map is clearly surjective, we have that M' and $M'p$ are isomorphic as $*$-algebras. Since being in the center is algebraic property, we have $\mathcal{Z}(M'p) = \mathcal{Z}(M')p = \mathbb{C}p$. Thus $M'p$ is a factor. Then by the equalities proved in Theorem 2.5.7, we have

$$\mathcal{Z}(pMp) = pMp \cap (pMp)' = (M'p)' \cap M'p = \mathcal{Z}(M'p) = \mathbb{C}p.$$

Thus pMp is also a factor. □

Chapter 3

Borel Functional Calculus

The material in this chapter is adapted from [3, Chapter 2.7] and [1, Chapter 2.9].

3.1 Projection-valued Measures

Lemma 3.1.1. *Let $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ be an increasing net of positive operators such that $\sup_\alpha \|x_\alpha\| < \infty$. Then this net converges in the SOT to some positive $x \in \mathcal{B}(\mathcal{H})$ satisfying $x \geq x_\alpha$ for all α.*

Proof. For $\alpha \leq \beta$, we have $x_\beta - x_\alpha > 0$. Thus for any $\xi \in \mathcal{H}$ we have

$$\|\sqrt{x_\beta}\xi\|^2 - \|\sqrt{x_\alpha}\xi\|^2 = \langle x_\beta\xi, \xi\rangle - \langle x_\alpha\xi, \xi\rangle = \langle (x_\beta - x_\alpha)\xi, \xi\rangle \geq 0.$$

On the other hand,

$$\|\sqrt{x_\alpha}\xi\|^2 \leq \|\sqrt{x_\alpha}\|^2\|\xi\|^2 = \|x_\alpha\|\|\xi\|^2 \leq (\sup_\alpha \|x_\alpha\|)\|\xi\|^2.$$

Hence $(\|\sqrt{x_\alpha}\xi\|^2)$ is a bounded, increasing net. Thus $\mathcal{H} \ni \xi \mapsto \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$ is a bounded quadratic form on $\mathcal{H}$. Consequently there is some $a \in \mathcal{B}(\mathcal{H})$ so that $\|a\xi\| = \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$ for all $\xi \in \mathcal{H}$. Define $x = a^*a$. Since

$$\|\sqrt{x}\xi\| = \|a\xi\| = \sup_\alpha \|\sqrt{x_\alpha}\xi\| \qquad \xi \in \mathcal{H},$$

it follows that $x \geq x_\alpha$ for all α. Note that $\sup_\alpha \|x - x_\alpha\| < \infty$. So for each $\xi \in \mathcal{H}$ we have

$$\begin{aligned}\|(x - x_\alpha)\xi\|^2 &\leq \|(x - x_\alpha)^{1/2}\|^2\|(x - x_\alpha)^{1/2}\xi\|^2 \\ &= \|x - x_\alpha\|\,\langle (x - x_\alpha)\xi, \xi\rangle \\ &= \|x - x_\alpha\|\left(\|\sqrt{x}\xi\|^2 - \|\sqrt{x_\alpha}\xi\|^2\right) \to 0.\end{aligned}$$

So (x_α) converges to x in the SOT. □

We leave the proof of the next corollary to reader, which follows easily from the above lemma.

Corollary 3.1.2. *Let $(p_n) \subset \mathcal{B}(\mathcal{H})$ be a sequence of pairwise orthogonal projections. Then the SOT limit of $\left(\sum_{n=1}^N p_n\right)_{N\in\mathbb{N}}$ exists and is denoted $\sum_{n=1}^\infty p_n$.* 1/25/2017

Definition 3.1.3. Let X be a set, Ω a σ-algebra on X, and $\mathcal{H}$ a Hilbert space. A **projection-valued measure** (or **spectral measure**) for $(X, \Omega, \mathcal{H})$ is a map $E\colon \Omega \to \mathcal{B}(\mathcal{H})$ such that

(i) $E(S)$ is a projection for all $S \in \Omega$.

(ii) $E(\emptyset) = 0$ and $E(X) = 1$.

(iii) $E(S \cap T) = E(S)E(T)$ for $S, T \in \Omega$.

(iv) If $(S_n) \subset \Omega$ is a sequence of pairwise disjoint sets, then

$$E\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} E(S_n).$$

where the summation is the SOT limit of the partial sums, as defined in Corollary 3.1.2.

Example 3.1.4. Let (X, Ω, μ) be a σ-finite measure space. For $\mathcal{H} = L^2(X, \mu)$, note that χ_S, for $S \in \Omega$, acting by pointwise multiplication is a projection in $\mathcal{B}(\mathcal{H})$. It follows that $E(S) = \chi_S$ defines a projection-valued measure.

Lemma 3.1.5. *Let E be a projection-valued measure for $(X, \Omega, \mathcal{H})$. Then for each $\xi, \eta \in \mathcal{H}$,*

$$E_{\xi,\eta} : \Omega \ni S \mapsto \langle E(S)\xi, \eta\rangle$$

defines a countably additive, complex-valued measure on Ω with total variation at most $\|\xi\|\|\eta\|$.

Proof. Fix $\xi, \eta \in \mathcal{H}$. The countable additivity follows immediately from the definition of a projection-valued measure. Recall that the total variation of a complex valued measure is

$$\|E_{\xi,\eta}\| = \sup_{\pi} \sum_{S \in \pi} |E_{\xi,\eta}(S)|,$$

where the supremum is over all partitions π of X into finite disjoint measurable subsets of X. Let $\pi = \{S_1, \ldots, S_n\}$ be one such partition. For each $j = 1, \ldots, n$, let $\alpha_j \in \mathbb{C}$ be such that $|\alpha_j| = 1$ and

$$\alpha_j E_{\xi,\eta}(S_j) = \alpha_j \langle E(S_j)\xi, \eta\rangle = |\langle E(S_j)\xi, \eta\rangle| = |E_{\xi,\eta}(S_j)|.$$

Thus

$$\sum_{j=1}^{n} |E_{\xi,\eta}(S_j)| = \left\langle \sum_{j=1}^{n} \alpha_j E(S_j)\xi, \eta \right\rangle \leq \left\| \sum_{j=1}^{n} \alpha_j E(S_j)\xi \right\| \|\eta\|.$$

Since the $E(S_j)$ are pairwise orthogonal projections we have

$$\left\| \sum_{j=1}^{n} \alpha_j E(S_j)\xi \right\|^2 = \sum_{j=1}^{n} \|E(S_j)\xi\|^2 = \left\| \sum_{j=1}^{n} E(S_j)\xi \right\|^2 = \left\| E\left(\bigcup_{j=1}^{n} S_j\right)\xi \right\|^2 \leq \|\xi\|^2$$

Thus $\sum |E_{\xi,\eta}(S_j)| \leq \|\xi\|\|\eta\|$, which implies $\|E_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$. □

Remark 3.1.6. Let E be a projection-valued measure for $(X, \Omega, \mathcal{H})$. For $\xi \in \mathcal{H}$ and $S \in \Omega$

$$E_{\xi,\xi}(S) = \langle E(S)\xi, \xi\rangle = \langle E(S)\xi, E(S)\xi\rangle = \|E(S)\xi\|^2.$$

Thus $E_{\xi,\xi}$ is a positive measure on X. If $\xi \in \mathcal{H}$ is a unit vector, then $E(X) = 1$ implies $E_{\xi,\xi}$ is a probability measure on X.

Let E be a projection-valued measure for $(X, \Omega, \mathcal{H})$. Observe that

$$\begin{aligned} E_{\xi+\alpha\zeta,\eta} &= E_{\xi,\eta} + \alpha E_{\zeta,\eta} \qquad \xi, \eta, \zeta \in \mathcal{H},\ \alpha \in \mathbb{C} \\ E_{\xi,\eta+\alpha\zeta} &= E_{\xi,\eta} + \bar{\alpha} E_{\xi,\zeta}. \end{aligned} \tag{3.1}$$

This combined with Lemma 3.1.5 implies that $(\xi, \eta) \mapsto \int_X 1 \, dE_{\xi,\eta}$ is a bounded sesquilinear form on $\mathcal{H}$. More generally, if $f \colon X \to \mathbb{C}$ is a bounded Ω-measurable function then $(\xi, \eta) \mapsto \int_X f \, dE_{\xi,\eta}$ is a sesquilinear form on $\mathcal{H}$ bounded by $\|f\|_\infty \|\xi\|\|\eta\|$. Thus, by the Riesz representation theorem (for bounded sesquilinear forms) there exists $x \in \mathcal{B}(\mathcal{H})$ with $\|x\| \leq \|f\|_\infty$ so that

$$\langle x\xi, \eta\rangle = \int_X f \, dE_{\xi,\eta}. \tag{3.2}$$

If $f = \chi_S$ for some $S \in \Omega$, then clearly $x = E(S)$. Otherwise, we have the following definition:

Definition 3.1.7. For E a projection-valued measure for $(X, \Omega, \mathcal{H})$ and $f: X \to \mathbb{C}$ a bounded Ω-measurable function, the operator $x \in \mathcal{B}(\mathcal{H})$ satisfying (3.2) is called the **integral of f with respect to** E and is denoted

$$x = \int_X f \, dE.$$

We denote by $B(X, \Omega)$ the set of $\mathbb{C}$-valued, bounded, Ω-measurable functions on X. If X is a topological space and $\Omega = \mathcal{B}_X$ is the Borel σ-algebra on X, then we write $B(X)$ for $B(X, \mathcal{B}_X)$.

Proposition 3.1.8. *Let E be a projection-valued measure for $(X, \Omega, \mathcal{H})$. Then*

$$\rho: B(X, \Omega) \ni f \mapsto \int_X f \, dE \in \mathcal{B}(\mathcal{H})$$

is a contractive $*$-homomorphism. If $(f_n) \subset B(X, \Omega)$ is an increasing sequence of non-negative functions with $f := \sup_n f_n \in B(X, \Omega)$, then $(\int_X f_n \, dE)$ converges to $\int_X f \, dE$ in the SOT.

In particular, if X is a compact Hausdorff space, then $\rho(B(X) \subset \rho(C(X))''$. Here $C(X)$ is the set of continuous functions from X to $\mathbb{C}$.

Proof. We have already seen that $\|\rho(f)\| \leq \|f\|_\infty$ for $f \in B(X, \Omega)$. It is also clear that ρ is linear and preserves the adjoint operation, so it remains to check that it is multiplicative. Recall that $\rho(\chi_S) = E(S)$ for $S \in \Omega$. Hence for $S, T \in \Omega$ we have

$$\rho(\chi_S)\rho(\chi_T) = E(S)E(T) = E(S \cap T) = \rho(\chi_{S\cap T}) = \rho(\chi_S \chi_T).$$

By the linearity of ρ, we have $\rho(f)\rho(g) = \rho(fg)$ for simple functions $f, g \in B(X, \Omega)$. Since arbitrary $f, g \in B(X, \Omega)$ are uniform limits of uniformly bounded sequences simple functions, we have $\rho(f)\rho(g) = \rho(fg)$. Hence ρ is a $*$-homomorphism. 1/27/2017

If $(f_n) \subset B(X, \Omega)$ is an increasing sequence of non-negative functions with $f = \sup_n f_n \in B(X, \Omega)$, then ρ being a $*$-homomorphism implies $(\rho(f_n))$ is an increasing sequence of positive operators such that

$$\sup_n \|\rho(f_n)\| \leq \sup_n \|f_n\|_\infty = \|f\|_\infty.$$

Thus Lemma 3.1.1 implies $(\rho(f_n))$ converges in the SOT to some $x \in \mathcal{B}(\mathcal{H})$. Hopefully $x = \rho(f)$, but this remains to be seen. Let $\xi, \eta \in \mathcal{H}$, then by the monotone convergence theorem we have

$$\langle \rho(f)\xi, \eta \rangle = \int_X f \, dE_{\xi,\eta} = \lim_{n\to\infty} \int_X f_n \, dE_{\xi,\eta} = \lim_{n\to\infty} \langle \rho(f_n)\xi, \eta \rangle.$$

Thus $\rho(f)$ is the WOT limit of $(\rho(f_n))$ and so we must have $x = \rho(f)$.

Now, let X be a compact Hausdorff space. Let $a \in \rho(C(X))'$. Fix $\xi, \eta \in \mathcal{H}$, then for any $f \in C(X)$ we have

$$0 = \langle (a\rho(f) - \rho(f)a)\xi, \eta \rangle = \langle \rho(f)\xi, a^*\eta \rangle - \langle \rho(f)(a\xi), \eta \rangle = \int_X f \, dE_{\xi,a^*\eta} - \int_X f \, dE_{a\xi,\eta}.$$

Thus $E_{\xi,a^*\eta} = E_{a\xi,\eta}$. This implies that a also commutes with operators of the form $\rho(g) = \int_X g \, dE$ for $g \in B(X)$. Hence $\rho(B(X)) \subset \rho(C(X))''$. □

Remark 3.1.9. It is natural to wonder when the $*$-homomorphism ρ from the previous proposition is an isometry, and the answer is never, except when X and E are rather trivial. However, there is an isometry induced by ρ. Let $N = \{S \in \Omega: E(S) = 0\}$, which we refer to as *$E$-null sets.* Define an equivalence relation on $B(X, \Omega)$ by saying $f \sim_E g$ if and only if $f = g$ except possibly on E-null sets. By first checking on simple functions in $B(X, \Omega)$, one can then show that $\ker(\rho) = \{f \in B(X, \Omega): f \sim_E 0\}$ and that $\|\rho(f)\| = \|f\|_{\infty,e}$, where $\|\cdot\|_{\infty,e}$ is the *essential supremum* of f:

$$\|f\|_{\infty,e} := \inf\{t \geq 0: E(\{x \in X: |f(x)| \geq t\}) = 0\}.$$

Hence if $L^\infty(X, E) = B(X, \Omega)/\sim_E$, then the essential supremum is a norm on this space and ρ factors through to an isometric $*$-homomorphism $\tilde{\rho}: L^\infty(X, E) \to \mathcal{B}(\mathcal{H})$.

Recall that for an abelian C^*-algebra A, its *spectrum* $\sigma(A)$ is the set of non-zero $*$-homomorphisms from A to $\mathbb{C}$, which is a locally compact Hausdorff space. Furthermore, the *Gelfand transform*

$$\Gamma\colon A \to C_0(\sigma(A))$$

is an isometric $*$-isomorphism. In particular, if A is unital, then $C_0(\sigma(A))$ is replaced with $C(\sigma(A))$.

Theorem 3.1.10. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a unital abelian C^*-algebra, and let $\mathcal{B}_{\sigma(A)}$ be the Borel σ-algebra on $\sigma(A)$. Then there is a unique projection-valued measure E for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ such that*

$$x = \int_{\sigma(A)} \Gamma(x)\ dE$$

for all $x \in A$.

Proof. For each $\xi, \eta \in \mathcal{H}$, $C(\sigma(A)) \ni f \mapsto \left\langle \Gamma^{-1}(f)\xi, \eta \right\rangle$ is a bounded linear functional. Thus, the Riesz–Markov representation theorem implies there exists a regular Borel measure $\mu_{\xi\,\eta}$ on $\sigma(A)$ so that

$$\left\langle \Gamma^{-1}(f)\xi, \eta \right\rangle = \int_{\sigma(A)} f\ d\mu_{\xi,\eta} \qquad f \in C(\sigma(A)).$$

We will show that $\mu_{\xi,\eta} = E_{\xi,\eta}$ for some projection-valued measure E. Note that for any $f, g \in C(\sigma(A))$, we have

$$\int_{\sigma(A)} fg\ d\mu_{\xi,\eta} = \left\langle \Gamma^{-1}(fg)\xi, \eta \right\rangle = \left\langle \Gamma^{-1}(f)\left(\Gamma^{-1}(g)\xi\right), \eta \right\rangle = \int_{\sigma(A)} f\ d\mu_{\Gamma^{-1}(g)\xi,\eta}.$$

Hence $g\ d\mu_{\xi,\eta} = d\mu_{\Gamma^{-1}(g)\xi,\eta}$, and similarly $g\ d\mu_{\xi,\eta} = d\mu_{\xi,\Gamma^{-1}(\bar{g})\eta}$. Also,

$$\int_{\sigma(A)} f\ d\bar{\mu}_{\xi,\eta} = \overline{\int_{\sigma(A)} \bar{f}\ d\mu_{\xi,\eta}} = \overline{\left\langle \Gamma^{-1}(\bar{f})\xi, \eta \right\rangle} = \left\langle \Gamma^{-1}(f)\eta, \xi \right\rangle = \int_{\sigma(A)} f\ d\mu_{\eta,\xi},$$

so that $\bar{\mu}_{\xi,\eta} = \mu_{\eta,\xi}$.

Now, for each $S \in \mathcal{B}_{\sigma(A)}$,

$$\mathcal{H} \times \mathcal{H} \ni (\xi, \eta) \mapsto \int_{\sigma(A)} \chi_S\ d\mu_{\xi,\eta}$$

defines a sesquilinear form on $\mathcal{H}$. Since

$$\left| \int_{\sigma(A)} f\ d\mu_{\xi,\eta} \right| = \left| \left\langle \Gamma^{-1}(f)\xi, \eta \right\rangle \right| \leq \|f\|_\infty \|\xi\| \|\eta\|$$

for all $f \in C(\sigma(A))$, we have that the above sesquilinear form is bounded. Thus there exists $E(S) \in \mathcal{B}(\mathcal{H})$ so that

$$\int_{\sigma(A)} \chi_S\ d\mu_{\xi,\eta} = \langle E(S)\xi, \eta \rangle.$$

Using $\bar{\mu}_{\eta,\xi} = \mu_{\xi,\eta}$, we have

$$\langle E(S)^*\xi, \eta \rangle = \overline{\langle E(S)\eta, \xi \rangle} = \overline{\int_{\sigma(A)} \chi_S\ d\mu_{\eta,\xi}} = \int_{\sigma(A)} \chi_S\ d\mu_{\xi,\eta} = \langle E(S)\xi, \eta \rangle.$$

Thus $E(S) = E(S)^*$.

From our previous observation, for any $f \in C(\sigma(A))$ we have

$$\left\langle \Gamma^{-1}(f)E(S)\xi, \eta \right\rangle = \int_{\sigma(A)} \chi_S\ d\mu_{\xi,\Gamma^{-1}(\bar{f})\eta} = \int_{\sigma(A)} \chi_S f\ d\mu_{\xi,\eta}.$$

Recall that $C(\sigma(A))$ is weak*-dense in $C(\sigma(A))^{**}$, which contains $B(\sigma(A))$. So, in particular, for $T \in \Omega$, there is a net $(f_\alpha) \subset C(\sigma(A))$ converging weak* to χ_T. This implies that $(\Gamma^{-1}(f_\alpha))$ converges in the WOT to $E(T)$. So using the above formula, we have

$$\begin{aligned}\langle E(T)E(S)\xi, \eta\rangle &= \lim_\alpha \langle \Gamma^{-1}(f_\alpha)E(S)\xi, \eta\rangle = \lim_\alpha \int_{\sigma(A)} \chi_S f_\alpha \, d\mu_{\xi,\eta} \\ &= \int_{\sigma(A)} \chi_S \chi_T \, d\mu_{\xi,\eta} = \int_{\sigma(A)} \chi_{S\cap T} \, d\mu_{\xi,\eta} = \langle E(S\cap T)\xi, \eta\rangle .\end{aligned}$$

Thus $E(T)E(S) = E(S \cap T)$ for any $S, T \in \mathcal{B}_{\sigma(A)}$. In particular, $E(S)^2 = E(S)$, so $E(S)$ is a projection. Clearly $E(\emptyset) = 0$ and $E(\sigma(A)) = \Gamma^{-1}(1) = 1$. Since each $\mu_{\xi,\eta}$ is a countably additive measure we have

$$\left\langle E\left(\bigcup_{n=1}^\infty S_n\right)\xi, \eta\right\rangle = \mu_{\xi,\eta}\left(\bigcup_{n=1}^\infty S_n\right) = \sum_{n=1}^\infty \mu_{\xi,\eta}(S_n) = \left\langle \sum_{n=1}^\infty E(S_n)\xi, \eta\right\rangle$$

for any sequence $(S_n) \subset \mathcal{B}_{\sigma(A)}$ of pairwise disjoint sets. Thus $E\left(\bigcup_n S_n\right) = \sum_n E(S_n)$ and so E is a projection-valued measure for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$.

Clearly, $E_{\xi,\eta} = \mu_{\xi,\eta}$ and so

$$\langle \Gamma^{-1}(f)\xi, \eta\rangle = \int_{\sigma(A)} f \, dE_{\xi,\eta},$$

implies $x = \int_{\sigma(A)} \Gamma(x) \, dE$ for all $x \in A$. 1/30/2017

Now, suppose E and E' are both projection-valued measures for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ satisfying the claimed formula. Then for each $\xi, \eta \in \mathcal{H}$ and all $f \in C(\sigma(A))$ we have

$$\int_{\sigma(A)} f \, dE_{\xi,\eta} = \langle \Gamma^{-1}(f)\xi, \eta\rangle = \int_{\sigma(A)} f \, dE'_{\xi,\eta}.$$

Thus $E_{\xi,\eta} = E'_{\xi,\eta}$ as elements of $C(\sigma(A))^*$. But then for all Borel subsets $S \subset \sigma(A)$ we have

$$\langle E(S)\xi, \eta\rangle = E_{\xi,\eta}(S) = E'_{\xi,\eta}(S) = \langle E'(S)\xi, \eta\rangle .$$

Since ξ and η were arbitrary, we have $E(S) = E'(S)$ and generally $E = E'$. Thus E is unique. □

Suppose $x \in \mathcal{B}(\mathcal{H})$ is a normal operator ($x^*x = xx^*$) and let A be the C^*-algebra generated by x and 1. Then $\sigma(A) = \sigma(x)$, and letting E be the projection-valued measure for $(\sigma(x), \mathcal{B}_{\sigma(x)}, \mathcal{H})$ from the previous theorem,

$$B(\sigma(x)) \ni f \mapsto \int_{\sigma(x)} f \, dE \in \mathcal{B}(\mathcal{H})$$

defines a contractive $*$-homomorphism by Proposition 3.1.8. In particular, if $f \in B(\sigma(x))$ is the identity function $f(z) = z$, then

$$\int_{\sigma(x)} f \, dE = \Gamma^{-1}(f) = x.$$

Thus, for general $f \in B(\sigma(x))$ we write

$$f(x) := \int_{\sigma(x)} f \, dE.$$

By Proposition 3.1.8, we know that $f(x) \in A'' = W^*(x)$ for all $f \in B(\sigma(x))$.

Definition 3.1.11. With $x \in \mathcal{B}(\mathcal{H})$ a normal operator, the map $B(\sigma(x)) \ni f \mapsto f(x) \in W^*(x)$ is called the **Borel functional calculus (for x)**.

We summarize the properties of the Borel functional calculus here:

Theorem 3.1.12 (Borel Functional Calculus)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with $x \in A$ a normal operator. The Borel functional calculus for x satisfies the following properties:*

(i) $B(\sigma(x)) \ni f \mapsto f(x) \in A$ *is a continuous* $*$*-homomorphism.*

(ii) For $f \in B(\sigma(x))$, $\sigma(f(x)) \subset f(\sigma(x))$.

(iii) If $f \in C(\sigma(x))$, *then* $f(x)$ *is the same operator given by the continuous functional calculus.*

We conclude with the following corollary which highlights the ubiquity of projections in a von Neumann algebra.

Corollary 3.1.13. *A von Neumann algebra is the operator norm closure of the span of its projections.*

Proof. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $x \in M$. By considering the real and imaginary parts of x ($\text{Re}\,(x) = \frac{1}{2}(x + x^*)$ and $\text{Im}\,(x) = \frac{i}{2}(x^* - x)$) we may assume x is self-adjoint. In particular, x is normal and hence $f(x) \in M$ for all $f \in B(\sigma(x))$ by the previous theorem. Note that for all Borel subsets $S \subset \sigma(x)$, $\chi_S(x)$ is a projection in M. Thus, approximating the identity function on $\sigma(x)$ uniformly by simple functions gives, via the Borel functional calculus, a uniform approximation of x by linear combinations of projections in M. □

Contrast this result with the fact that there exists C^*-algebras with no non-trivial projections. Indeed, if X is compact Hausdorff space, and X is connected, then $C(X)$ has exactly two projections: 0 and 1. Non-commutative examples exist as well. 1/31/2017

3.2 Abelian von Neumann Algebras

In this section we examine certain abelian von Neumann algebras; namely, those with a "cyclic" vector $\xi \in \mathcal{H}$. This will not be a comprehensive study, but for now will offer a satisfactory explanation for earlier claims that abelian von Neumann algebras are of the form $L^\infty(X, \mu)$ for a σ-finite measure space (X, Ω, μ). We will also see how this is connected to the results from the previous section.

We begin with some adjectives for vectors in $\mathcal{H}$.

Definition 3.2.1. Let $A \subset \mathcal{B}(\mathcal{H})$ be a subalgebra. A vector $\xi \in \mathcal{H}$ is said to be **cyclic for** A if the subspace $A\xi$ is dense in $\mathcal{H}$. We say ξ is **separating for** A if $x\xi = 0$ implies $x = 0$ for $x \in A$.

Example 3.2.2. Let Γ be a discrete group and let $L(\Gamma)$ be the group von Neumann algebra acting on $\ell^2(\Gamma)$. Also let $\lambda, \rho \colon \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be the left and right regular representations of Γ. Then $\delta_e \in \ell^2(\Gamma)$ is clearly a cyclic vector for $\mathbb{C}[\lambda(\Gamma)]$ and $\mathbb{C}[\rho(\Gamma)]$, and consequently is cyclic for $L(\Gamma)$ and $R(\Gamma)$. The following proposition implies that δ_e is also separating for both $L(\Gamma)$ and $R(\Gamma)$.

Proposition 3.2.3. *Let* $A \subset \mathcal{B}(\mathcal{H})$ *be a subalgebra. If* $\xi \in \mathcal{H}$ *is cyclic for* A*, then it is separating for its commutant* A'*. If* A *is a unital* $*$*-subalgebra and* ξ *is separating for* A'*, then* ξ *is cyclic for* A*.*

In particular, for $M \subset \mathcal{B}(\mathcal{H})$ *a von Neumann algebra,* $\xi \in \mathcal{H}$ *is cyclic for* M *if and only if it is separating for* M'*, and* ξ *is separating for* M *if and only if it is cyclic for* M'*.*

Proof. Suppose $\xi \in \mathcal{H}$ is cylic for A. Let $y \in A'$ be such that $y\xi = 0$. For any $\eta \in \mathcal{H}$, we can find a sequence $(x_n) \subset A$ such that $\|\eta - x_n\xi\| \to 0$. We then have

$$y\eta = \lim_{n\to\infty} yx_n\xi = \lim_{n\to\infty} x_n y\xi = 0.$$

Thus $y = 0$, and ξ is separating for A'.

Now suppose A is a unital $*$-subalgebra and ξ is separating for A'. Let p be the projection of $\mathcal{H}$ onto $\mathcal{K} := (A\xi)^\perp$, so that our goal is to show $p = 0$. For $x_1, x_2 \in A$ and $\eta \in \mathcal{K}$ we have

$$\langle x_1\eta, x_2\xi\rangle = \langle \eta, x_1^* x_2\xi\rangle = 0,$$

since $x_1^*x_2 \in A$. Thus $x_1\eta \in \mathcal{K}$, and hence $A\mathcal{K} \subset \mathcal{K}$. That is, $\mathcal{K}$ is reducing for A and so Lemma 2.2.3 implies $p \in A'$. Note that $\xi \in A\xi$ since A is unital, and hence $p\xi = 0$. Since ξ is separating for A', this implies $p = 0$. The final observations follow from M being a unital $*$-subalgebra and $M = (M')'$. □

Corollary 3.2.4. *If $A \subset \mathcal{B}(\mathcal{H})$ is an abelian algebra, then every cyclic vector for A is also separating for A.*

Proof. If $\xi \in \mathcal{H}$ is cyclic for A, then by the proposition it is separating for A'. In particular, it is separating for $A \subset A'$. □

Theorem 3.2.5. *Let $A \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra with a cyclic vector $\xi_0 \in \mathcal{H}$. For any SOT dense unital C^*-subalgebra $A_0 \subset A$ there exists a finite regular Borel measure μ on $\sigma(A_0)$ and an isomorphism*

$$\Gamma^*\colon A \to L^\infty(\sigma(A_0), \mu),$$

extending the Gelfand transform $\Gamma\colon A_0 \to C(\sigma(A_0))$. This isomorphism is spatial*: it is implemented via conjugation by a unitary operator $U\colon \mathcal{H} \to L^2(\sigma(A_0), \mu)$.*

Proof. Since A_0 is a unital abelian C^*-algebra, the Gelfand transform $\Gamma : A_0 \to C(\sigma(A_0))$ is an isometric $*$-isomorphism. Define $\varphi\colon A \to \mathbb{C}$ by $\varphi(x) = \langle x\xi_0, \xi_0\rangle$ for $x \in A$. Then $\varphi \circ \Gamma^{-1}$ is a continuous linear functional on $C(\sigma(A_0))$, and so the Riesz–Markov theorem implies there is a regular Borel measure μ on $\sigma(A_0)$ so that

$$\varphi \circ \Gamma^{-1}(f) = \int_{\sigma(A_0)} f\ d\mu.$$

Observe that for a positive function $f \in C(\sigma(A_0))$, we have

$$\int_{\sigma(A_0)} f\ d\mu = \int_{\sigma(A_0)} \sqrt{f}^2\ d\mu = \varphi \circ \Gamma^{-1}(\sqrt{f}^2)) = \left\langle \Gamma^{-1}(\sqrt{f}^2)\xi_0, \xi_0 \right\rangle = \left\| \Gamma^{-1}(\sqrt{f})\xi_0 \right\|^2 \geq 0.$$

Hence μ is a positive measure. Also, μ is finite: $\mu(\sigma(A_0)) = \varphi(1) = \|\xi_0\|^2 < \infty$. We also claim that $\operatorname{supp}(\mu) = \sigma(A_0)$. If not, then there is a non-empty open subset $S \subset \sigma(A_0)$ so that $\mu(S) = 0$. Let $f \in C(\sigma(A_0))$ be a non-zero, positive, and supported on S. Then by the above computation,

$$\left\| \Gamma^{-1}(\sqrt{f})\xi_0 \right\|^2 = \int_{\sigma(A_0)} f\ d\mu = \int_S f\ d\mu = 0,$$

or $\Gamma^{-1}(\sqrt{f})\xi_0 = 0$. Since ξ_0 is cyclic for A, it is also separating by Corollary 3.2.4. Hence $\Gamma^{-1}(\sqrt{f}) = 0$, but this contradicts f being non-zero. Thus $\operatorname{supp}(\mu) = \sigma(A_0)$.

Define $U_0\colon A_0\xi_0 \to C(\sigma(A_0)) \subset L^2(\sigma(A_0), \mu)$ by

$$U_0(x\xi_0) = \Gamma(x) \qquad x \in A_0.$$

Since ξ_0 is separating for A, this is well-defined. Moreover, for $x, y \in A + -$

$$\langle U_0(x\xi_0), U_0(y\xi_0)\rangle_2 = \langle \Gamma(x), \Gamma(y)\rangle_2 = \int_{\sigma(A_0)} \overline{\Gamma(y)}\Gamma(x)\ d\mu = \int_{\sigma(A_0)} \Gamma(y^*x)\ d\mu = \varphi(y^*x) = \langle x\xi_0, y\xi_0\rangle\,.$$

Thus U_0 is an isometry on $A_0\xi_0$. Note that ξ_0 is cyclic for A_0 because it is cyclic for A and A_0 is SOT dense in A. Hence $A_0\xi_0$ is dense in $\mathcal{H}$ and so we can extend U_0 to a unitary $U\colon \mathcal{H} \to L^2(\sigma(A_0), \mu)$ (recall that $C(\sigma(A_0))$ is dense in $L^2(\sigma(A_0), \mu)$).

Define $\Gamma^*\colon A \to \mathcal{B}(L^2(\sigma(A_0), \mu))$ via $\Gamma^*(x) = UxU^*$. Then Γ^* is an isometric $*$-homomorphism. For $x \in A_0$ and $g \in C(\sigma(A_0))$ we have

$$\Gamma^*(x)g = UxU^*g = Ux\left(\Gamma^{-1}(g)\xi_0\right) = U\Gamma^{-1}(\Gamma(x)g)\xi_0 = \Gamma(x)g = m_{\Gamma(x)}g.$$

By the density of $C(\sigma(A_0)) \subset L^2(\sigma(A_0), \mu)$, it follows that $\Gamma^*(x) = m_{\Gamma(x)}$. Thus

$$\Gamma^*(A_0) = \{m_f\colon f \in C(\sigma(A_0))\} \subset \{m_f\colon f \in L^\infty(\sigma(A_0), \mu)\}.$$

Since Γ^* is SOT continuous we have

$$\Gamma^*(A) = \Gamma^*\left(\overline{A_0}^{SOT}\right) \subset \overline{\Gamma^*(A_0)}^{SOT} \subset \overline{L^\infty(\sigma(A_0), \mu)}^{SOT} = L^\infty(\sigma(A_0), \mu).$$

On the other hand, we claim that $\Gamma^*(A) \supset \overline{\Gamma^*(A_0)}^{WOT}$. Indeed, suppose $(\Gamma^*(x_\alpha)) \subset \Gamma^*(A_0)$ converges in the WOT to $T \in \mathcal{B}(L^2(\sigma(A_0), \mu))$. Then for all $\xi, \eta \in \mathcal{H}$ we have

$$\langle TU\xi, U\eta \rangle = \lim_\alpha \langle Ux_\alpha U^* U\xi, U\eta \rangle = \lim_\alpha \langle x_\alpha \xi, \eta \rangle .$$

Thus (x_α) converges in the WOT to $U^*TU \in \mathcal{B}(\mathcal{H})$. Since $A = \overline{A_0}^{W0T}$, $x := U^*TU \in A$ and clearly $\Gamma^*(x) = T$. So the claimed inclusion holds.

Since the WOT for $\{m_f : f \in L^\infty(\sigma(A_0), \mu)\}$ is simply the weak*-topology for $L^\infty(\sigma(A_0), \mu)$ (by Proposition 2.4.1), and since $C(\sigma(A_0))$ is weak* dense in $L^\infty(\sigma(A_0), \mu)$, we have

$$\{m_f : f \in L^\infty(\sigma(A_0), \mu)\} = \overline{\{m_f : f \in C(\sigma(A_0))\}}^{WOT} = \overline{\Gamma^*(A_0)}^{WOT} \subset \Gamma^*(A).$$

Thus $\Gamma^*(A) = \{m_f : f \in L^\infty(\sigma(A_0), \mu)\}$. □

2/1/2017

Remark 3.2.6. Observe that if we take $A_0 = A$ in the proof of the previous theorem, then it follows that

$$\{m_f : f \in L^\infty(\sigma(A), \mu)\} = \Gamma^*(A) = \{m_f : f \in C(\sigma(A))\}.$$

That is, the essentially bounded functions coincide with the continuous functions on $\sigma(A)$. This should be taken as an indication that the spectrum of a C^*-algebra A is strange when A is also a von Neumann algebra. See [4, Section III.1] (specifically Theorem III.1.18) for more information.

Consider now a normal operator $x \in \mathcal{B}(\mathcal{H})$. Between the Borel functional calculus and the above theorem, we have two different ways to associate operators to (essentially) bounded measurable functions. We will now demonstrate that these methods coincide.

Suppose there exists a cyclic vector $\xi_0 \in \mathcal{H}$ for $W^*(x)$. Let $C^*(x, 1)$ be the C^*-algebra generated by x and 1. Then $C^*(x, 1) \cong C(\sigma(x))$, and applying the theorem to $C^*(x, 1) \subset W^*(x)$ yields a regular Borel measure μ on $\sigma(x)$ and an isometric $*$-isomorphism

$$\Gamma^* : W^*(x) \to L^\infty(\sigma(x), \mu),$$

which extends the Gelfand transform $\Gamma : C^*(x, 1) \to C(\sigma(x))$. For $S \in \mathcal{B}_{\sigma(x)}$, define $E(S) := (\Gamma^*)^{-1}(\chi_S)$. Note that if $S, T \in \mathcal{B}_{\sigma(x)}$ differ by a μ-null set, then $E(S) = E(T)$, but this is not an issue. One can easily check that E is a projection-valued measure for $(\sigma(x), \mathcal{B}_{\sigma(x)}, \mathcal{H})$. We claim that

$$\int_{\sigma(x)} f \, dE = (\Gamma^*)^{-1}(f) \qquad f \in L^\infty(\sigma(x), \mu).$$

By definition of E, this holds for simple functions, and so by approximating $f \in L^\infty(\sigma(x), \mu)$ uniformly by simple functions we obtain the claimed equality. In particular, for $f \in C(\sigma(x))$ we have

$$\int_{\sigma(x)} f \, dE = \Gamma^{-1}(f).$$

Therefore the uniqueness of the projection-valued measure in Theorem 3.1.10 implies E is exactly the projection-valued measure which arises from the Borel functional calculus for x.

Observe that

$$E_{\xi_0,\xi_0}(S) = \langle E(S)\xi_0, \xi_0 \rangle = \left\langle (\Gamma^*)^{-1}(\chi_S)\xi_0, \xi_0 \right\rangle = \int_{\sigma(x)} \chi_S \, d\mu = \mu(S).$$

Thus $E_{\xi_0,\xi_0} = \mu$. Note also that the E-null sets are precisely the μ-null sets, and consequently $L^\infty(\sigma(x), E) = L^\infty(\sigma(x), \mu)$.

Example 3.2.7. For some $n \in \mathbb{N}$, $n \geq 2$, let

$$\Gamma := \mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}.$$

Since Γ is an abelian group, $L(\Gamma)$ is an abelian von Neumann algebra. From Example 3.2.2, we know δ_e is a cyclic vector for $L(\Gamma)$. Let $x \in L(\Gamma)$ be the unitary operator corresponding to the group generator $1 \in \mathbb{Z}/n\mathbb{Z}$. Thus $L(\Gamma) = W^*(x)$. Since $xx^* = 1 = x^*x$ (i.e. x is normal), from the above discussion we know

$$L(\Gamma) \cong L^\infty(\sigma(x), \mu)$$

for μ a regular Borel measure. One easily computes $\sigma(x) = \{\exp(\frac{2\pi ik}{n}) \colon k = 0, 1, \ldots, n-1\}$. Denote $\zeta_k = \exp(\frac{2\pi ik}{n})$, $k = 0, 1, \ldots, n-1$, then

$$e_k := \frac{1}{\sqrt{n}} \left(\delta_0 + \zeta_k^{-1}\delta_1 + \cdots + \zeta_k^{-(n-1)}\delta_{n-1} \right)$$

is a unit eigenvector of x with eigenvalue ζ_k. If E is the projection-valued measure given by Theorem 3.1.10, then $E(\{\zeta_k\})$ is clearly the projection onto the eigenspace spanned by e_k. From the discussion preceding this example, we know $\mu(\{\zeta_k\}) = E_{\delta_0,\delta_0}(\{\zeta_k\})$. Thus

$$\mu(\{\zeta_k\}) = \langle E(\{\zeta_k\})\delta_0, \delta_0 \rangle = \langle \langle \delta_0, e_k \rangle e_k, \delta_0 \rangle = |\langle e_k, \delta_0 \rangle|^2 = \frac{1}{n}.$$

Thus μ is the uniform probability distribution on $\{\zeta_k \colon k = 0, 1, \ldots, n-1\}$.

Example 3.2.8. Consider the abelian von Neumann algebra $L(\mathbb{Z})$. As in the previous example, $\delta_0 \in \ell^2(\mathbb{Z})$ is a cyclic vector for $L(\mathbb{Z})$. Let $x \in L(\mathbb{Z})$ be the unitary operator corresponding to $1 \in \mathbb{Z}$. Let

$$\mathbb{T} = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\},$$

then $\mathbb{Z}$ and $\mathbb{T}$ are Pontryagin duals to each other via

$$\mathbb{Z} \times \mathbb{T} \ni (n, \zeta) \mapsto \zeta^n.$$

This duality allows us to define a unitary $U \colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}, m)$ (where m is the normalized Lebesgue measure on $\mathbb{T}$) via

$$[U(\xi)](\zeta) = \sum_{n \in \mathbb{Z}} \xi(n)\zeta^n \qquad \xi \in \ell^2(\mathbb{Z}),\ \zeta \in \mathbb{T}.$$

It is then easy to check that $UxU^* = f$ where $f \colon L^\infty(\mathbb{T}, m)$ is the identity function $f(\zeta) = \zeta$. Thus by Proposition 2.4.1 and the weak*-density of the polynomials in $L^\infty(\mathbb{T}, m)$ we have

$$UL(\mathbb{Z})U^* = U\overline{\mathbb{C}\langle x \rangle}^{WOT} U^* = \overline{U\mathbb{C}\langle x \rangle U^*}^{WOT} = \overline{\mathbb{C}\langle f \rangle}^{WOT} = \overline{\mathbb{C}\langle f \rangle}^{wk*} = L^\infty(\mathbb{T}, m).$$

Thus $L(\mathbb{Z})$ is spatially isomorphic with $L^\infty(\mathbb{T}, m)$. 2/6/2017

Chapter 4

The Predual

4.1 The Polar Decomposition

Recall that for $x \in \mathcal{B}(\mathcal{H})$, $|x| = (x^*x)^{\frac{1}{2}}$.

Theorem 4.1.1. *Let $x \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry v so that $x = v|x|$ and $\ker(v) = \ker(x)$. This decomposition is unique in that if $x = wy$ for $y \geq 0$ and w a partial isometry with $\ker(w) = \ker(y)$, then $w = v$ and $y = |x|$. Moreover, $v \in W^*(x)$ and $I(v) = \overline{ran(x^*)}$ and $F(v) = \overline{ran(x)}$.*

Proof. Define $v_0 \colon \operatorname{ran}(|x|) \to \operatorname{ran}(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Since $\||x|\xi\| = \|x\xi\|$ for all $\xi \in \mathcal{H}$, v_0 is well-defined and can be extended to an isometry $v \colon \overline{\operatorname{ran}(|x|)} \to \overline{\operatorname{ran}(x)}$. Defining v to be zero on $\overline{\operatorname{ran}(|x|)}^\perp = \ker(|x|) = \ker(x)$, we obtain a partial isometry with $\ker(v) = \ker(x)$. By definition we have $v|x| = x$.

Suppose $x = wy$ for some $y \geq 0$ and w a partial isometry with $\ker(w) = \ker(y)$. Then $I(w) = \ker(y)^\perp = \overline{\operatorname{ran}(y)}$. Thus $w^*wy = y$, since w^*w is the projection onto $I(w)$. Consequently,

$$|x|^2 = x^*x = yw^*wy = y^2,$$

which implies $|x| = y$. Thus $\ker(w) = \ker(y) = \ker(|x|)$ and $w|x|\xi = wy\xi = x\xi$ for all $\xi \in \mathcal{H}$. That is, $w = v$.

To see that $v \in W^*(x)$, suppose $y \in W^*(x)'$. Then for all $\xi \in \mathcal{H}$ we have

$$yv|x|\xi = yx\xi = xy\xi = v|x|y\xi = vy|x|\xi,$$

so that $yv = vy$ on $\overline{\operatorname{ran}(|x|)}$. For $\xi \in \overline{\operatorname{ran}(|x|)}^\perp = \ker(|x|) = \ker(x)$, note that $y\xi \in \ker(x) = \ker(v)$. Thus $vy\xi = 0 = yv\xi$. Thus $v \in W^*(x)'' = W^*(x)$.

Finally, we have

$$I(v) = \ker(v)^\perp = \ker(x)^\perp = \overline{\operatorname{ran}(x^*)}.$$

Also, $F(v) = vI(v) = v(\ker(|x|))^\perp = v\overline{\operatorname{ran}(|x|)} = \overline{\operatorname{ran}(x)}$. □

Corollary 4.1.2. *Let $x \in \mathcal{B}(\mathcal{H})$ with polar decomposition $x = v|x|$. Then $x = |x^*|v$ and $x^* = v^*|x^*|$.*

Proof. By taking adjoints, it suffices to show $x = |x^*|v$. Towards that end, we claim that $v|x|v^* = |x^*|$. Indeed, observe

$$xx^* = v|x||x|v^* = vx^*xv^*.$$

Let (p_n) be a sequence of polynomials uniformly approximating $f(t) = \sqrt{t}$ on $\sigma(x^*x) \cup \sigma(xx^*) \subset [0, \infty)$. Then

$$v|x|v^* = \lim_{n\to\infty} vp_n(x^*x)v^* = \lim_{n\to\infty} p_n(xx^*) = |x^*|.$$

Thus $v|x|v^* = |x^*|$. Since v^*v is the projection onto

$$I(v) = \overline{\operatorname{ran}(x^*)} = \ker(x)^\perp = \ker(|x|)^\perp = \overline{\operatorname{ran}(|x|)},$$

we have $v^*v|x| = |x|$, and taking adjoints $|x|v^*v = |x|$. Thus $|x^*|v = v|x|v^*v = v|x| = x$. □

4.2 Trace Class Operators

Fix an orthonormal basis $\{\xi_i\}$ for a Hilbert space $\mathcal{H}$. For a positive operator $x \in \mathcal{B}(\mathcal{H})$, define

$$\mathrm{Tr}(x) = \sum_i \langle x\xi_i, \xi_i \rangle ,$$

which we note is potentially infinite. We will see (*cf.* Corollary 4.2.4) that this value does not actually depend on the chosen basis.

Definition 4.2.1. We say $x \in \mathcal{B}(\mathcal{H})$ is **trace class** if

$$\|x\|_1 := \mathrm{Tr}(|x|) < \infty.$$

The set of trace class operators is denoted $L^1(\mathcal{B}(\mathcal{H}))$.

Example 4.2.2. For each $n \in \mathbb{N}$, let $p_n \in \mathcal{B}(\ell^2(\mathbb{N}))$ be the projection onto δ_n. Let (c_n) be a sequence of positive numbers. Then

$$\sum_{n\in\mathbb{N}} c_n p_n$$

is trace class if and only if $(c_n) \in \ell^1(\mathbb{N})$.

Lemma 4.2.3. *If $x \in \mathcal{B}(\mathcal{H})$, then $Tr(x^*x) = Tr(xx^*)$.*

Proof. By Parseval's identity we have for each i,

$$\langle x^*x\xi_i, \xi_i\rangle = \|x\xi_i\|^2 = \sum_j |\langle x\xi_i, \xi_j\rangle|^2 = \sum_j |\langle x^*\xi_j, \xi_i\rangle|^2.$$

Then by Fubini's theorem we have

$$\mathrm{Tr}(x^*x) = \sum_i \langle x^*x\xi_i, \xi_i\rangle = \sum_i\sum_j |\langle x^*\xi_j, \xi_i\rangle|^2 = \sum_j\sum_i |\langle x^*\xi_j, \xi_i\rangle|^2 = \sum_j \langle xx^*\xi_j, \xi_j\rangle = \mathrm{Tr}(xx^*),$$

as claimed. □

Corollary 4.2.4. *If $x \in \mathcal{B}(\mathcal{H})$ is a positive oeprator and $u \in \mathcal{B}(\mathcal{H})$ is a unitary, then*

$$Tr(u^*xu) = Tr(x).$$

In particular, the trace is independent of the chosen orthonormal basis.

Proof. Write $x = y^*y$. Then by the previous lemma we have

$$\mathrm{Tr}(x) = \mathrm{Tr}(y^*y) = \mathrm{Tr}(yy^*) = \mathrm{Tr}(yuu^*y^*) = \mathrm{Tr}(u^*y^*yu) = \mathrm{Tr}(u^*xu),$$

as claimed. □ 2/10/2017

Example 4.2.5. Let $p \in \mathcal{B}(\mathcal{H})$ be a projection onto a closed subspace $\mathcal{K} \subset \mathcal{H}$. Let $\{\xi_i\}$ and $\{\eta_j\}$ be orthonormal bases for $\mathcal{K}$ and $\mathcal{K}^\perp$, respectively. Then

$$\mathrm{Tr}(p) = \sum_i \langle p\xi_i, \xi_i\rangle + \sum_j \langle p\eta_j, \eta_j\rangle = \sum_i \|\xi_i\|^2 = \dim(\mathcal{K}).$$

Thus p is trace class if and only if $\mathcal{K}$ is finite dimensional.

We next wish to extend Tr to not necessarily positive elements of $L^1(\mathcal{B}(\mathcal{H}))$. In order to see that this is well-defined, we have the following lemma.

Lemma 4.2.6. *Let $x \in \mathcal{B}(\mathcal{H})$ with polar decomposition $x = v|x|$. Then*

$$|\langle x\xi, \xi\rangle| \leq \frac{1}{2}\langle |x|\xi, \xi\rangle + \frac{1}{2}\langle |x|v^*\xi, v^*\xi\rangle \qquad \forall \xi \in \mathcal{H}.$$

Proof. Fix $\xi \in \mathcal{H}$ and let $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$. Then

$$\begin{aligned} 0 \leq \||x|^{1/2}\xi - \lambda|x|^{1/2}v^*\xi\|^2 &= \||x|^{1/2}\xi\|^2 - 2\text{Re}\ \left(\bar{\lambda}\left\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi\right\rangle\right) + \||x|^{1/2}v^*\xi\|^2 \\ &= \langle |x|\xi, \xi\rangle - 2\text{Re}\ \left(\bar{\lambda}\langle v|x|\xi, \xi\rangle\right) + \langle |x|v^*\xi, v^*\xi\rangle. \end{aligned}$$

Since $v|x| = x$, if we choose

$$\lambda = \frac{\langle x\xi, \xi\rangle}{|\langle x\xi, \xi\rangle|},$$

then the result follows. ∎

Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ with polar decomposition $x = v|x|$. From Lemma 4.2.3, we see that

$$\text{Tr}(v|x|v^*) = \text{Tr}(|x|^{1/2}v^*v|x|^{1/2}) = \text{Tr}(|x|) = \|x\|_1, \tag{4.1}$$

since v^*v is the projection onto $\overline{\text{ran}(|x|)}$. Thus

$$\sum_i \left(\frac{1}{2}\langle |x|\xi_i, \xi_i\rangle + \frac{1}{2}\langle |x|v^*\xi_i, v^*\xi_i\rangle\right) = \frac{1}{2}\text{Tr}(|x|) + \frac{1}{2}\text{Tr}(v|x|v^*) = \|x\|_1 < \infty.$$

This and the previous lemma imply that the series $\sum \langle x\xi_i, \xi_i\rangle$ is absolutely convergent.

Definition 4.2.7. Fix an orthonormal basis $\{\xi_i\}$ for $\mathcal{H}$. For $x \in L^1(\mathcal{B}(\mathcal{H}))$, we define the **trace of** x as

$$\text{Tr}(x) := \sum_i \langle x\xi, \xi\rangle.$$

From the discussion preceding the definition, we immediately obtain

$$|\text{Tr}(x)| \leq \|x\|_1$$

for all $x \in L^1(\mathcal{B}(\mathcal{H}))$.

Theorem 4.2.8. *On $L^1(\mathcal{B}(\mathcal{H}))$, $\|\cdot\|_1$ is a norm satisfying*

$$\|x\| \leq \|x\|_1.$$

$L^1(\mathcal{B}(\mathcal{H}))$ is a self-adjoint ideal in $\mathcal{B}(\mathcal{H})$ such that $\|x^*\|_1 = \|x\|_1$ and

$$\|axb\|_1 \leq \|a\|\|b\|\|x\|_1 \qquad a, b \in \mathcal{B}(\mathcal{H}),\ x \in L^1(\mathcal{B}(\mathcal{H})).$$

$L^1(\mathcal{B}(\mathcal{H}))$ is spanned by positive operators with finite trace and consequently the trace on $L^1(\mathcal{B}(\mathcal{H}))$ is independent of the chosen basis. Furthermore,

$$Tr(ax) = Tr(xa) \qquad a \in \mathcal{B}(\mathcal{H}),\ x \in L^1(\mathcal{B}(\mathcal{H})).$$

Proof. For $x, y \in L^1(\mathcal{B}(\mathcal{H}))$ let $x + y = w|x + y|$ be the polar decomposition. Note that

$$|w^*x|^2 = x^*ww^*x \leq \|ww^*\||x|^2 \leq |x|^2.$$

Consequently $w^*x \in L^1(\mathcal{B}(\mathcal{H}))$, and similarly $w^*y \in L^1(\mathcal{B}(\mathcal{H}))$. Also note that this implies $\|w^*x\|_1 \leq \|x\|_1$ and $\|w^*y\|_1 \leq \|y\|$. Thus

$$\text{Tr}(|x + y|) = \sum_i \langle |x + y|\xi_i, \xi_i\rangle = \sum_i \langle w^*(x + y)\xi, \xi\rangle = \text{Tr}(w^*x) + \text{Tr}(w^*y) < \infty.$$

Thus $x + y \in L^1(\mathcal{B}(\mathcal{H}))$ with

$$\|x + y\|_1 \leq \operatorname{Tr}(w^*x) + \operatorname{Tr}(w^*y) \leq \|w^*x\|_1 + \|w^*y\|_1 \leq \|x\|_1 + \|y\|_1.$$

So $L^1(\mathcal{B}(\mathcal{H}))$ is a subspace of $\mathcal{B}(\mathcal{H})$ on which, $\|\cdot\|_1$ is a norm. Since

$$\|x\| = \||x|\| = \sup_{\|\xi\|=1} \langle |x|\xi, \xi \rangle ,$$

and since $\|x\|_1 = \operatorname{Tr}(|x|)$ is independent of the chosen basis, we obtain $\|x\| \leq \|x\|_1$.

Recall from the proof of Corollary 4.1.2 that $v|x|v^* = |x^*|$. Using (4.1) we have

$$\operatorname{Tr}(|x^*|) = \operatorname{Tr}(v|x|v^*) = \|x\|_1.$$

Thus $x^* \in L^1(\mathcal{B}(\mathcal{H}))$ with $\|x^*\|_1 = \|x\|_1$.

Let $a \in \mathcal{B}(\mathcal{H})$ and $x \in L^1(\mathcal{B}(\mathcal{H}))$. Since

$$|ax| \leq \|a\||x|$$

one sees that

$$\operatorname{Tr}(|ax|) \leq \|a\|\operatorname{Tr}(|x|) < \infty.$$

Thus $ax \in L^1(\mathcal{B}(\mathcal{H}))$ with $\|ax\|_1 \leq \|a\|\|x\|_1$, so that $L^1(\mathcal{B}(\mathcal{H}))$ is a left-ideal. Since it is self-adjoint, it is also a right-ideal. In particular, for $b \in \mathcal{B}(\mathcal{H})$ we have

$$\|xb\|_1 = \|b^*x^*\|_1 \leq \|b\|\|x^*\|_1 = \|b\|\|x\|_1.$$

Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ with polar decomposition $x = v|x|$. Then

$$x = v|x| = \frac{1}{4}\sum_{k=0}^{3} i^k (v + i^k)|x|(v + i^k)^*.$$

Observe that

$$\operatorname{Tr}\left((v + i^k)|x|(v + i^k)^*\right) = \operatorname{Tr}\left(|x|^{1/2}|v + i^k|^2|x|^{1/2}\right) \leq \|v + i^k\|^2 \operatorname{Tr}(|x|) < \infty.$$

Thus x is a linear combination of positive elements with finite trace, and consequently,

$$\operatorname{Tr}(x) = \frac{1}{4}\sum_{k=0}^{3} i^k \operatorname{Tr}\left((v + i^k)|x|(v + i^k)^*\right).$$

The trace is therefore independent of the chosen basis since it is independent on positive elements.

Let $u \in \mathcal{B}(\mathcal{H})$ be an unitary operator. Since Tr is independent of the chosen basis, we have

$$\operatorname{Tr}(xu) = \sum_i \langle xu\xi_i, \xi_i \rangle = \sum_i \langle ux(u\xi_i), u\xi_i \rangle = \operatorname{Tr}(ux).$$

For $a \in \mathcal{B}(\mathcal{H})$, using the continuous functional calculus we can write a as a linear combination of four unitaries and hence the above implies $\operatorname{Tr}(xa) = \operatorname{Tr}(ax)$. □

Proposition 4.2.9. *$L^1(\mathcal{B}(\mathcal{H}))$ endowed with the norm $\|\cdot\|_1$ is a Banach space.*

Proof. We need only show that $L^1(\mathcal{B}(\mathcal{H}))$ is complete. Suppose $(x_n) \subset L^1(\mathcal{B}(\mathcal{H}))$ is a Cauchy sequence with respect to $\|\cdot\|_1$. As this norm dominates the operator norm, we see that (x_n) converges in operator norm to some $x \in \mathcal{B}(\mathcal{H})$. Consequently, $(|x_n|)$ converges in operator norm to $|x|$. Thus, for any finite, orthonormal set $\{\eta_1, \ldots, \eta_k\}$ we have

$$\sum_{i=1}^{k} \langle |x|\eta_i, \eta_i \rangle = \lim_{n\to\infty} \sum_{i=1}^{k} \langle |x_n|\eta_i, \eta_i \rangle \leq \lim_{n\to\infty} \|x_n\|_1 < \infty.$$

Thus $x \in L^1(\mathcal{B}(\mathcal{H}))$.

Now, we need to show that x is the $\|\cdot\|_1$-norm limit of (x_n). Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $\|x_n - x_N\|_1 < \frac{\epsilon}{3}$ for all $n \geq N$. There exists a finite dimensional subspace $\mathcal{K} \subset \mathcal{H}$ so that if p is the projection from $\mathcal{H}$ to $\mathcal{K}$ then $\|x_N(1-p)\|_1$ and $\|x(1-p)\|_1$ are less than $\frac{\epsilon}{3}$. Then for $n \geq N$ we have

$$\|x - x_n\|_1 \leq \|(x - x_n)p\|_1 + \|(x - x_N)(1-p)\|_1 + \|(x_N - x_n)(1-p)\|_1 \leq \|(x - x_n)p\|_1 + \epsilon.$$

Since $\mathcal{K}$ is finite dimensional, $\|x - x_n\| \to 0$ implies $\|(x - x_n)p\|_1 \to 0$. Since $\epsilon > 0$ was arbitrary, we see that $\|x - x_n\|_1 \to 0$. $\square$

Fix $a \in \mathcal{B}(\mathcal{H})$. Then for any $x \in L^1(\mathcal{B}(\mathcal{H}))$ we have

$$|\mathrm{Tr}(ax)| \leq \|ax\|_1 \leq \|a\|\|x\|_1.$$

Thus the map

$$\psi_a : L^1(\mathcal{B}(\mathcal{H})) \ni x \mapsto \mathrm{Tr}(ax) \in \mathbb{C} \tag{4.2}$$

is a continuous linear functional on $L^1(\mathcal{B}(\mathcal{H}))$ with $\|\psi_a\| \leq \|a\|$. That is, $\psi_a \in L^1(\mathcal{B}(\mathcal{H}))^*$. We will see that $\mathcal{B}(\mathcal{H}) \cong L^1(\mathcal{B}(\mathcal{H}))^*$, but we must first discuss a dense subclass of trace class operators. 2/13/2017

4.2.1 Finite Rank Operators

Definition 4.2.10. We say $x \in \mathcal{B}(\mathcal{H})$ is a **finite rank operator** if $\overline{\mathrm{ran}(x)}$ is finite dimensional. In this case, the **rank of** x is $\dim(\overline{\mathrm{ran}(x)})$. The set of finite rank operators is denoted $\mathcal{FR}(\mathcal{H})$.

Observe that if $x \in \mathcal{FR}(\mathcal{H})$, then

$$\ker(x)^\perp = \overline{\mathrm{ran}(x^*)} = \overline{\mathrm{ran}(x^* \mid_{\ker(x*)^\perp})} = \overline{\mathrm{ran}(x^* \mid_{\overline{\mathrm{ran}(x)}})}$$

is finite dimensional as the image of a finite dimensional subspace. Letting p and q be the projections onto the finite dimensional subspaces $\overline{\mathrm{ran}(x)}$ and $\ker(x)^\perp$, respectively, we have that $pxq = x$. Moreover, this implies $x^* \in \mathcal{FR}(\mathcal{H})$.

Let $\overline{\mathcal{H}}$ denote the Banach space dual to $\mathcal{H}$. Then $\mathcal{H}$ and $\overline{\mathcal{H}}$ are naturally anti-isomorphic, and we denote this isomorphism by $\xi \mapsto \bar{\xi}$. Hence $\overline{z\xi} = \bar{z}\bar{\xi}$ for $z \in \mathbb{C}$ and

$$\langle \bar{\xi}, \bar{\eta} \rangle = \langle \eta, \xi \rangle \qquad \xi, \eta \in \mathcal{H}.$$

Given $\xi, \eta \in \mathcal{H}$, define $\xi \otimes \bar{\eta} \in \mathcal{FR}(\mathcal{H})$ by

$$(\xi \otimes \bar{\eta})(\zeta) = \langle \zeta, \eta \rangle \, \xi.$$

Since $x \in \mathcal{FR}(\mathcal{H})$ satisfies $pxq = x$ for finite rank projections p and q, it is easily checked that $\mathcal{FR}(\mathcal{H}) = \mathrm{span}\{\xi \otimes \bar{\eta} \colon \xi, \eta \in \mathcal{H}\}$. It is also easy to check that $\xi \otimes \bar{\eta}$ is a rank one operator with

$$\mathrm{Tr}(\xi \otimes \bar{\eta}) = \langle \xi, \eta \rangle$$

and

$$\|\xi \otimes \bar{\eta}\| \leq \|\xi\|\|\eta\|,$$

that $(\xi \otimes \bar{\eta})^* = \eta \otimes \bar{\xi}$, and that $x(\xi \otimes \bar{\eta})y = (x\xi) \otimes (\overline{y^*\eta})$ for $x, y \in \mathcal{B}(\mathcal{H})$.

Proposition 4.2.11. *$\mathcal{FR}(\mathcal{H})$ is a dense subset of $L^1(\mathcal{B}(\mathcal{H}))$.*

Proof. It is easily checked that $\mathcal{FR}(\mathcal{H}) \subset L^1(\mathcal{B}(\mathcal{H}))$. Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ and let $\epsilon > 0$. Then there exists a finite dimensional subspace $\mathcal{K} \subset \mathcal{H}$ so that if p is the projection onto $\mathcal{K}$, then

$$\|x^*(1-p)\|_1 < \epsilon.$$

Then $px \in \mathcal{FR}(\mathcal{H})$ and

$$\|x - px\|_1 = \|(1-p)x\|_1 = \|x^*(1-p)\|_1 < \epsilon.$$

Thus $\mathcal{FR}(\mathcal{H})$ is dense in $L^1(\mathcal{B}(\mathcal{H}))$. $\square$

4.2.2 The Predual of $\mathcal{B}(\mathcal{H})$

Theorem 4.2.12. *The map*

$$\psi\colon \mathcal{B}(\mathcal{H}) \ni a \mapsto \psi_a \in L^1(\mathcal{B}(\mathcal{H}))^*,$$

with ψ_a defined by (4.2), is a Banach space isomorphism.

Proof. For $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$, the map

$$\mathcal{H} \times \mathcal{H} \ni (\xi, \eta) \mapsto \varphi(\xi \otimes \bar{\eta})$$

defines a bounded sesquilinear form on $\mathcal{H}$. Thus there exists $a \in \mathcal{B}(\mathcal{H})$ so that

$$\varphi(\xi \otimes \bar{\eta}) = \langle a\xi, \eta\rangle = \mathrm{Tr}((a\xi) \otimes \bar{\eta}) = \mathrm{Tr}(a(\xi \otimes \bar{\eta})) = \psi_a(\xi \otimes \bar{\eta}).$$

Thus $\varphi = \psi_a$ on $\mathcal{FR}(\mathcal{H})$, and so the density of the finite rank operators implies $\varphi = \psi_a$ on $L^1(\mathcal{B}(\mathcal{H}))$. Thus ψ is onto.

For $a \in \mathcal{B}(\mathcal{H})$ we have

$$\|a\| = \sup_{\|\xi\|=\|\eta\|=1} |\langle a\xi, \eta\rangle| = \sup_{\|\xi\|=\|\eta\|=1} |\psi_a(\xi \otimes \bar{\eta})| \leq \|\psi_a\|.$$

Since we have already seen that ψ is a contraction, this implies it is isometric. In particular is is injective. □

Remark 4.2.13. Note that for $a \in L^1(\mathcal{B}(\mathcal{H}))$, $\psi_a \in \mathcal{B}(\mathcal{H})^*$ and we claim $\|\psi_a\| = \|a\|_1$. Indeed, if $a = v|a|$ is a the polar decomposition then

$$\|a\|_1 = \mathrm{Tr}(|a|) = \mathrm{Tr}(v^*v|a|) = \mathrm{Tr}(v^*a) \leq \|\psi_a\|\|v^*\| = \|\psi_a\|.$$

On the other hand, for $x \in \mathcal{B}(\mathcal{H})$ we have

$$|\psi_a(x)| = |\mathrm{Tr}(xa)| \leq \|xa\|_1 \leq \|x\|\|a\|_1,$$

so that $\|\psi_a\| \leq \|a\|_1$. Thus $\|a\|_1 = \|\psi_a\|$. Since $(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$ is complete, it follows that $\{\psi_a\colon a \in L^1(\mathcal{B}(\mathcal{H}))\}$ is a closed subspace of $\mathcal{B}(\mathcal{H})^*$.

Remark 4.2.14. So far we have considered three topologies on $\mathcal{B}(\mathcal{H})$: the uniform (operator norm) topology, the SOT, and the WOT. Viewing $\mathcal{B}(\mathcal{H})$ as the dual space of $L^1(\mathcal{B}(\mathcal{H}))$, we can consider the weak* topology on $\mathcal{B}(\mathcal{H})$: $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ converges weak* to $x \in \mathcal{B}(\mathcal{H})$ if

$$\mathrm{Tr}(xy) = \lim_\alpha \mathrm{Tr}(x_\alpha y) \qquad \forall y \in L^1(\mathcal{B}(\mathcal{H})).$$

If $y \in \mathcal{FR}(\mathcal{H})$ with $\{\xi_1, \ldots, \xi_n\}$ an orthonormal basis for $\ker(y)^\perp$, then

$$\mathrm{Tr}(xy) = \sum_{i=1}^n \langle xy\xi_i, \xi_i\rangle .$$

Thus if (x_α) converges in the WOT to x, then $\mathrm{Tr}(x) = \lim \mathrm{Tr}(x_\alpha y)$. However, this need not be true for arbitrary $y \in L^1(\mathcal{B}(\mathcal{H}))$.

4.3 Hilbert–Schmidt Operators

Definition 4.3.1. We say $x \in \mathcal{B}(\mathcal{H})$ is a **Hilbert–Schmidt operator** if

$$\|x\|_2 := \sqrt{\mathrm{Tr}(|x|^2)} < \infty;$$

that is, if $|x|^2 \in L^1(\mathcal{B}(\mathcal{H}))$. We denote by $L^2(\mathcal{B}(\mathcal{H}))$ the set of Hilbert–Schmidt operators.

Lemma 4.3.2. *$L^2(\mathcal{B}(\mathcal{H}))$ is a self-adjoint ideal in $\mathcal{B}(\mathcal{H})$ such that $\|x^*\|_2 = \|x\|_2$ and*

$$\|axb\|_2 \leq \|a\|\|b\|\|x\|_2 \qquad a, b \in \mathcal{B}(\mathcal{H}),\ x \in L^2(\mathcal{B}(\mathcal{H})).$$

For $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, we have $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$ with

$$Tr(xy) = Tr(yx),$$

Proof. First note that $L^2(\mathcal{B}(\mathcal{H}))$ is a subspace since

$$|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2).$$

Since $|ax|^2 \leq \|a\|^2|x|^2$ for $a \in \mathcal{B}(\mathcal{H})$, we see that $\|ax\|_2 \leq \|a\|\|x\|_2$ and $L^2(\mathcal{B}(\mathcal{H}))$ is an ideal. Since $|x^*|^2 = v|x|^2v^*$ where $x = v|x|$ is the polar decomposition, we have

$$\mathrm{Tr}(|x^*|^2) = \mathrm{Tr}(v|x|^2v^*) = \mathrm{Tr}(v^*v|x|^2) = \mathrm{Tr}(|x|^2),$$

Hence $x^* \in L^2(\mathcal{B}(\mathcal{H}))$ with $\|x^*\|_2 = \|x\|_2$. Thus $L^2(\mathcal{B}(\mathcal{H}))$ is a self-adjoint ideal.

Now, for $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ we have

$$yx = \frac{1}{4}\sum_{k=0}^{3} i^k|x + i^k y^*|^2$$

by the polarization identity. Consequently $yx \in L^1(\mathcal{B}(\mathcal{H}))$, and by our previous observation

$$\mathrm{Tr}(yx) = \frac{1}{4}\sum_{k=0}^{3} i^k\|x + i^k y^*\|_2 = \frac{1}{4}\sum_{k=0}^{3} i^k\|x^* - i^k y\|_2 = \mathrm{Tr}\left(\frac{1}{4}\sum_{k=0}^{3} i^k|y + i^k x^*|^2\right) = \mathrm{Tr}(xy).$$

□

The previous lemma allows us to define a sesquilinear form on $L^2(\mathcal{B}(\mathcal{H}))$:

$$\langle x, y\rangle_2 := \mathrm{Tr}(y^*x) \qquad x, y \in L^2(\mathcal{B}(\mathcal{H})).$$

Note that $\langle x, x\rangle_2 = \|x\|_2^2$.

Proposition 4.3.3. *On $L^2(\mathcal{B}(\mathcal{H}))$, $\|\cdot\|_2$ is a norm satisfying*

$$\|x\| \leq \|x\|_2 \leq \|x\|_1,$$

and $L^2(\mathcal{B}(\mathcal{H}))$ is a Banach space equipped with the norm $\|\cdot\|_2$.

Proof. Since $\|\cdot\|_2$ is given by a sesquilinear form, standard arguments imply it satisfies the triangle inequality and hence is a norm. That $L^2(\mathcal{B}(\mathcal{H}))$ is complete and hence a Banach space follows by the same arguments which showed $L^1(\mathcal{B}(\mathcal{H}))$ is complete.

Letting $\{\xi_i\}$ be an orthonormal basis for $\mathcal{H}$, note that

$$\|x\|_2 = \mathrm{Tr}(|x|^2) = \sum_i \langle x^*x\xi, \xi\rangle = \sum_i \|x\xi_i\|^2.$$

Thus $\|x\| \leq \|x\|_2$ is clear. If $\|x\|_1 = \infty$, the last inequality is immediate. Otherwise $x \in L^2(\mathcal{B}(\mathcal{H})) \cap L^1(\mathcal{B}(\mathcal{H}))$ and we have

$$\|x\|_2 = \left|\mathrm{Tr}\left(\frac{x^*}{\|x\|_2}x\right)\right| \leq \left\|\frac{x^*}{\|x\|_2}x\right\|_1 \leq \left\|\frac{x^*}{\|x\|_2}\right\|\|x\|_1 \leq \|x\|_1,$$

where we have used $\|x^*\| = \|x\| \leq \|x\|_2$ in the last inequality. □ 2/15/2017

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Any $x \in \mathcal{B}(\mathcal{H},\mathcal{K})$ can be extended to $\tilde{x} \in \mathcal{B}(\mathcal{H}\oplus\mathcal{K})$ by

$$\tilde{x}(\xi,\eta) := (0, x\xi).$$

That is,

$$\tilde{x} = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}.$$

Definition 4.3.4. The **Hilbert–Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$**, denoted $\mathrm{HS}(\mathcal{H},\mathcal{K})$, are the $x \in \mathcal{B}(\mathcal{H},\mathcal{K})$ so that $\tilde{x} \in L^2(\mathcal{B}(\mathcal{H}\oplus\mathcal{K}))$. We denote $\mathrm{HS}(\mathcal{H}) := \mathrm{HS}(\mathcal{H},\mathcal{H})$.

Let $\{\xi_i\}$ and $\{\eta_j\}$ be orthonormal basis for $\mathcal{H}$ and $\mathcal{K}$, respectively. Then $\{(\xi_i,0),(0,\eta_j)\}$ is an orthonormal basis for $\mathcal{H}\oplus\mathcal{K}$. Thus for $x \in \mathcal{B}(\mathcal{H},\mathcal{K})$ we have

$$\|\tilde{x}\|_2 = \sqrt{\sum_{i,j} \|\tilde{x}(\xi_i,0)\|^2 + \|\tilde{x}(0,\eta_j)\|^2} = \sqrt{\sum_i \|x\xi_i\|^2}.$$

In particular, if $\mathcal{K} = \mathcal{H}$ then for $x \in \mathcal{B}(\mathcal{H})$ we have $\|\tilde{x}\|_2 = \|x\|_2$. Thus $\mathrm{HS}(\mathcal{H}) = L^2(\mathcal{B}(\mathcal{H}))$.

For $x, y \in \mathrm{HS}(\mathcal{H},\mathcal{K})$ we define a sesquilinear form by

$$\langle x, y\rangle_{\mathrm{HS}} := \langle \tilde{x}, \tilde{y}\rangle_2 .$$

By checking that the image of $\mathrm{HS}(\mathcal{H},\mathcal{K})$ under $x \mapsto \tilde{x}$ is a closed subspace of $L^2(\mathcal{B}(\mathcal{H}\oplus\mathcal{K}))$, we see that $\mathrm{HS}(\mathcal{H},\mathcal{K})$ is complete with respect to the norm induced by this sesquilinear form. Hence it is a Hilbert space.

Lemma 4.3.5. *$HS(\mathcal{H},\mathcal{K})$ is an algebraic $\mathcal{B}(\mathcal{K})$-$\mathcal{B}(\mathcal{H})$ bimodule such that*

$$\|axb\|_{HS} \leq \|a\|\|b\|\|x\|_{HS} \qquad a\mathcal{B}(\mathcal{K}),\ b \in \mathcal{B}(\mathcal{H}),\ x \in HS(\mathcal{H},\mathcal{K}).$$

For $x \in HS(\mathcal{H},\mathcal{K})$, $x^* \in HS(\mathcal{K},\mathcal{H})$ with $\|x^*\|_{HS} = \|x\|_{HS}$. Also, for $x, y \in HS(\mathcal{H},\mathcal{K})$ we have $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ and $xy^* \in L^1(\mathcal{B}(\mathcal{K}))$ with

$$\langle x, y\rangle_{HS} = Tr(y^*x) = Tr(xy^*).$$

Proof. For $x \in \mathrm{HS}(\mathcal{H},\mathcal{K})$, $a \in \mathcal{B}(\mathcal{K})$, and $b \in \mathcal{B}(\mathcal{H})$ we have

$$\widetilde{axb} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$$

Thus by Lemma 4.3.2 we have

$$\|axb\|_{\mathrm{HS}} = \|\widetilde{axb}\|_2 \leq \left\|\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right\|\left\|\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right\| \|\tilde{x}\|_2 = \|a\|\|b\|\|x\|_{\mathrm{HS}}.$$

Next we have

$$\widetilde{x^*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\tilde{x}^*\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $\|\tilde{x^*}\|_2 = \|\tilde{x}^*\|_2 = \|\tilde{x}\|_2 < \infty$. Thus $x^* \in \mathrm{HS}(\mathcal{K},\mathcal{H})$ with $\|x^*\|_{\mathrm{HS}} = \|x\|_{\mathrm{HS}}$.

Now, for $x \in \mathrm{HS}(\mathcal{H},\mathcal{K})$, we have $x^*x \in \mathcal{B}(\mathcal{H})$ and

$$\mathrm{Tr}(x^*x) = \sum_i \langle x^*x\xi_i, \xi_i\rangle = \sum_i \langle x\xi_i, x\xi_i\rangle = \sum_i \langle \tilde{x}(\xi,0), \tilde{x}(\xi_i,0)\rangle = \|\tilde{x}\|_2^2 < \infty.$$

Thus $x^*x \in L^1(\mathcal{B}(\mathcal{H}))$ with $\mathrm{Tr}(x^*x) = \|x\|_{\mathrm{HS}}^2$, and consequently

$$\mathrm{Tr}(x^*x) = \|x\|_{\mathrm{HS}}^2 = \|x^*\|_{\mathrm{HS}}^2 = \mathrm{Tr}(xx^*).$$

So, from the polarization identity it follows that $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ and $xy^* \in L^1(\mathcal{B}(\mathcal{K}))$ with

$$\langle x, y\rangle_{\mathrm{HS}} = \mathrm{Tr}(y^*x) = \mathrm{Tr}(xy^*)$$

for any $y \in \mathrm{HS}(\mathcal{H},\mathcal{K})$. ∎

Theorem 4.3.6. *As Hilbert spaces, $HS(\mathcal{H},\mathcal{K}) \cong \mathcal{K}\otimes\overline{\mathcal{H}}$. In particular, $L^2(\mathcal{B}(\mathcal{H})) \cong \mathcal{H}\otimes\overline{\mathcal{H}}$.*

Proof. Recall that $\mathcal{K}\otimes\overline{\mathcal{H}}$ is the completion of $\mathcal{K}\odot\overline{\mathcal{H}} := \operatorname{span}(\eta\otimes\overline{\xi}\colon \eta\in\mathcal{K},\ \overline{\xi}\in\overline{\mathcal{H}}\}$ under the norm induced by the inner product

$$\left\langle \sum_i \eta_i\otimes\overline{\xi}_i, \sum_j \zeta_j\otimes\overline{\gamma}_j \right\rangle = \sum_{i,j} \langle \eta_i,\zeta_j\rangle \langle \overline{\xi}_i,\overline{\gamma}_j\rangle .$$

For $\eta\otimes\overline{\xi}\in\mathcal{K}\odot\overline{\mathcal{H}}$, define $x_{\eta,\overline{\xi}}\in\mathcal{B}(\mathcal{H},\mathcal{K})$ by

$$x_{\eta,\overline{\xi}}(\zeta) := \langle \zeta,\xi\rangle\,\eta \qquad \zeta\in\mathcal{H}.$$

Then $x_{\xi,\eta}\in \mathrm{HS}(\mathcal{H},\mathcal{K})$ and for $\zeta\otimes\overline{\gamma}\in\mathcal{K}\odot\overline{\mathcal{H}}$ we have (for $\{\xi_i\}$ an orthonormal basis for $\mathcal{H}$)

$$\begin{aligned}\left\langle x_{\eta,\overline{\xi}}, y_{\zeta,\overline{\gamma}}\right\rangle_{\mathrm{HS}} &= \left\langle \widetilde{x_{\eta,\overline{\xi}}}, \widetilde{y_{\zeta,\overline{\gamma}}}\right\rangle_2 = \sum_i \left\langle x_{\eta,\overline{\xi}}\xi_i, y_{\zeta,\overline{\gamma}}\xi_i\right\rangle_2 = \sum_i \langle \xi_i,\xi\rangle\langle\eta,\zeta\rangle\langle\gamma,\xi_i\rangle\\ &= \langle\gamma,\xi\rangle\langle\eta,\zeta\rangle = \langle\overline{\xi},\overline{\gamma}\rangle\langle\eta,\zeta\rangle = \langle \eta\otimes\overline{\xi},\zeta\otimes\overline{\gamma}\rangle .\end{aligned}$$

Thus the linear extension of $\eta\otimes\overline{\xi}\mapsto x_{\eta,\overline{\xi}}\in\mathrm{HS}(\mathcal{H},\mathcal{K})$ to $\mathcal{K}\odot\overline{\mathcal{H}}$ can be further extended to an isomorphism $\mathcal{K}\otimes\overline{\mathcal{H}}\cong\mathrm{HS}(\mathcal{H},\mathcal{K})$. Since $L^2(\mathcal{B}(\mathcal{H})) = \mathrm{HS}(\mathcal{H})$, the isomorphism with $\mathcal{H}\otimes\overline{\mathcal{H}}$ follows. □

Recall that for $\xi,\eta\mathcal{H}$ we had defined $\xi\otimes\overline{\eta}\in\mathcal{FR}(\mathcal{H})$ by

$$\xi\otimes\overline{\eta}(\zeta) = \langle\zeta,\eta\rangle\,\xi \qquad \zeta\in\mathcal{H}.$$

That is, $\xi\otimes\overline{\eta} = x_{\xi,\overline{\eta}}$ from the proof of the above theorem.

Remark 4.3.7. For $a,b\in\mathrm{HS}(\mathcal{H},\mathcal{K})$, let $\xi,\eta\in\mathcal{K}\otimes\overline{\mathcal{H}}$ be the corresponding vectors from the isomorphism in Theorem 4.3.6. Then for $x\in\mathcal{B}(\mathcal{K})$ one can check that

$$\langle xa,b\rangle_{\mathrm{HS}} = \langle (x\otimes 1)\xi,\eta\rangle .$$

4.4 The σ-Topologies

In this section we introduce two new topologies in $\mathcal{B}(\mathcal{H})$, one of which will coincide with the weak* topology (*cf.* Remark 4.2.14).

Definition 4.4.1. Let $(x_\alpha)\subset\mathcal{B}(\mathcal{H})$ be a net of bounded operators, and let $x\in\mathcal{B}(\mathcal{H})$. We say that (x_α) **converges σ-strongly to** x if for any sequence $(\xi_n)\in\ell^2(\mathbb{N},\mathcal{H})$

$$\lim_\alpha \sum_{n=1}^\infty \|(x-x_\alpha)\xi_n\|^2 = 0.$$

The topology induced by this convergence is called the **σ-strong operator topology** (or **σ-SOT**). This is also sometimes called the **ultrastrong topology**.

Definition 4.4.2. Let $(x_\alpha)\subset\mathcal{B}(\mathcal{H})$ be a net of bounded operators, and let $x\in\mathcal{B}(\mathcal{H})$. We say that (x_α) **converges σ-weakly to** x if for any pair of sequences $(\xi_n),(\eta_n)\in\ell^2(\mathbb{N},\mathcal{H})$ we have

$$\lim_\alpha \sum_{n=1}^\infty \langle (x-x_\alpha)\xi_n,\eta_n\rangle = 0.$$

The topology induced by this convergence is called the **σ-weak operator topology** (or **σ-WOT**). This is also sometimes called the **ultraweak topology**.

Remark 4.4.3. The σ-SOT (resp. σ-WOT) is the topology on $\mathcal{B}(\mathcal{H})$ defined as the pull back of the SOT (resp. WOT) on $\mathcal{B}(\mathcal{H}\otimes\ell^2\mathbb{N})$ under the map $\mathrm{id}\otimes 1$.

We note that on bounded sets, the σ-SOT (resp. σ-WOT) agrees with the SOT (resp. WOT).

Proposition 4.4.4. *Let $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ be a uniformly bounded net.*

(i) (x_α) converges strongly if and only if it converges σ-strongly.

(ii) (x_α) converges weakly if and only if it converges σ-weakly.

Proof. We prove (ii), the proof of (i) being similar. That (x_α) converges weakly if it converges σ-weakly is clear. So assume (x_α) converges to $x \in \mathcal{B}(\mathcal{H}))$ σ-weakly. Fix $\xi = (\xi_n), \eta = (\eta_n) \in \ell^2(\mathbb{N}, \mathcal{H})$. Let $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that

$$\left(\sum_{n\geq N} \|\xi_n\|^2\right)^{\frac{1}{2}} \left(\sum_{n\geq N} \|\eta_n\|^2\right)^{\frac{1}{2}} < \frac{\epsilon}{\sup_\alpha \|x - x_\alpha\|}.$$

Then

$$\left|\sum_{n=1}^{\infty} \langle (x - x_\alpha)\xi_n, \eta_n\rangle\right| \leq \sum_{n=1}^{N-1} |\langle (x - x_\alpha)\xi_n, \eta_n\rangle| + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the weak convergence of (x_α) to x implies σ-weak convergence. □

For $\xi = (\xi_n), \eta = (\eta_n) \in \ell^2(\mathbb{N}, \mathcal{H})$, define $\omega_{\xi,\eta} \colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by

$$\omega_{\xi,\eta}(x) \sum_{n=1}^{\infty} \langle x\xi_n, \eta_n\rangle \qquad x \in \mathcal{B}(\mathcal{H}).$$

Then $\omega_{\xi,\eta} \in \mathcal{B}(\mathcal{H})^*$ with $\|\omega_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$. Moreover, $\omega_{\xi,\eta}$ is clearly σ-WOT continuous.

We have the following relationship between these new topologies and the three we have previously considered:

$$\begin{array}{ccccc} \sigma\text{-WOT} & \prec & \sigma\text{-SOT} & \prec & \text{Uniform} \\ \curlyvee & & \curlyvee & & \\ \text{WOT} & \prec & \text{SOT} & & \end{array}$$

where '$\prec$' points from the finer to the coarser topology. We will show that the σ-WOT is equivalent to the weak* topology on $\mathcal{B}(\mathcal{H})$. Indeed, suppose a net $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ converges to $x \in \mathcal{B}(\mathcal{H})$ in the σ-WOT. Let $y \in L^1(\mathcal{B}(\mathcal{H}))$, with polar decomposition $y = v|y|$. Then by Lemma 4.3.2, for any $a \in \mathcal{B}(\mathcal{H}))$ we have $av|y|^{1/2}, |y|^{1/2} \in L^2(\mathcal{B}(\mathcal{H}))$ with

$$\mathrm{Tr}(ay) = \mathrm{Tr}(av|y|^{1/2}|y|^{1/2}) = \mathrm{Tr}(|y|^{1/2}av|y|^{1/2}) = \sum_i \left\langle av|y|^{1/2}\xi_i, |y|^{1/2}\xi_i\right\rangle,$$

where $\{\xi_i\}$ is some orthonormal basis for $\mathcal{H}$. Now, since $\mathrm{Tr}(|y|) < \infty$ and $\ker(|y|) = \ker(|y|^{1/2})$, at most countably many ξ_i have $|y|^{1/2}\xi_i \neq 0$, so let us relabel them $\{\xi_n\}_{n\in\mathbb{N}}$. Define

$$\eta_n := v|y|^{1/2}\xi_n \qquad \zeta_n := |y|^{1/2}\xi_n \qquad n \in \mathbb{N}.$$

Then $\|v|y|^{1/2}\xi_n\|^2 = \||y|^{1/2}\xi_n\|^2 = \langle |y|\xi_n, \xi_n\rangle$ implies $\eta := (\eta_n), \zeta := (\zeta_n) \in \ell^2(\mathbb{N}, \mathcal{H})$. We have

$$\mathrm{Tr}(ay) = \sum_{n=1}^{\infty} \langle a\eta_n, \zeta_n\rangle = \omega_{\eta,\zeta}(a),$$

Thus by the σ-WOT convergence of (x_α) to x and the σ-WOT continuity of $\omega_{\eta,\zeta}$ we have

$$\mathrm{Tr}(xy) = \omega_{\eta,\zeta}(x) = \lim_\alpha \omega_{\eta,\zeta}(x_\alpha) = \lim_\alpha \mathrm{Tr}(x_\alpha y).$$

Thus (x_α) converges to x in the weak* topology. To see the other direction of the equivalence we first need a lemma. 2/17/2017

Lemma 4.4.5. *Let $\varphi\colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a continuous linear functional. Then the following are equivalent:*

(i) There exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$ so that $\varphi(x) = \sum_{i=1}^n \langle x\xi_i, \eta_i\rangle$.

(ii) φ is WOT continuous.

(iii) φ is SOT continuous.

Proof. (i)⇒(ii)⇒(iii) are clear, so it suffices to prove (iii)⇒(i). Suppose φ is SOT continuous. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Then $\varphi^{-1}(\mathbb{D})$ contains an open neighborhood (in the SOT) of the zero operator. Consequently there exists $\epsilon > 0$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$ so that $\varphi(x) \in \mathbb{D}$ whenever

$$\sqrt{\sum_{i=1}^n \|x\xi_i\|^2} \leq \epsilon.$$

The linearity of φ then implies

$$\left|\varphi\left(\frac{\epsilon x}{\sqrt{\sum_{i=1}^n \|x\xi_i\|^2}}\right)\right| < 1 \qquad \forall x \in \mathcal{B}(\mathcal{H}).$$

Thus

$$|\varphi(x)| \leq \frac{1}{\epsilon}\sqrt{\sum_{i=1}^n \|x\xi_i\|^2} \qquad \forall x \in \mathcal{B}(\mathcal{H}).$$

This implies that

$$(x\xi_1, \ldots, x\xi_n) \mapsto \varphi(x).$$

is a well-defined, continuous map on $\mathcal{K}$, the closure of $\{(x\xi_1, \ldots, x\xi_n) \in \mathcal{H}^{\oplus n}\colon x \in \mathcal{B}(\mathcal{H})\}$. Thus, the Riesz representation theorem implies there exists $(\eta_1, \ldots, \eta_n) \in \mathcal{H}^{\oplus n}$ such that

$$\varphi(x) = \sum_{i=1}^n \langle x\xi_i, \eta_i\rangle$$

for all $x \in \mathcal{B}(\mathcal{H})$. □

Corollary 4.4.6. *For $K \subset \mathcal{B}(\mathcal{H})$ convex, the SOT and WOT closures coincide.*

Proof. $\bar{K}^{SOT} \subset \bar{K}^{WOT}$ always holds. By the lemma, the dual spaces of K under the two topologies coincide, and hence the reverse inclusion follows from the Hahn–Banach separation theorem, □

Theorem 4.4.7. *Let $\varphi\colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a continuous linear functional. Then the following are equivalent:*

(i) There exists $a \in L^1(\mathcal{B}(\mathcal{H}))$ so that $\varphi(x) = Tr(xa)$.

(ii) There exists $\xi, \eta \in \ell^2(\mathbb{N}, \mathcal{H})$ so that $\varphi(x) = \omega_{\xi,\eta}(x)$.

(iii) φ is σ-WOT continuous.

(iv) φ is σ-SOT continuous.

Proof. (i)⇒(ii) was established by the discussion preceding Lemma 4.4.5. (ii)⇒(iii)⇒(iv) are clear, so it suffices to check (iv)⇒(i). Suppose φ is σ-SOT continuous. Define Φ on the image of $\mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$ under the map id $\otimes\, 1$ by

$$\Phi(x \otimes 1) = \varphi(x) \qquad x \in \mathcal{B}(\mathcal{H}).$$

Then Φ is SOT continuous and by the Hahn–Banach theorem there exists an SOT continuous extension to all of $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$, which we still denote by Φ. Then by Lemma 4.4.5 we have that there exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H} \otimes \ell^2\mathbb{N}$ so that

$$\Phi(x) = \sum_{i=1}^n \langle x\xi_i, \eta_i\rangle \qquad x \in \mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N}).$$

In particular,

$$\varphi(x) = \sum_{i=1}^{n} \langle (x \otimes 1)\xi_i, \eta_i \rangle \qquad x \in \mathcal{B}(\mathcal{H}).$$

Now, for each $i = 1, \ldots, n$ let $a_i, b_i \colon \overline{\ell^2\mathbb{N}} \to \mathcal{H}$ be the Hilbert–Schmidt operators corresponding to ξ_i and η_i, respectively, via the isomorphism $\mathrm{HS}(\overline{\ell^2\mathbb{N}}, \mathcal{H}) \cong \mathcal{H} \otimes \ell^2\mathbb{N}$. Define

$$a := \sum_{i=1}^{n} a_i b_i^*.$$

By Lemma 4.3.5, $a \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\mathrm{Tr}(xa) = \sum_{i=1}^{n} \mathrm{Tr}(xa_i b_i^*) = \sum_{i=1}^{n} \mathrm{Tr}(b_i^* x a_i) = \sum_{i=1}^{n} \langle x a_i, b_i \rangle_{\mathrm{HS}} = \sum_{i=1}^{n} \langle (x \otimes 1)\xi_i, \eta_i \rangle = \varphi(x),$$

for all $x \in \mathcal{B}(\mathcal{H})$). ☐

From (ii)⇒(i) in the previous theorem, it follows that convergence in the weak* topology implies convergence in the σ-WOT. Thus these two topologies coincide.

Corollary 4.4.8. *The σ-WOT on $\mathcal{B}(\mathcal{H})$ is equivalent to the weak* topology under the identification $\mathcal{B}(\mathcal{H}) \cong (L^1(\mathcal{B}(\mathcal{H})))^*$.*

By the Banach-Alaoglu theorem we also obtain the following corollary:

Corollary 4.4.9. *The unit ball in $\mathcal{B}(\mathcal{H})$ is compact in the σ-WOT.*

We also have the analogue of Corollary 4.4.6

Corollary 4.4.10. *For $K \subset \mathcal{B}(\mathcal{H})$ convex, the σ-SOT and σ-WOT closures coincide.*

4.5 The Predual of a von Neumann Algebra

Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Nuemann algebra. Observe that for σ-WOT continuous $\varphi \in M^*$, the Hahn–Banach extension theorem implies there exists $\Phi \in \mathcal{B}(\mathcal{H})$ such that $\Phi\mid_M = \varphi$ and Φ is σ-WOT continuous. Thus, Theorem 4.4.7 implies $\varphi = \mathrm{Tr}(a\cdot) = \omega_{\xi,\eta}$ for some $a \in L^1(\mathcal{B}(\mathcal{H}))$ and $\xi, \eta \in \ell^2(\mathbb{N}, \mathcal{H})$.

Definition 4.5.1. For M a von Nuemann algebra, $\varphi \in M^*$ is said to be **normal** if it satisfies any (hence all) of the properties in Theorem 4.4.7. Denote the set of normal linear functionals on M by M_*.

Proposition 4.5.2. *For $M \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra, M_* is a closed subspace of M^*.*

Proof. Let $(\psi_\alpha) \subset M_*$ be a net converging in M^* to ψ. By the discussion preceding the above definition, $\psi_\alpha = \mathrm{Tr}(a_\alpha\cdot)$ for some $a_\alpha \in L^1(\mathcal{B}(\mathcal{H}))$. By Remark 4.2.13, $\psi = Tr(a\cdot)$ for some $a \in L^1(\mathcal{B}(\mathcal{H}))$ and in particular ψ is normal. ☐ 2/22/2017

Theorem 4.5.3. *As Banach spaces, $(M_*)^* \cong M$.*

Proof. Identifying $L^1(\mathcal{B}(\mathcal{H}))$ in $\mathcal{B}(\mathcal{H})^*$, the remarks at the beginning of this section imply

$$M_* \cong L^1(\mathcal{B}(\mathcal{H}))/{}^\perp M,$$

where

$${}^\perp M = \{a \in L^1(\mathcal{B}(\mathcal{H})) \colon \mathrm{Tr}(ax) = 0\ \forall x \in M\}.$$

From general Banach space theory (*cf.* [2, Theorem 1.10.17]), it follows that

$$(M_*)^* \cong \left(L^1(\mathcal{B}(\mathcal{H}))/M^\perp\right)^* = ({}^\perp M)^\perp,$$

where

$$({}^\perp M)^\perp = \{x \in \mathcal{B}(\mathcal{H}) \colon \mathrm{Tr}(ax) = 0\ a \in^\perp M\}.$$

Furthermore, $({}^\perp M)^\perp$ is the weak* closure of M (*cf.* [2, Proposition 2.6.6]). Since the weak* topology coincides with the σ-WOT, M is weak* closed because it is closed in the WOT. Thus $(M_*)^* = M$. ☐

In light of the above theorem, we make the following definition.

Definition 4.5.4. For a von Neumann algebra M, M_* is called the **predual of** M.

The theorem also implies that the weak* topology (identifying $M \cong (M_*)^*$) is exactly the σ-WOT restricted to M. Thus by the Banach-Alaoglu theorem we obtain the following:

Corollary 4.5.5. *The unit ball in M is compact in the σ-WOT.*

4.6 Positive Normal Linear Functionals

In this section we show that the positive, normal linear functionals have a much nicer characterization than σ-WOT continuity. Recall that M^* is an M-M bimodule:

$$(a \cdot \varphi \cdot b)(x) = \varphi(bxa) \qquad a, b, x \in M.$$

Lemma 4.6.1. *Let $\varphi, \psi \in M^*$ be positive. Suppose there exists a projection $p \in M$ such that $p \cdot \psi \cdot p$ is normal and $\varphi(p) < \psi(p)$. Then there exists a non-zero projection $q \in M$ such that $q \leq p$ and $\varphi(x) < \psi(x)$ for all $x \in qMq$, $x > 0$.*

Proof. Consider the set S of operators $x \in M$ such that $0 \leq x \leq p$ and $\phi(x) \geq \psi(x)$. If $(x_\alpha) \subset S$ is an increasing net, then the net is uniformly bounded by p and hence by Lemma 3.1.1 (x_α) converges strongly to $x = \sup_\alpha x_\alpha$. Note that $0 \leq x \leq p$ and $x \geq x_\alpha$. By Lemma 4.4.4, (x_α) converges σ-strongly to x and consequently converges σ-weakly. Since $p \cdot \psi \cdot p$ is normal, we therefore have

$$\psi(x) = \psi(pxp) = \lim_\alpha \psi(px_\alpha p) = \lim_\alpha \psi(x_\alpha)$$

By $x \geq x_\alpha$ we have $\varphi(x) \geq \varphi(x_\alpha) \geq \psi(x_\alpha)$. Thus $\varphi(x) \geq \psi(x)$, and $x \in S$. By Zorn's Lemma there exists a maximal operator $x_0 \in S$. Note that $\varphi(p) < \psi(p)$ and $\varphi(x_0) \geq \psi(x_0)$ imply $x_0 < p$. Consequently, there exists $\epsilon > 0$ such that

$$q := \chi_{[\epsilon,1]}(p - x_0) \neq 0.$$

We have $q \leq p$, and if $0 < y \leq \epsilon q$ then $x_0 < x_0 + y \leq x_0 + \epsilon q \leq p$. By maximality of x_0,

$$\varphi(x_0 + y) < \psi(x_0 + y) \leq \varphi(x_0) + \psi(y).$$

Thus $\varphi(y) < \psi(y)$. For arbitrary $y \in qMq$ positive, we have $\frac{\epsilon}{\|y\|} y \leq \epsilon q$ and so by rescaling we obtain $\varphi(y) < \psi(y)$. ∎

Theorem 4.6.2. *Let M be a von Neumann algebra and $\varphi \in M^*$ be positive. Then the following are equivalent:*

(i) φ is normal.

(ii) There exists $\xi = (\xi_n) \in \ell^2(\mathbb{N}, \mathcal{H})$ such that $\varphi = \omega_{\xi,\xi}$.

(iii) There exists a positive $a \in L^1(\mathcal{B}(\mathcal{H}))$ so that $\varphi = Tr(a\cdot)$.

(iv) If $(x_\alpha) \subset M$ is a bounded, increasing net then

$$\varphi(\sup_\alpha x_\alpha) = \sup_\alpha \varphi(x_\alpha).$$

(v) If $\{p_i\}_{i \in I} \subset M$ is a family of pairwise orthogonal projections, then

$$\varphi(\sum_{i \in I} p_i) = \sum_{i \in I} \varphi(p_i).$$

Proof. **(i)⇒(ii):** From Theorem 4.4.7 we have $\varphi = \omega_{\eta,\zeta}$ for $\eta, \zeta \imath \ell^2(\mathbb{N}, \mathcal{H})$. Since φ is positive, it follows that $\omega_{\eta,\zeta} = \omega_{\zeta,\eta}$ and consequently,

$$\varphi = \frac{1}{4}\omega_{\eta+\zeta,\eta+\zeta} - \frac{1}{4}\omega_{\eta-\zeta,\eta-\zeta}.$$

But then $\varphi \leq \frac{1}{4}\omega_{\eta+\zeta,\eta+\zeta}$, and so letting $\mathcal{K} = \overline{(M \otimes \mathbb{C})(\eta+\zeta)} \subset \mathcal{H} \otimes \ell^2\mathbb{N}$,

$$\mathcal{K} \times \mathcal{K} \ni ((x \otimes 1)(\eta+\zeta), (y \otimes 1)(\eta+\zeta)) \mapsto \varphi(y^*x)$$

defines a boudned sesquilinear form on $\mathcal{K}$. Thus there exists a positive operator $T \in \mathcal{B}(\mathcal{K})$ such that

$$\varphi(y^*x) = \langle T(x \otimes 1)(\eta+\zeta), (y \otimes 1)(\eta+\zeta)\rangle .$$

Note that for $x, y, z \in M$ we have

$$\begin{aligned}\langle (z \otimes 1)T(x \otimes 1)(\eta+\zeta), (y \otimes 1)(\eta+\zeta)\rangle &= \langle T(x \otimes 1)(\eta+\zeta), ((z^*y) \otimes 1)(\eta+\zeta)\rangle \\ &= \varphi(y^*zx) \\ &= \langle T(zx \otimes 1)(\eta+\zeta), (y \otimes 1)(\eta+\zeta)\rangle \\ &= \langle T(z \otimes 1)(x \otimes 1)(\eta+\zeta), (y \otimes 1)(\eta+\zeta)\rangle .\end{aligned}$$

Thus $T \in (M \otimes \mathbb{C})'$, and therefore

$$\varphi(y^*x) = \left\langle (x \otimes 1)T^{1/2}(\eta+\zeta), T^{1/2}(\eta+\zeta)\right\rangle .$$

In particular, letting $\xi = T^{1/2}(\eta+\zeta) \in \ell^2(\mathbb{N}, \mathcal{H})$, we have $\varphi = \omega_{\xi,\xi}$.

(ii)⇒(iii): Using the isomorphism $\mathcal{H} \otimes \ell^2\mathbb{N} \cong \mathrm{HS}(\overline{\ell^2\mathbb{N}}, \mathcal{H})$, let $b \in \mathcal{B}(\overline{\ell^2\mathbb{N}}, \mathcal{H})$ be the Hilbert-Schmidt operator corresponding to $\xi \in \mathcal{H} \otimes \ell^2\mathbb{N}$. Then $a := bb^* \in \mathcal{B}(\mathcal{H})$ is positive, and by Lemma 4.3.5 $a \in L^1(\mathcal{B}(\mathcal{H}))$ with

$$\mathrm{Tr}(xa) = \mathrm{Tr}(b^*xb) = \langle xb, b\rangle_{\mathrm{HS}} = \langle (x \otimes 1)\xi, \xi\rangle = \omega_{\xi,\xi}(x) = \varphi(x),$$

for $x \in M$.

(iii)⇒(iv): Recall that by Lemma 3.1.1, (x_α) converges strongly to $x := \sup_\alpha x_\alpha$. Because the x_α are uniformly bounded in norm, Lemma 4.4.4 implies (x_α) converges to x in the σ-SOT. Consequently (x_α) also converges to x in the σ-WOT. Since $\mathrm{Tr}(a\cdot)$ is σ-WOT continuous, we have

$$\varphi(x) = \mathrm{Tr}(ax) = \sup_\alpha \mathrm{Tr}(ax_\alpha) = \sup_\alpha \varphi(x_\alpha).$$

(iv)⇒(v): This is clear. 2/24/2017

(v)⇒(i): We will show φ is normal by comparing it to $x \mapsto \langle x\xi, \xi\rangle$ for some $\xi \in \mathcal{H}$, which is clearly normal.

Let $p \in M$ be a non-zero projection. Then there exists $\xi \in \mathcal{H}$ such that $\varphi(p) < \langle p\xi, \xi\rangle$. Define $\psi \in M^*$ by $\psi(x) = \langle x\xi, \xi\rangle$. Clearly ψ is normal and hence so is $p \cdot \psi \cdot p$. Let $q \in M$ be the projection guaranteed by Lemma 4.6.1. Then for any $x \in M$ we have

$$|\varphi(xq)|^2 = |\langle xq, 1\rangle_\varphi|^2 \leq \varphi(qx^*xq)\varphi(1) \leq \psi(qx^*xq)\varphi(1) = \langle qx^*xq\xi, \xi\rangle \varphi(1) = \|xq\xi\|^2\varphi(1).$$

Observe that this implies $q \cdot \varphi$ is SOT continuous and hence normal.

Now, by Zorn's Lemma we can find a maximal family $\{q_i\}_{i\in I}$ of pairwise orthogonal projections such that $q_i \cdot \varphi$ is normal for each $i \in I$. By maximality, $\sum q_i = 1$ and so by (v) we have

$$\varphi(1) = \sum_{i\in I} \varphi(q_i).$$

Thus for any $\epsilon > 0$ we can find a finite subcollection $J \subset I$ such that if $q = \sum_{i\in J} q_i$ then $\varphi(q) > \varphi(1) - \epsilon$, or $\varphi(1-q) < \epsilon$. For $x \in M$ we then have

$$|(\varphi - q \cdot \varphi)(x)|^2 = |\varphi(x(1-q))|^2 = |\langle 1-q, x^*\rangle_\varphi|^2 \leq \varphi(1-q)\varphi(xx^*) \leq \epsilon\|x\|^2\varphi(1).$$

Thus the finite partial sums of $\sum_{i\in I} q_i \cdot \varphi$ converge uniformly to φ. These finite partial sums are normal as finite sums of normal linear functionals, and consequently φ is normal by Proposition 4.5.2. □

4.7 Kaplansky's Density Theorem

We use the approach of [3, Section 2.6] to prove Kaplansky's density theorem, which roughly says the following: Let $A \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra and $B = \overline{A}^{SOT}$. Clearly any $b \in B$ can be realized as the SOT limit of a net $(a_\alpha) \subset A$, but in fact we can actually demand that the net is uniformly bounded by $\|b\|$.

Lemma 4.7.1. *Let $(x_\alpha), (y_\alpha) \subset \mathcal{B}(\mathcal{H})$ be strongly convergent nets. If (x_α) is uniformly bounded, then $(x_\alpha y_\alpha)$ is strongly convergent.*

Proof. Let $x, y \in \mathcal{B}(\mathcal{H})$ be the respective strong limits of (x_α) and (y_α), and let $R = \sup \|x_\alpha\|$. For $\xi \in \mathcal{H}$ we have

$$\|xy\xi - x_\alpha y_\alpha \xi\| \leq \|(x - x_\alpha)y\xi\| + \|x_\alpha(y - y_\alpha)\xi\| \leq \|(x - x_\alpha)y\xi\| + R\|(y - y_\alpha)\xi\|.$$

Thus $(x_\alpha y_\alpha)$ converges strongly to xy. ☐

Proposition 4.7.2. *If $f \in C(\mathbb{C})$, then the map $x \mapsto f(x)$ on normal operators in $\mathcal{B}(\mathcal{H})$ is SOT continuous on bounded subsets.*

Proof. Let $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ be a net of uniformly bounded normal operators converging strongly to $x \in \mathcal{B}(\mathcal{H})$. Let $R = \sup_\alpha \|x_\alpha\|$. The Stone–Weierstrass theorem allows us to approximate f uniformly on $\{z \in \mathbb{C} \colon |z| \leq R\}$ by a sequence polynomials $(p_n(z, \bar{z}))$. The previous lemma and the fact that taking adjoints is SOT continuous for normal operators imply that for each $n \in \mathbb{N}$ the net $(p_n(x_\alpha, x_\alpha^*))$ converges strongly to $p_n(x, x^*)$. Since the sequences $(p_n(x, x^*))$ and $(p_n(x_\alpha, x_\alpha^*))$ converge uniformly to $f(x)$ and $f(x_\alpha)$, respectively, we see that $f(x_\alpha)$ converges strongly to $f(x)$. ☐

For $x \in \mathcal{B}(\mathcal{H})$ self-adjoint, the unitary operator

$$(x - i)(x + i)^{-1} = (x + i)^{-1}(x - i)$$

is called the *Cayley transform of x*.

Proposition 4.7.3. *The Cayley transform is SOT continuous.*

Proof. Let $(x_\alpha) \subset \mathcal{B}(\mathcal{H})$ be a net of self-adjoint operators converging to a self-adjoint operator $x \in \mathcal{B}(\mathcal{H})$. Note that by the spectral mapping theorem $\|(x_\alpha + i)^{-1}\| \leq 1$ for all α. For $\xi \in \mathcal{H}$ we have

$$\begin{aligned}
\|(x - i)(x + i)^{-1}\xi - (x_\alpha - i)(x_\alpha + i)^{-1}\xi\| &= \|(x_\alpha + i)^{-1}\left[(x_\alpha + i)(x - i) - (x_\alpha - i)(x + i)\right](x - i)^{-1}\xi\| \\
&= \|2i(x_\alpha + i)^{-1}\left[x - x_\alpha\right](x - i)^{-1}\xi\| \\
&\leq 2\|[x - x_\alpha](x - i)^{-1}\xi\|.
\end{aligned}$$

Thus strong convergence of (x_α) to x implies the strong convergence of their respective Cayley transforms. ☐

Corollary 4.7.4. *If $f \in C_0(\mathbb{R})$, then the map $x \mapsto f(x)$ on self-adjoint operators is SOT continuous.*

Proof. Since f vanishes at infinity,

$$g(z) := \begin{cases} 0 & \text{if } z = 1 \\ f\left(i\frac{1+z}{1-z}\right) & \text{otherwise} \end{cases}$$

defines a continuous function on $\mathbb{T} \subset \mathbb{C}$. By Proposition 4.7.2, $x \mapsto g(x)$ is SOT continuous on the set of unitary operators. Then using Proposition 4.7.3, we see that $x \mapsto g((x - i)(x + i)^{-1}) = f(x)$ is SOT continuous as the composition of two SOT continuous maps. ☐

For $S \subset \mathcal{B}(\mathcal{H})$, write $S_{s.a.}$ for the self-adjoint operators in S, and write $(S)_1$ for the operators in S with norm at most one.

Theorem 4.7.5 (Kaplansky's density theorem)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra and let $B = \overline{A}^{SOT}$. Then:*

(i) The SOT closure of $A_{s.a.}$ is $B_{s.a.}$.

(ii) The SOT closure of $(A)_1$ is $(B)_1$.

Proof. Note that we assume A is uniformly closed; that is, that A is a C^*-algebra. Using that SOT convergence implies WOT convergence, it is easy to check that $\overline{A_{s.a.}}^{SOT} \subset B_{s.a.}$. Now let $x \in B_{s.a.}$, then there exists a net $(x_\alpha) \subset A$ converging strongly to x. Since taking adjoints is WOT continuous, we have that $\left(\frac{x_\alpha + x_\alpha^*}{2}\right)$ converges weakly to x. Since $A_{s.a.}$ is convex, Corollary 4.4.6 implies $\overline{A_{s.a.}}^{SOT} = \overline{A_{s.a.}}^{WOT} \ni x$. Thus $B_{s.a.} = \overline{A_{s.a.}}^{SOT}$.

Now, using the principle of uniform boundedness, we see that $\overline{(A)_1}^{SOT} \subset (B)_1$. If $(x_\alpha) \subset A_{s.a.}$ converges strongly to $x \in B_{s.a}$, then taking $f \in C_0(\mathbb{R})$ with $f(t) = t$ for $t \leq \|x\|$, and $|f(t)| \leq \|x\|$ otherwise, we have that $(f(x_\alpha))$ converges strongly to $f(x) =$ by Corollary 4.7.4. Thus $(A_{s.a.})_1$ is SOT dense in $(B_{s.a.})_1$. So, for $x \in (B)_1$ let us consider

$$\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(B))_1.$$

Note that $\tilde{x}$ is self-adjoint. Since $M_2(A)$ is SOT dense in $M_2(B)$, the above implies there exists a net $(\tilde{x}_\alpha) \subset (M_2(A)_{s.a.})_1$ converging strongly to $\tilde{x}$. Let x_α be the $(1,2)$-entry of $\tilde{x}_\alpha$. Then $x_\alpha \in (A)_1$ and (x_α) converges strongly to x. □

Observe that if $A = M$ is a von Neumann algebra, then the second part of the above theorem implies $(M)_1$ is SOT closed. Conversely, for a unital $*$-subalgebra $A \subset \mathcal{B}(\mathcal{H})$, if $(A)_1$ is SOT closed, then it easily follows that A is SOT closed and hence A is a von Neumann algebra. Using Corollary 4.4.6, Proposition 4.4.4, and Corollary 4.4.10, we can make the same assertion in the other topologies. This yields the following corollary.

Corollary 4.7.6. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a unital $*$-subalgebra. The following are equivalent:*

(i) A is a von Neumann algebra.

(ii) $(A)_1$ is SOT closed.

(iii) $(A)_1$ is WOT closed.

(iv) $(A)_1$ is σ-SOT closed.

(v) $(A)_1$ is σ-WOT closed.

Chapter 5

Types of von Neumann Algebras

The material in this chapter is adapted from [3, Chapter 3].

5.1 Lattice of Projections

Definition 5.1.1. Let $\{p_i\}_{i\in I} \subset \mathcal{B}(\mathcal{H})$ be a family of projections (not necessarily pairwise orthogonal). The **infimum of** $\{p_i\}_{i\in I}$ is the projection $\left[\bigcap_{i\in I} p_i\mathcal{H}\right]$ and is denoted $\bigwedge_{i\in I} p_i$. The **supremum of** $\{p_i\}_{i\in I}$ is the projection $\left[\bigcup_{i\in I} p_i\mathcal{H}\right]$ and is denoted $\bigvee_{i\in I} p_i$.

Observe that for a family of projections $\{p_i\}_{i\in I}$,

$$\bigvee_{i\in I} p_i = 1 - \bigwedge_{i\in I}(1-p_i)$$

and

$$\bigwedge_{i\in I} p_i = 1 - \bigvee_{i\in I}(1-p_i)$$

For each $j \in I$ we have $\bigwedge_{i\in I} p_i \leq p_j \leq \bigvee_{i\in I} p_i$.

Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $\mathcal{P}(M)$ denote the set of projections in M. If $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$, then $\bigwedge_i p_i, \bigvee_i p_i \in \mathcal{P}(M)$. Indeed, it is easy to see that

$$\overline{\operatorname{span}}\left(\bigcap_{i\in I} p_i\mathcal{H}\right) \qquad \text{and} \qquad \overline{\operatorname{span}}\left(\bigcup_{i\in I} p_i\mathcal{H}\right)$$

is reducing for M', and so $\bigwedge_i p_i$ and $\bigvee_i p_i$ belong to $M'' = M$ by Lemma 2.2.3. 2/28/2017

Definition 5.1.2. For $p, q \in \mathcal{P}(M)$, we say that p **is sub-equivalent to** q **(in** M**)** and write $p \preceq q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. We say p **is equivalent to** q **(in** M**)** and write $p \sim q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. If $p \preceq q$ but $p \not\sim q$, we write $p \prec q$.

Remark 5.1.3. If $p, q \in \mathcal{P}(M)$ are such that $p \leq q$, then $p \preceq q$ with partial isometry $v = p$.

Proposition 5.1.4. *For $M \subset \mathcal{B}(\mathcal{H})$, the relation $\preceq$ is a partial ordering on $\mathcal{P}(M)$, and the relation $\sim$ is an equivalence relation on $\mathcal{P}(M)$.*

Proof. Let $p, q, r \in \mathcal{P}(M)$ with $p \preceq q$ and $q \preceq r$. Then there exists partial isometries $u, v \in M$ so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. It follows that

$$qu = quu^*u = uu^*u = u,$$

so that

$$(vu)^*(vu) = u^*v^*vu = u^*qu = u^*u = p$$

and

$$(vu)(vu)^* = vuu^*v \leq vqv^* = v(v^*v)v^* = vv^* \leq r.$$

Thus $p \preceq r$. The same argument shows that $\sim$ is transitive, and $\sim$ is clearly reflexive and symmetric. $\square$

For M a von Neumann algebra, suppose $\tau \in M^*$ is a tracial state. Let $p, q \in M$ be projections such that $p \preceq q$ with partial isometry $v \in M$ satisfying $v^*v = p$ and $vv^* \leq q$. Then

$$\tau(p) = \tau(v^*v) = \tau(vv^*) \leq \tau(q).$$

Moreover, if $p \sim q$ then $\tau(p) = \tau(q)$. We will eventually see that when M is a factor with a faithful tracial state τ, then $p \sim q$ if and only if $\tau(p) = \tau(q)$. That is, the equivalence classes of projections are parametrized by images under τ. Let us explore the notion of equivalence for projections further.

Proposition 5.1.5. *For $M \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra, $p \preceq q$ and $q \preceq p$ for $p, q \in \mathcal{P}(M)$ implies $p \sim q$.*

Proof. Let $u, v \in M$ be partial isometries so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq p$. Set $p_0 = p - vv^*$, $q_0 = up_ou^*$, and inductively define sequences of orthogonal projections (check this) $(p_n), (q_n)$ by

$$p_n = vq_{n-1}v^* \qquad \text{and} \qquad q_n = up_nu^*.$$

Note that $p_n \leq vv^* \leq p$ for all $n \geq 1$, while $p_0 \perp vv^*$ and $p_0 \leq p$, and $q_n \leq uu^* \leq q$ for all $n \geq 0$. Also define

$$p_\infty := p - \sum_{n=0}^{\infty} p_n \qquad \text{and} \qquad q_\infty := q - \sum_{n=0}^{\infty} q_n.$$

Let

$$w = \sum_{n=1}^{\infty} up_n + v^*p_\infty.$$

We claim $w^*w = p$ and $ww^* = q$. We will break the argument up into smaller claims.

Claim 1: $(up_n)^*(up_m) = \delta_{n=m}p_n$ and $(up_n)(up_m)^* = \delta_{n=m}q_n$.

We compute

$$(up_n)^*(up_m) = p_nu^*up_m = p_npp_m = \delta_{n=m}p_n.$$

Also

$$(up_n)(up_m)^* = up_np_mu^* = \delta_{n=m}up_nu^* = \delta_{n=m}q_n.$$

Claim 2: $(v^*p_\infty)^*(v^*p_\infty) = p_\infty$ and $(v^*p_\infty)(v^*p_\infty)^* = q_\infty$.

Let $v_k = v^*\left(p - \sum_{n=0}^{k} p_n\right)$. Then

$$v_kv_k^* = v^*\left(p - \sum_{n=0}^{k} p_n\right)v = v^*pv - \sum_{n=0}^{k} v^*p_nv = q - \sum_{n=1}^{k} q_{n-1} = q - \sum_{n=0}^{k-1} q_n,$$

where we are using Proposition 1.4.2 to assert $v^*pv = q$ and $v^*p_nv = q_{n-1}$. Also, since $p_0vv^* = 0$ we have

$$v_k^*v_k = \left(p - \sum_{n=0}^{k} p_n\right)vv^*\left(p - \sum_{n=0}^{k} p_n\right) = vv^* - \sum_{n=1}^{k} p_n = p - p_0 - \sum_{n=1}^{k} p_n = p - \sum_{n=0}^{k} p_n.$$

Thus

$$(v^*p_\infty)^*(v^*p_\infty) = \lim_{k\to\infty} v_k^*v_k = \lim_{k\to\infty} p - \sum_{n=0}^{k} p_n = p_\infty,$$

and

$$(v^*p_\infty)(v^*p_\infty)^* = \lim_{k\to\infty} v_k v_k^* = \lim_{k\to\infty} q - \sum_{n=0}^{k-1} q_n = q_\infty,$$

where the limits are in the SOT.

Claim 3: $(up_n)^*(v^*p_\infty) = 0$, $(v^*p_\infty)^*(up_n) = 0$, $(v^*p_\infty)(up_n)^* = 0$, and $(up_n)(v^*p_\infty)^* = 0$ for all $n \in \mathbb{N}$.

First note that by taking adjoints, the second equality follows from the first and the fourth from the third. Now, Proposition 1.4.2 implies $p_n u^* = u^* q_n$ and $v^* p_\infty = q_\infty v^*$. Thus

$$(up_n)^*(v^*p_\infty) = p_n u^* v^* p_\infty = u^* q_n q_\infty v^* = 0.$$

Also

$$(v^*p_\infty)(up_n)^* = v^* p_\infty p_n u^* = 0,$$

and so the claim follows.

From Claims 1-3 we have $w^*w = \sum_{n=1}^\infty p_n + p_\infty = p$ and $ww^* = \sum_{n=1}^\infty q_n + q_\infty = q$. Thus $p \sim q$. □

Lemma 5.1.6. *Let M be a von Neumann algebra. If $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ are two families of pairwise orthogonal projections such that $p_i \preceq q_i$ for each $i \in I$, then $\sum_{i\in I} p_i \preceq \sum_{i\in I} q_i$. In particular, if $p_i \sim q_i$ for each $i \in I$, then $\sum_{i\in I} p_i \sim \sum_{i\in I} q_i$.*

Proof. Let $u_i \in M$ be a partial isometry such that $u_i^* u_i = p_i$ and $u_i u_i^* \le q_i$. Write $r_i = u_i u_i^*$ and note that $\{r_i\}_{i\in I}$ is pairwise orthogonal because $\{q_i\}_{i\in I}$ is. We have for $i \ne j$

$$u_i^* u_j = u_i^* u_i u_i^* u_j u_j^* u_j = u_i^* r_i r_j u_j = 0,$$

and

$$u_i u_j^* = u_i u_i^* u_i u_j^* u_j u_j^* = u_i p_i p_j u_j^* = 0.$$

Consequently,

$$\left(\sum_{i\in I} u_i\right)^* \left(\sum_{j\in I} u_j\right) = \sum_{i\in I} u_i^* u_i = \sum_{i\in I} p_i$$

and

$$\left(\sum_{i\in} u_i\right) \left(\sum_{j\in I} u_j\right)^* = \sum_{i\in} u_i u_i^* = \sum_{i\in I} r_i \le \sum_{i\in I} q_i.$$

Thus $\sum p_i \preceq \sum q_i$. The last assertion follows from the above and Proposition 5.1.5. □ 3/1/2017

Lemma 5.1.7. *For $M \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra and $x \in M$, $[x\mathcal{H}], [x^*\mathcal{H}] \in M$ and $[x\mathcal{H}] \sim [x^*\mathcal{H}]$.*

Proof. That these projections lie in M follows from Lemma 2.2.3 since the subspaces $\overline{x\mathcal{H}}$ and $\overline{x^*\mathcal{H}}$ are clearly reducing for M'. However, we also show this direclty. Let $x = v|x|$ be the polar decomposition. Recall that $v \in M$ and $v^*v = [x^*\mathcal{H}]$ and $vv^* = [x\mathcal{H}]$. Thus the projections are equivalent. □

Proposition 5.1.8 (Kaplansky's formula)**.** *For $M \subset \mathcal{B}(\mathcal{H})$ a von Neumman algebra and $p, q \in \mathcal{P}(M)$,*

$$(p \vee q - p) \sim (q - p \wedge q)$$

Proof. Consider $x = (1-p)q$. Then $\ker(x) = \ker(q) \oplus (p\mathcal{H} \cap q\mathcal{H})$. Thus

$$[x^*\mathcal{H}] = 1 - ((1-q) + p \wedge q) = q - p \wedge q$$

Since $x^* = p(1-q)$, symmetry implies

$$[x\mathcal{H}] = (1-p) - (1-q) \wedge (1-p) = 1 - p - (1 - p \vee q) = p \vee q - p.$$

The equivalence then follows from Lemma 5.1.7. □

Definition 5.1.9. For $x \in M$, the **central support of** x, denoted $z(x)$, is the infimum of all central projections $z \in \mathcal{P}(\mathcal{Z}(M))$ satisfying $xz = zx = x$. We say $p, q \in \mathcal{P}(M)$ are **centrally orthogonal** if $z(p)$ and $z(q)$ are orthogonal.

Note that for $p \in \mathcal{P}(M)$, $p \leq z(p)$. Thus if $p, q \in \mathcal{P}(M)$ are centrally orthogonal, then they are also orthogonal.

Lemma 5.1.10. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. The central support of $p \in \mathcal{P}(M)$ is*

$$z(p) = \bigvee_{x \in M} [xp\mathcal{H}] = [Mp\mathcal{H}].$$

Proof. Let $z = [Mp\mathcal{H}]$. Since M is unital, we have $p \leq z$. Because $Mp\mathcal{H}$ is clearly reducing for M and M', we have that $z \in M \cap M' = \mathcal{Z}(M)$. Thus $z(p) \leq z$. Conversely, for any $x \in M$ we have

$$xp\mathcal{H} = xz(p)p\mathcal{H} = z(p)xp\mathcal{H},$$

which implies $[xp\mathcal{H}] \leq z(p)$. Since this holds for all $x \in M$, we have $z \leq z(p)$. □

Proposition 5.1.11. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p, q \in \mathcal{P}(M)$ satisfy $p \preceq q$ then $z(p) \leq z(q)$.*

Proof. Let $v \in M$ be such that $v^*v = p$ and $vv^* \leq q$. Recall that $vv^* = F(v) = [v\mathcal{H}]$. We have by Lemma 5.1.10

$$z(p) = [Mp\mathcal{H}] = [Mv^*v\mathcal{H}] \leq [Mv\mathcal{H}] = [Mvv^*\mathcal{H}] \leq [Mq\mathcal{H}] = z(q).$$

□

Proposition 5.1.12. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}(M)$, the following are equivalent:*

(i) p and q are centrally orthogonal.

(ii) $pMq = \{0\}$.

(iii) There does not exist non-zero projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim q_0$.

Proof. We first show (i) and (ii) are equivalent. If p and q are centrally orthogonal, then for any $x \in M$ we have

$$pxq = pz(p)xz(q)q = pxz(p)z(q)q = 0.$$

Thus $pMq = \{0\}$. Conversely, if $pMq = \{0\}$, then by Lemma 5.1.10 $pz(q) = p[Mq\mathcal{H}] = 0$. This implies $p \leq 1 - z(q)$, and since $1 - z(q) \in \mathcal{Z}(M)$ we have $z(p) \leq 1 - z(q)$. That is, $z(p)z(q) = 0$. Thus (i) and (ii) are equivalent.

Next we show (ii) and (iii) are equivalent. Suppose (ii) does not hold and let $x \in M$ be such that $pxq \neq 0$. Then $qx^*p \neq 0$ and consequently, $p_0 := [pxq\mathcal{H}]$ and $q_0 := [qx^*p\mathcal{H}]$ are non-zero projections. Clearly $p_0 \leq p$ and $q_0 \leq q$, and by Lemma 5.1.7 $p_0 \sim q_0$. Conversely, suppose (iii) does not hold and $p_0 \leq p$ and $q_0 \leq q$ are non-zero projections such that $p_0 \sim q_0$. Let $v \in M$ be a partial isometry so that $v^*v = p_0$ and $vv^* = q_0$. Then $v^* = p_0v^*q_0$ so that

$$pv^*q = pp_0v^*q_0q = p_0v^*q_0 = v^* \neq 0.$$

Thus $pMq \neq \{0\}$, and we see that (ii) and (iii) are equivalent. □

Theorem 5.1.13 (Comparison theorem)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}(M)$, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$*

$$zp \preceq zq \qquad \textit{and} \qquad (1 - z)q \preceq (1 - z)p.$$

Proof. By Zorn's lemma there exists maximal families $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ of pairwise orthogonal projections such that $p_i \sim q_i$ for all $i \in I$ and

$$p_0 := \sum_{i\in I} p_i \leq p$$
$$q_0 := \sum_{i\in I} q_i \leq q.$$

Note that $p_0 \sim q_0$ by Lemma 5.1.6. Denote $z := z(q - q_0)$. By maximality of the families, Proposition 5.1.12 yields $z(p - p_0)z = 0$. Consequently, $(p - p_0)z = 0$, or $pz = p_0 z$. Now, if $v \in M$ is such that $v^*v = p_0$ and $vv^* = q_0$, then one easily checks that $p_0 z \sim q_0 z$ via the partial isometry vz. Thus

$$pz = p_0 z \sim q_0 z \leq qz.$$

Similarly, $p_0(1 - z) \sim q_0(1 - z)$ and since $q - q_0 \leq z$ we have

$$q(1 - z) = q_0(1 - z) \sim p_0(1 - z) \leq p(1 - z).$$ □

Corollary 5.1.14. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If M is a factor, then for $p, q \in \mathcal{P}(M)$ exactly one of the following holds:*

$$p \prec q \qquad p \sim q \qquad q \prec p.$$

Proof. By the comparison theorem, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ so that $pz \preceq qz$ and $p(1-z) \preceq q(1-z)$. Since $\mathcal{Z}(M) = \mathbb{C}$, we have either $z = 0$ or $z = 1$ and the result follows. □

5.2 Types of Projections

Definition 5.2.1. For $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $p \in \mathcal{P}(M)$ is said to be

- **minimal** if $p \neq 0$ and $q \leq p$ implies $q = 0$ or $q = p$; equivalently, $\dim(pMp) = 1$.
- **abelian** if pMp is abelian.
- **finite** if $q \leq p$ and $q \sim p$ implies $p = q$.
- **semi-finite** if there exists a family $\{p_i\}_{i\in I}$ of pairwise orthogonal, finite projections such that $p = \sum_{i\in I} p_i$.
- **purely infinite** if $p \neq 0$ and there does not exist any non-zero finite projections $q \leq p$.
- **properly infinite** if $p \neq 0$ and for all non-zero $z \in \mathcal{P}(\mathcal{Z}(M))$ the projection zp is not finite.

Furthermore, M is said to be **finite, semi-finite, purely infinite**, or **properly infinite** if $1 \in M$ has the corresponding property. 3/3/2017

For a projection $p \in \mathcal{P}(M)$ we immediately have the following implications:

$$\text{minimal} \Longrightarrow \text{abelian} \Longrightarrow \text{finite} \Longrightarrow \text{semi-finite} \Longrightarrow \text{not purely infinite},$$

and

$$\text{purely infinite} \Longrightarrow \text{properly infinite}$$

In the following section, we will use the notions in the above definition to classify von Neumann algebras by "types." We first further develop the theory of projections by examining which operations preserve the above properties. For example, we shall see that being abelian, finite, semi-finite, or purely infinite is passed to subequivalent projections.

Proposition 5.2.2. *A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is finite if and only if all isometries are unitaries.*

Proof. Suppose M is finite and let $v \in M$ be an isometry: $v^*v = 1$. Then $vv^* \leq 1$ and so by finiteness $vv^* = 1$. That is, v is a unitary. Conversely, suppose every isometry is a unitary. Suppose $p \leq 1$ satisfies $p \sim 1$. Let $v \in M$ satisfy $v^*v = 1$ and $vv^* = p$. Then v is an isometry and hence a unitary, and therefore $p = vv^* = 1$. Thus 1 is finite. □

The proposition implies that $\mathcal{B}(\mathcal{H})$ is finite if and only if $\mathcal{H}$ is finite dimensional. When $\mathcal{H}$ is infinite dimensional, letting $\{\xi_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ we have $1 = \sum_{i\in i}[\mathbb{C}\xi_i]$ so that $\mathcal{B}(\mathcal{H})$ is semi-finite.

Lemma 5.2.3. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Let $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ be a family of pairwise centrally orthogonal projections. If p_i is abelian (resp. finite) for each $i \in I$, then $p = \sum_{i\in I} p_i$ is abelian (resp. finite).*

Proof. Suppose each p_i, $i \in I$, is abelian. By Proposition 5.1.12, we have $p_ixp_j = 0$ for $i, j \in I$ with $i \neq j$. Thus
$$pxp = \sum_{i\in I} p_ixp_i.$$
Since p_iMp_i is abelian we have
$$(pxp)(pyp) = \sum_{i\in I} p_ixp_iyp_i = \sum_{i\in I} p_iyp_ixp_i = (pyp)(pxp).$$
That is, pMp is abelian.

Suppose each p_i, $i \in I$, is finite. Let $v \in M$ be such that $v^*v = p$ and $vv^* \leq p$. Then for $u_i := vz(p_i)$, $i \in I$, we have
$$u_i^*u_i = z(p_i)pz(p_i) = p_i$$
and
$$u_iu_i^* = vz(p_i)v^* = z(p_i)vv^* \leq z(p_i)p = p_i.$$
By finiteness, we have $u_iu_i^* = p_i$ for each $i \in I$. Now, because $\{p_i\}_{i\in I}$ are pairwise centrally orthogonal, we have $z(p) = \sum_{i\in I} z(p_i)$. Also note that since $vp = v$, we have $vz(p) = vpz(p) = vp = v$. Thus
$$vv^* = vz(p)v^* = \sum_{i\in I} vz(p_i)v^* = \sum_{i\in I} u_iu_i^* = \sum_{i\in I} p_i = p.$$
Therefore p is finite. □

Proposition 5.2.4. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Let $p, q \in \mathcal{P}(M)$ be non-zero projections that satisfy $p \preceq$. If q is abelian, then p is abelian.*

Proof. If $p \leq q$, then pMp is a subalgebra of pMp. In particular it abelian.

If $p \sim q$, let $v \in M$ be such that $v^*v = p$ and $vv^* = q$. Then $p = p^2 = v^*qv$. Thus for $x, y \in M$ we have
$$(pxp)(pyp) = (v^*qvxv^*qv)(v^*qvyv^*v) = v^*(qvxv^*q)(qvyv^*q)v = v^*(qvyv^*q)(qvxv^*q)v = (pyp)(pxp).$$
Thus pMp is abelian.

Finally, if $p \preceq q$ then $p \sim q_0 \leq q$ for some projection q_0. Then by the above, q_0 is abelian and therefore so is p. □

Proposition 5.2.5. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Let $p, q \in \mathcal{P}(M)$ be non-zero projections that satisfy $p \preceq q$. If q is finite (resp. purely infinite), then p is also finite (resp. purely infinite).*

Proof. Suppose q is finite, and further suppose $p \sim q$. Let $v \in M$ be such that $v^*v = p$ and $vv^* = q$. If $u \in M$ satisfies $u * u = p$ and $uu^* \leq p$, then
$$(vuv^*)^*(vuv^*) = vu^*v^*vuv^* = vu^*puv^* = vu^*uv^* = vpv^* = vv^* = q$$
and
$$(vuv^*)(vuv^*)^* = vuv^*vu^*v^* = vupu^*v^* = vuu^*v^* \leq vpv^* = q.$$

Since q is finite, we must have $(vuv^*)(vuv^*)^* = q$. But then

$$uu^* = pupu^*p = v^*(vuv^*)(vuv^*)^*v = v^*qv = p.$$

Thus p is finite.

Now assume $p \leq q$. If $u \in M$ is such that $u^*u = p$ and $uu^* \leq p$, then for $w = u + (q - p)$ we have

$$w^*w = u^*u + u^*(q-p) + (q-p)u + (q-p) = p + (q-p) = q,$$

and

$$ww^* = uu^* + u(q-p) + (q-p)u^* + (q-p) = uu^* + (q-p) \leq q.$$

Since q is finite, we have $uu^* + (q-p) = q$ or $uu^* = p$. Thus p is finite. In general, if $p \preceq q$, then there exists $q_0 \in \mathcal{P}(M)$ such that $p \sim q_0 \leq q$. By the two previous arguments we see that p is finite.

Finally, if q is purely infinite then it has no finite subprojections. If $p \preceq q$ had a finite subprojection $p_0 \leq p$, then $p_0 \preceq q$. In particular, $p_0 \sim q_0 \leq q$, which is finite by the above arguments, a contradiction. □ 3/6/2017

Proposition 5.2.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A projection $p \in \mathcal{P}(M)$ is semi-finite if and only if it is a supremum of finite projections. In particular, the supremum of semi-finite projections is again semi-finite. Moreover, any subprojection of a semi-finite projection is also semi-finite.*

Proof. If $p \in \mathcal{P}(M)$ is semi-finite, then by definition it is the sum (hence supremum) of an pairwise orthogonal finite projections. Conversely, suppose $p = \bigvee_i p_i$ for $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ finite projections. Let $\{q_j\}_{j\in J}$ be a maximal family of pairwise orthogonal finite subprojections of p. Suppose, towards a contradiction, that $q := p - \sum_{j\in J} q_j \neq 0$. Then, by definition of the supremum, there exists $i \in I$ so that q and p_i are not orthogonal. In particular, they are not centrally orthogonal and so by Proposition 5.1.12 there exists non-zero $q_0 \leq q$ so that $q_0 \preceq p_i$. Thus q_0 is finite by Proposition 5.2.5, which contradicts the maximality of $\{q_j\}_{j\in J}$. The final observation follows from the fact that the above argument also works if $p \leq \bigvee_i p_i$. □

Corollary 5.2.7. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}(M)$, if q is semi-finite and $p \preceq q$, then p is semi-finite.*

Proof. Suppose $p \sim q$. Let $v \in M$ be such that $v^*v = p$ and $vv^* = q$. Since q is semi-finite, there exists a family $\{q_i\}_{i\in I}$ of pairwise orthogonal finite projections such that $q = \sum_i q_i$. For each $i \in I$, define $p_i := v^*q_iv$. It easily follows that $\{p_i\}_{i\in I}$ is a pairwise orthogonal family such that $p = \sum_i p_i$. For each $i \in I$ we have that $p_i \sim q_i$ via the partial isometry vp_i, and consequently p_i is finite by Proposition 5.2.5. Thus p is semi-finite.

If $p \prec q$, then $p \sim q_0$ for some $q_0 \leq q$. By Proposition 5.2.6, q_0 is semi-finite and so by the above p is semi-finite. □

Corollary 5.2.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p \in \mathcal{P}(M)$ is semi-finite (resp. purely infinite), then the central support $z(p)$ is also semi-finite (resp. purely infinite).*

Proof. Suppose p is semi-finite. By Lemma 5.1.7, $[xp\mathcal{H}] \sim [px^*\mathcal{H}] \leq p$ for all $x \in M$. Thus $[xp\mathcal{H}]$ is semi-finite by Corollary 5.2.7. Since

$$z(p) = \bigvee_{x\in M} [xp\mathcal{H}]$$

by Lemma 5.1.10, $z(p)$ is semi-finite as the supremum of semi-finite projections.

Now suppose p is purely infinite. If $q \leq z(p)$ is a non-zero finite projection, then p and q are not centrally orthogonal. Thus Proposition 5.1.12 implies there exists non-zero $p \geq p_0 \sim q_0 \leq q$. But then p_0 is finite by Proposition 5.2.5, contradicting p being purely infinite. Thus $z(p)$ has no non-zero finite subprojections; that is, $z(p)$ is purely infinite. □

Lemma 5.2.9. *If $M \subset \mathcal{B}(\mathcal{H})$ is a properly infinite von Neumann algebra, then there exists $p \in \mathcal{P}(M)$ such that $p \sim (1-p) \sim 1$.*

Proof. We first claim that for any non-zero $z \in \mathcal{P}(\mathcal{Z}(M))$, there exists $r \in \mathcal{P}(M)$ such that $r \leq z$ and $r \sim (z(r) - r) \sim z(r)$.

Fix a non-zero $z \in \mathcal{P}(\mathcal{Z}(M))$. By definition, z is not finite so there exists $v \in M$ such that $v^*v = z$ and $vv^* < 1$. Define $p_0 := z - vv^*$, and $p_n := v^n p_0 (v^*)^n$ for all $n \in \mathbb{N}$. It easily follows that $\{p_n\}_{n\geq 0}$ are pairwise orthogonal, equivalent projections. Let $\{q_i\}_{i\in I}$ be a maximal family of pairwise orthogonal, equivalent subprojections of z extending $\{p_n\}_{n\geq 0}$. Let $J \subset I$ be a subset with the same cardinality as I. Fix $j_0 \in J$, and note that by Lemma 5.1.6

$$\sum_{i\in I} q_i \sim \sum_{j\in J} q_j \sim \sum_{j\in J\setminus\{j_0\}} q_j$$

since J is infinite. Define $q_0 := z - \sum_{i\in I} q_i$, then by the comparison theorem there exists $z' \in \mathcal{P}(\mathcal{Z}(M))$ such that $q_0 z' \preceq q_{j_0} z'$ and $q_{j_0}(1 - z') \preceq q_0(1 - z')$. Observe that if $z' = 0$, then there exists $q_\infty \leq q_0$ such that $q_\infty \sim q_{j_0}$, which means $\{q_\infty\} \cup \{q_i\}_{i\in I}$ contradicts the maximality of $\{q_i\}_{i\in I}$. Thus $z' \neq 0$ and so

$$zz' = q_0 z' + \sum_{i\in I} q_i z' \sim q_0 z' + \sum_{j\in J\setminus\{j_0\}} q_j z' \preceq q_{j_0} z' + \sum_{j\in J\setminus\{j_0\}} q_j z' = \sum_{j\in J} q_j z' \leq zz'.$$

Consequently, $zz' \sim \sum_{j\in J} q_j z'$ by Proposition 5.1.5. Choosing J so that $I \setminus J$ also has the same cardinality as I, we have $r \sim (zz' - r) \sim zz'$ for $r := \sum_{j\in J} q_j z'$. We also have that $z(r) = zz'$. Indeed, $r \leq zz'$ so $z(r) \leq zz'$. On the other hand, there exists $u \in M$ so that $u^*u = r$ and $uu^* = zz'$. Thus

$$zz' = [zz'\mathcal{H}] = [uu^*\mathcal{H}] = [uru^*\mathcal{H}] \leq [ur\mathcal{H}] \leq z(r).$$

So $r \sim (z(r) - r) \sim z(r)$.

Let $\{r_k\}_{k\in K}$ be a maximal family of centrally orthogonal projections such that $r_k \sim (z(r_k) - r_k) \sim z(r_k)$. We must have $\sum_{k\in K} z(r_k) = 1$, since otherwise the above claim applied to $1 - \sum_k z(r_k)$ would contradict maximality. So $p := \sum_k r_k$ is the desired projection by Lemma 5.1.6. □ 3/10/2017

Proposition 5.2.10. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For finite $p, q \in \mathcal{P}(M)$, $p \vee q$ is finite.*

Proof. By Kaplansky's formula,

$$p \vee q - q \sim p - p \wedge q.$$

Thus $p \vee q - q \preceq p$ and is therefore finite by Proposition 5.2.5. In light of this, it suffices to consider the case when p and q are orthogonal, in which case $p \vee q = p + q$. By considering the compression $(p + q)M(p + q)$, it further suffices to suppose $p + q = 1$.

Let $\{z_i\}_{i\in I} \subset \mathcal{P}(\mathcal{Z}(M))$ be a maximal family of centrally orthogonal, finite, central projections. Let $z_0 = \sum_i z_i$, which is finite by Lemma 5.2.3. If $z_0 = 1$ then we are done, so by considering $(1 - z_0)p$ and $(1 - z_0)q$ in the compression $(1 - z_0)M$, we may assume $z_0 = 0$. That is, we may assume M is properly infinite.

Consequently, Lemma 5.2.9 implies there is a projection $r \in \mathcal{P}(M)$ such that $r \sim (1 - r) \sim 1$. By the comparison theorem, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that

$$z \cdot (p \wedge r) \preceq z \cdot (q \wedge (1 - r)) \qquad \text{and} \qquad (1 - z) \cdot (q \wedge (1 - r)) \preceq (1 - z) \cdot (p \wedge r)$$

Thus

$$zr = z \cdot (r - p \wedge r) + z \cdot (p \wedge r) \preceq z(r - p \wedge r) + z \cdot (q \wedge (1 - r)) = z(r - (p + q) \wedge r + q) = zq.$$

Thus $z \sim zr \preceq zq \leq q$ implies z is finite. Since we are assuming M is properly infinite, it follows that $z = 0$. Thus the comparison theorem above actually implies

$$(q \wedge (1 - r)) \preceq (p \wedge r),$$

so that

$$1 - r = [(1 - r) - q \wedge (1 - r)] + q \wedge (1 - r) \preceq [(1 - r) - q \wedge (1 - r)] + p \wedge r = 1 - q = p.$$

Thus $1 \sim (1 - r) \preceq p$ implies 1 is finite, a contradiction. □

Proposition 5.2.11. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $p, q \in \mathcal{P}(M)$ be equivalent finite projections. Then $1-p$ and $1-q$ are equivalent. Consequently, there exists a unitary $u \in M$ so that $upu^* = q$.*

Proof. By Proposition 5.2.10, $p \vee q$ is finite. By considering $(p\vee q)M(p\vee q)$, we may assume M is finite. By the comparison theorem, there exists $z \in \mathcal{Z}(\mathcal{P}(M))$ and $p_1, q_1 \in \mathcal{P}(M)$ such that

$$(1-p)z \sim q_1 \leq (1-q)z \qquad \text{and} \qquad (1-q)(1-z) \sim p_1 \leq (1-p)(1-z).$$

Then

$$z = (1-p)z + pz \sim q_1 + qz \leq (1-q)z + qz = z,$$

and

$$1-z = (1-q)(1-z) + q(1-z) \sim p_1 + p(1-z) \leq (1-p)(1-z) + p(1-z) = 1-z.$$

Since z and $1-z$ are finite, we must have $q_1 = (1-q)z$ and $p_1 = (1-p)(1-z)$. Thus

$$\begin{aligned} 1-p &= (1-p)z + (1-p)(1-z) = (1-p)z + p_1 \\ &\sim q_1 + (1-q)(1-z) = (1-q)z + (1-q)(1-z) = 1-q. \end{aligned}$$

Finally, let $v, u \in M$ be such that $v^*v = p$, $vv^* = q$, $u^*u = 1-p$, and $uu^* = (1-q)$. Then one can check that $v+u$ is a unitary element and that $(v+u)p(v+u)^* = vpv^* = q$. □

Definition 5.2.12. For $M \subset \mathcal{B}(\mathcal{H})$ a von Neuamnn algebra, $p \in \mathcal{P}(M)$ is **countably decomposable** if every family of non-zero pairwise orthogonal subprojections is countable. We say M is **countably decomposable** if $1 \in M$ is countably decomposable.

Note that if $\mathcal{H}$ is a separable Hilbert space, then $1 \in \mathcal{B}(\mathcal{H})$ is separable. Also note that if p is countably separable, then so is any subprojection.

Proposition 5.2.13. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a countably decomposable von Neumann algebra. If $p, q \in \mathcal{P}(M)$ are properly infinite projections such that $z(p) \leq z(q)$, then $p \preceq q$. In particular, if M is a factor then all properly infinite projections are equivalent.* 3/14/2017

Proof. We first make several reductions. By the comparison theorem, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that

$$zp \preceq zq \qquad \text{and} \qquad (1-z)q \preceq (1-z)p.$$

Thus it suffices to show $(1-z)q \sim (1-z)p$. Note that $(1-z)q$ and $(1-z)p$ are properly infinite with $z((1-z)p) = (1-z)z(p) \leq (1-z)z(q) = z((1-z)q)$. So by considering $(1-z)M(1-z)$, we may assume $q \preceq p$. Let $q_0 \in \mathcal{P}(M)$ be such that $q \sim q_0 \leq p$. Then q_0 is also properly infinite by Proposition 5.2.5, and $z(p) \leq z(q) = z(q_0)$ by Proposition 5.1.11. Thus we may assume $q \leq p$. Finally, by considering pMp we may assume $p = 1$.

Thus we assume M is a properly infinite, countably decomposable von Neumann algebra and let $q \in M$ be a properly infinite projection with $z(q) = 1$. We must show $q \sim 1$. By Lemma 5.2.9, there exists $q_0 \leq q$ such that $q_0 \sim (q - q_0) \sim q$. Let $v \in M$ be such that $v^*v = q$ and $vv^* = q - q_0$. Define $q_n := v^n q_0 (v^*)^n$ for each $n \in \mathbb{N}$. Then it is easy to see that $\{q_n\}_{n\geq 0}$ is a family or pairwise orthogonal, equivalent subprojections of q. Let $\{r_n\}_{n\geq 0}$ be a maximal family of pairwise orthogonal projections such that $r_n \preceq q$, which we note is countable by the assumption that M is countably decomposable. By Proposition 5.1.12 and the maximality of $\{r_n\}_{n\geq 0}$, we must have that $1 - \sum_n r_n$ and q are centrally orthogonal. Since $z(q) = 1$, we must have $\sum_n r_n = 1$. Since $r_n \preceq q \sim q_0 \sim q_n$ for each $n \in \mathbb{N}$, we have

$$1 = \sum_{n=0}^{\infty} r_n \preceq \sum_{n=0}^{\infty} q_n \leq q.$$

Hence $q \sim 1$ by Proposition 5.1.5. □

5.3 The Type Decomposition of a von Neumann Algebra

Definition 5.3.1. A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is said to be

- **type** I if every non-zero projection has a non-zero abelian subprojection.
- **type** II if it is semi-finite and has no non-zero abelian projections.
- **type** III if it is purely infinite.

Theorem 5.3.2. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then there exists unique, central, pairwise orthogonal projections $Z_{\mathrm{I}}, Z_{\mathrm{II}}, Z_{\mathrm{III}} \in \mathcal{Z}(M)$ such that $Z_{\mathrm{I}}+Z_{\mathrm{II}}+Z_{\mathrm{III}} = 1$ and MZ_T is type T for $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.*

Proof. Let Z_{I} be the supremum of all abelian projections in M. Since a unitary conjuation of an abelian projection is still an abelian projection, we have $uZ_{\mathrm{I}}u^* = Z_{\mathrm{I}}$ for all unitaries $u \in M$. Thus $Z_{\mathrm{I}} \in \mathcal{Z}(M)$. If $p \leq Z_{\mathrm{I}}$ is non-zero, then by definition of the supremum there exists an abelian projection $r \in M$ so that $pr \neq 0$. Consequently, p and r are not centrally orthogonal so there exists non-zero $p \geq p_0 \sim r_0 \leq r$. Proposition 5.2.5 implies that p_0 is abelian. Thus MZ_{I} is type I.

Next, let Z_{II} be the supremum of all finite $p \in \mathcal{P}(M)$ such that $p \leq 1 - Z_{\mathrm{I}}$. By the same argument as above, we have $Z_{\mathrm{II}} \in \mathcal{Z}(M)$. Also, Z_{II} is semi-finite by Proposition 5.2.6. Since $Z_{\mathrm{II}} \leq 1 - Z_{\mathrm{I}}$, it has no non-zero abelian subprojections. Thus MZ_{II} is type II.

Finally, we let $Z_{\mathrm{III}} = 1 - Z_{\mathrm{I}} - Z_{\mathrm{II}}$, which has no finite subprojections because it lies under $1 - Z_{\mathrm{II}}$. Thus MZ_{III} is type III.

Towards showing this decomposition is unique, suppose $P_{\mathrm{I}}, P_{\mathrm{II}}, P_{\mathrm{III}}$ are central, pairwise orthogonal projections summing to one and such that MP_R is type R for $R \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Then $P_{\mathrm{III}}Z_{\mathrm{I}}$ and $P_{\mathrm{III}}Z_{\mathrm{II}}$ are both finite and purely infinite by Proposition 5.2.5. That is, $P_{\mathrm{III}}Z_{\mathrm{I}} = P_{\mathrm{III}}Z_{\mathrm{II}} = 0$, and consequently $P_{\mathrm{III}} \leq Z_{\mathrm{III}}$. Reversing the roles of Z and P yields $P_{\mathrm{III}} = Z_{\mathrm{III}}$. Next, $P_{\mathrm{II}}Z_{\mathrm{I}} = 0$ as an abelian subprojection of P_{II}. Thus $P_{\mathrm{II}} \leq Z_{\mathrm{II}}$ and by symmetry we have $P_{\mathrm{II}} = Z_{\mathrm{II}}$. Finally

$$P_{\mathrm{I}} = 1 - P_{\mathrm{II}} - P_{\mathrm{III}} = 1 - Z_{\mathrm{II}} - Z_{\mathrm{III}} = Z_{\mathrm{I}}.$$

So the decomposition is unique. □

Corollary 5.3.3. *A factor is either type* I*, type* II*, or type* III. 3/15/2017

Theorem 5.3.4. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For $p \in \mathcal{P}(M)$ with $z(p) = 1$ and $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, M is type T if and only if pMp is type T.*

Proof. Suppose M is type I. Since $\mathcal{P}(pMp) \subset \mathcal{P}(M)$, we immediately have that pMp is type I. Conversely, suppose pMp is type I. If $q \in \mathcal{P}(M)$, then $z(p) = 1$ implies p and q are not centrally orthogonal. Thus there exists non-zero projections $p \geq p_0 \sim q_0 \leq q$. Since $p_0 \in pMp$, it has a an abelian subprojection and consequently so does q_0. This abelian projection is also a subprojection of q, and so we have that M is type I.

Suppose M is type II. Then $p \leq 1$ is semi-finite and has no non-zero abelian subprojections. Thus pMp is type II. Conversely, suppoe pMp is type II. Then p is semi-finite, and consequently so is $1 = z(p)$ by Corollary 5.2.8. By the same argument as above, $q \in \mathcal{P}(M)$ cannot have any abelian subprojections because p does not have any. Thus M is type II.

Finally, suppose M is type III. Then $p \leq 1$ has no non-zero finite subprojections and so pMp is type III. Conversely, suppose pMp is type III. Corollary 5.2.8 once more implies $1 = z(p)$ is purely finite and so M is type III. □

We next want to show that a von Neumann algebra and its commutant always have the same type. We first require some technical results.

Lemma 5.3.5. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic vector $\xi \in \mathcal{H}$. Then for each $\eta \in \mathcal{H}$, there exists $x, y \in M$, with $x \geq 0$, and $\zeta \in \overline{x\mathcal{H}}$ such that $x\zeta = \xi$ and $y\zeta = \eta$.*

Proof. Fix $\eta \in \mathcal{H}$ and assume $\|\xi\|, \|\eta\| \leq 1$. As ξ is cyclic, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset M$ so that

$$\left\| \eta - \sum_{n=1}^{k} x_n \xi \right\| \leq \frac{1}{4^k}.$$

Define

$$h_k := \sqrt{1 + \sum_{n=1}^{k} 4^n x_n^* x_n}.$$

Then (h_k) is an increasing sequence of positive, invertible elements so that (h_k^{-1}) is a uniformly bounded, decreasing sequence of positive elements. By Lemma 3.1.1, (h_k^{-1}) converges strongly to some positive x such that $x \leq h_k^{-1}$ for all $k \in \mathbb{N}$. We will show this is the desired x.

Note that for $n \in \mathbb{N}$,

$$\|x_n\xi\| \leq \left\| \eta - \sum_{\ell=1}^{n} x_\ell \xi \right\| + \left\| \sum_{\ell=1}^{n-1} x_\ell \xi - \eta \right\| \leq \frac{1}{4^n} + \frac{1}{4^{n-1}} < \frac{2}{4^{n-1}}.$$

Thus for $k \in \mathbb{N}$ we have

$$\|h_k\xi\|^2 = \left\langle h_k^2 \xi, \xi \right\rangle = \|\xi\|^2 + \sum_{n=1}^{k} 4^n \|x_n \xi\|^2 \leq 1 + \sum_{n=1}^{k} \frac{1}{4^{n-3}} \leq 1 + \frac{64}{3}$$

Thus $\{h_k\xi\}_{k\in\mathbb{N}}$ is bounded and therefore (by the Banach–Alaoglu theorem) has a weak cluster point $\zeta \in \mathcal{H}$.

Let us show $x\zeta = \xi$. Fix $\xi_0 \in \mathcal{H}$ and $\epsilon > 0$, and let $k \in \mathbb{N}$ be such that $|\langle \zeta - h_k\xi, x\xi_0 \rangle| < \frac{\epsilon}{2}$ and $\|(x - h_k^{-1}))\xi_0\| < \frac{\epsilon}{2\sup_k \|h_k\xi\|}$. Then

$$\begin{aligned}
|\langle x\zeta - \xi, \xi_0 \rangle| &= |\langle \zeta - h_k\xi + h_k\xi, x\xi_0 \rangle - \langle \xi, \xi_0 \rangle| \\
&\leq \frac{\epsilon}{2} + |\langle h_k\xi, x\xi_0 \rangle - \left\langle h_k\xi, h_k^{-1}\xi_0 \right\rangle| \\
&= \frac{\epsilon}{2} + |\left\langle h_k\xi, (x - h_k^{-1})\xi_0 \right\rangle| \\
&\leq \frac{\epsilon}{2} + \|h_k\xi\| \frac{\epsilon}{2\sup_k \|h_k\xi\|} < \epsilon.
\end{aligned}$$

Thus $x\zeta = \xi$. Note that since x is self-adjoint, we have $\overline{\operatorname{ran}(x)} = \ker(x)^\perp$. So if we replace ζ with $[x\mathcal{H}]\zeta$, we still have $x\zeta = \xi$.

Since (h_m^{-1}) is uniformly bounded, $(h_m^{-1} 4^k x_k^* x_k h_m^{-1})_m$ converges strongly to $4^k x x_k^* x_k x$ for each $k \in \mathbb{N}$ by Lemma 4.7.1. Also, for $m > k \in \mathbb{N}$ we have

$$0 \leq h_m^{-1} 4^k x_k^* x_k h_m^{-1} \leq h_m^{-1} \left(1 + \sum_{n=1}^{m} 4^n x_n^* x_n \right) h_m^{-1} = 1.$$

Thus $\|x_k x\|^2 = \|x x_k^* x_k x\| \leq 4^{-k}$. We therefore define

$$y := \sum_{k=1}^{\infty} x_k x,$$

so that

$$y\zeta = \sum_{k=1}^{\infty} x_k x \eta = \sum_{k=1}^{\infty} x_k \xi = \eta.$$

□

Proposition 5.3.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with two cyclic vectors $\xi, \eta \in \mathcal{H}$. Then $[M'\xi] \sim [M'\eta]$ in M.*

Proof. We first note that these projections lie in M since the subspaces are reducing for M'. By the previous lemma there exists $x, y \in M$, with $x \geq 0$, and $\zeta \in \overline{x\mathcal{H}}$ so that $x\zeta = \xi$ and $y\zeta = \eta$. Let $p = [M'\zeta]$, then $\zeta \in \overline{px\mathcal{H}}$ since $\zeta \in \overline{x\mathcal{H}}$ and $p\zeta = \zeta$. It follows that

$$p \leq [M'px\mathcal{H}] \leq [px\mathcal{H}] \leq [p\mathcal{H}] = p.$$

Thus $p = [px\mathcal{H}] \sim [xp\mathcal{H}] = [xM'\zeta] = [M'\xi]$. Now, $[M'\eta] = [M'y\zeta] = [yM'\zeta] = [yp\mathcal{H}] \sim [py\mathcal{H}] \leq p \sim [M'\xi]$. By symmetry we also have $[M'\xi] \preceq [M'\eta]$ and therefore obtain equivalence. □

Proposition 5.3.7. *Let $M \subset \mathcal{B}(\mathcal{H})$. For any $\xi, \eta \in \mathcal{H}$, $[M'\xi] \preceq [M'\eta]$ in M if and only if $[M\xi] \preceq [M\eta]$ in M'.*

Proof. Note that by the bicommutant theorem, it suffices to proof the "only if" direction. We first show this holds for equivalence, rather than subequivalence. Suppose there exists $v \in M$ so that $v^*v = [M'\xi]$ and $vv^* = [M'\eta]$. Then

$$[M'\eta] = F(v) = [v\mathcal{H}] = [vv^*v\mathcal{H}] = [vM'\xi] = [M'v\xi].$$

and

$$[Mv\xi] \leq [M\xi] = [Mv^*v\xi] \leq [Mv\xi],$$

so that $[Mv\xi] = [M\xi]$. So by replacing ξ with $v\xi$, we may assume $[M'\eta] = [M'\xi]$.

Now, the central support z of $[M\eta]$ in M' is

$$\bigvee_{y \in M'} [yM\eta] = [M'M\eta] = [MM'\eta] = [MM'\xi] = [M'M\xi].$$

That is, z is the central support of $[M\eta]$ and $[M\xi]$ in M', and the central support of $[M'\eta] = [M'\xi]$ in M. Thus, replacing M with Mz, we may assume all the relevant projections have central support equal to 1. 3/17/2017

Denote $p := [M'\xi] = [M'\eta] \in M$. Since $z(p) = 1$ we have, that $M' \ni y \mapsto yp \in M'p$ is a $*$-algebra isomorphism. Indeed, if $yp = 0$ then for any $x \in M$ and $z \in M'$ we have

$$yxz\xi = yxpz\xi = xypz\xi = 0,$$

and vectors of the form $xz\xi$ are dense since $z(p) = 1$. Consequently, $[M\xi]$ and $[M\eta]$ are equivalent in M' if and only if $[M\xi]p$ and $[M\eta]p$ are equivalent in $M'p$. Observe that $p\xi$ and $p\eta$ are cyclic vectors for $M'p$, and so we obtain the desired equivalence by the previous proposition.

Now suppose $[M'\xi] \sim q \leq [M'\eta]$ for $q \in M$. Then $q = [M'q\eta]$ and so by the above we get $[M\xi] \sim [Mq\eta] \leq [M\eta]$. □

Lemma 5.3.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra with a cyclic and separating vector. Then M' is also finite.*

Proof. Let $\xi \in \mathcal{H}$ be a cyclic and separating vector for M. Then by Proposition 3.2.3, ξ is cyclic and separating for M'. We proceed by contrapositive and assume that M' is not finite.

Let $\{z_i\}_{i\in I} \subset \mathcal{P}(\mathcal{Z}(M'))$ be a maximal family of pairwise orthogonal finite projections. If $z_0 := \sum_i z_i$, then z_0 is a maximal, central, finite projection. Indeed, if $z \in \mathcal{P}(\mathcal{Z}(M'))$ is finite, then $z - zz_0 \leq z$ is a finite projection orthogonal to z_0, so we must have $z - zz_0 = 0$ otherwise the maximality of $\{z_i\}_{i\in I}$ is contradicted. Thus $z = zz_0 \leq z_0$ and so z_0 is maximal. Now, $(1 - z_0)\xi$ is cyclic for both $M(1 - z_0)$ and $M'(1 - z_0)$. Therefore $(1 - z_0)\xi$ is cyclic and separating for $M(1 - z_0)$ by Proposition 3.2.3. Thus, by replacing M with $M(1 - z_0)$ we may assume $\mathcal{Z}(M')$ contains no finite projections; that is, M' is properly infinite.

We next claim that there are no non-zero $z \in \mathcal{P}(\mathcal{Z}(M))$ such that Mz is abelian. Indeed, for such a projection $z\xi$ is cyclic for Mz, so by Theorem 3.2.5 $Mz \cong L^\infty(X, \mu)$ for some finite measure space (X, μ). In particular, as shown in Section 2.4.2, $Mz = (Mz)'$, which is $M'z$ by the proof of Theorem 2.5.7. In particular, $z \in \mathcal{P}(\mathcal{Z}(M'))$ is an abelian projection which contradicts M' being properly infinite. Now, let $\{p_i\}_{i\in I} \subset M$ be a maximal family of centrally orthogonal projections such that $p_i < z(p_i)$. Let $p := \sum_i p_i$, which we note satisfies $p < \sum_i z(p_i) = z(p)$. By maximality, every subprojection of $1 - z(p)$ in M is equal to its own central support. That is, all $M(1 - z(p))$ is abelian. So by the previous argument, it follows that $z(p) = 1$.

Let $q = [Mp\xi] \in \mathcal{P}(M')$. Since ξ is separating for M, it is easily seen that 1 is countably decomposable. So by Proposition 5.2.13 we have

$$q \sim z(q) = [M'Mp\xi] = [MpM'\xi] = [Mp\mathcal{H}] = z(p) = 1.$$

Thus $[Mp\xi] \sim 1 = [M\xi]$ in M'. So by Proposition 5.3.7 we have $p = [pM'\xi] = [M'p\xi \sim [M'\xi] = 1$ in M. Since $p < 1$, this means M is not finite. $\square$

Theorem 5.3.9. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, M is type T if and only if M' is type T.*

Proof. By the bicommutant theorem, it suffices to prove the "only if" direction for each T. First suppose M is type II, and let $q \in \mathcal{P}(M')$ be non-zero. Let $\xi \in \ker(q)^{\perp}$ and take a non-zero finite projection $p \leq [qM'q\xi] \in Mq$. Observe that $p\xi = pq\xi$ is cyclic and separating vector for pMp and $M'p$. If $M'p$ is abelian, then (as in the proof of the above lemma) we have that $M'p = (M'p)' = pMp$. However, M is type II and so p cannot be an abelian projection. In particular, $qM'q$ cannot be abelian otherwise $qM'qp = M'p$ would be. Furthermore, by Lemma 5.3.8, since pMp is finite so is $M'p$. Let $z(p)$ be the central support of p in Mq. Then we claim $M'z(p)$ finite. Note that by the usual argument,

$$M'z(p) \ni yz(p) \mapsto yp \in M'p$$

is an $*$-isomorphism. So $v \in M'z(p)$ satisfies $v^*v = z(p)$ and $vv^* \leq z(p)$, then $(vp)^*vp = p$ and $vp(vp)^* \leq p$. Since p is finite in $M'p$, we have $vp(vp)^* = p$. In particular, $(z(p) - vv^*)p = 0$ and so $z(p) - vv^* = 0$. Thus $z(p)$ is finite in $M'z(p)$, and we have shown q has a finite subprojection. We have shown that M' has no abelian projections (so it has no type I direct summand) and that any projection has a non-zero finite subprojection (so it has no type III direct summand). Thus it must be that M' is type II. 3/21/2017

Next, suppose M is type III. If $q \in M'$ is a non-zero finite projection, then the above argument (applied to $qM'q$ and $M'q$ rather than M and M') would imply that Mq is semi-finite, a contradiction. Thus M' has no non-zero finite projections and is therefore type III.

Finally, if M is type I, then the above arguments imply that M' cannot have a non-trivial type II or type III direct summand. Thus M is type I. $\square$

We now proceed to examine type I and II von Neumann algebras more closely, producing a further type decomposition in each of these classes. We will not be able to cover it in these notes, but the class of type III factors can also be further decomposed into types III_λ for $\lambda \in [0, 1]$. This classification requires very deep results known as *Tomita–Takesaki theory*. Essentially, von Neumann algebras of this type can be written as crossed products via the non-commutative analogue of the group measure space construction, and this crossed product reveals the parameter $\lambda \in [0, 1]$.

5.3.1 Type I von Neumann Algebras

Definition 5.3.10. For a cardinal n, a type I von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is said to be **type** I_n if $1 \in M$ is a sum of n equivalent non-zero abelian projections. We say M is **type** I_∞ if it is type I_n for an infinite cardinal n.

In this section we will previous a decomposition theorem and classification theorem for type I von Neumann algebras. We must first prove some additional results about abelian projections.

Lemma 5.3.11. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{P}(M)$ with central support $z(p)$ in M. Then*

$$M'z(p) \ni yz(p) \mapsto yp \in M'p$$

is a $*$-algebra isomorphism. In particular, $\mathcal{Z}(pMp) = \mathcal{Z}(Mz(p))p$.

Proof. Recall that $z(p) = [Mp\mathcal{H}]$. If $yp = 0$, then for all $x \in M$ and $\xi \in \mathcal{H}$ we have

$$yxp\xi = xyp\xi = 0.$$

Thus $yz(p) = 0$. So the map $yz(p) \mapsto yp$ is injective, and it easily follows that it is a $*$-algebra isomorphism.

Now, using the proof of Theorem 2.5.7 we have

$$\mathcal{Z}(pMp) = \mathcal{Z}(M'p).$$

By the first part of the proof, $yp \in \mathcal{Z}(M'p)$ if and only if $yz(p) \in \mathcal{Z}(M'z(p)) = \mathcal{Z}(Mz(p))$. So the claimed equality follows. $\square$

We have the following immediate corollary:

Corollary 5.3.12. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p \in \mathcal{P}(M)$ has $z(p) = 1$, then $\mathcal{Z}(pMp) = \mathcal{Z}(M)p$.*

Lemma 5.3.13. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p, q \in \mathcal{P}(M)$ are abelian projections such that $z(p) = z(q)$, then $p \sim q$.*

Proof. First, assume $p \preceq q$ and let $v \in M$ be such that $v^*v = p$ and $vv^* = q_0 \leq q$. Since q is abelian we have, by Lemma 5.3.11, that

$$qMq = \mathcal{Z}(qMq) = \mathcal{Z}(Mz(q))q$$

Since $q_0 \in qMq$, we know $q_0 = zq$ for some $z \in \mathcal{Z}(Mz(q))$. Using Proposition 5.1.11 we have

$$z(p) = z(q_0) = zz(q) \leq z(q) = z(p).$$

Thus $z(q_0) = z(p) = z(q)$, so that $zz(q) = z(q_0) = z(q)$. That is, $z(q) \leq z$. Thus $z = z(q)$ and $q_0 = q$, which means $p \sim q$.

Now, without assuming $p \preceq q$, the comparison theorem implies there exists $z \in \mathcal{Z}(M)$ such that

$$pz \preceq qz \qquad \text{and} \qquad q(1-z) \preceq p(1-z).$$

Note that $z(pz) = z(p)z = z(q)z = z(qz)$, and so the above argument implies $pz \sim qz$. Similarly, $q(1-z) \sim p(1-z)$. Thus $p \sim q$. $\square$

Lemma 5.3.14. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If n and m are cardinals such that M is both type I_n and I_m, then $n = m$.*

Proof. Let $\{p_i\}_{i\in I}$ and $\{q_j\}_{j\in J}$ be families of pairwise orthogonal, equivalent, abelian projections such

$$\sum_{i\in I} p_i = \sum_{j\in J} q_j = 1.$$

Let $n = |I|$ and $m = |J|$. Recall that $z(p_i)$ constant for $i \in I$ by Proposition 5.1.11. Thus

$$1 - z(p_i) = (1 - z(p_i)) \sum_{k\in I} p_k = 0,$$

so $z(p_i) = 1$ for all $i \in I$. Similarly, $z(q_j) = 1$ for all $j \in J$. So Lemma 5.3.13 implies $p_i \sim q_j$ for all $i \in I$, $j \in J$.

Now, suppose $n < \infty$ and fix $i_0 \in I$. For each $i \in I$ and $j \in J$, let $v_i, u_j \in M$ be such that $v_i^* v_i = u_j^* u_j = p_{i_0}$, $v_i v_i^* = p_i$, and $u_j u_j^* = q_j$. Consider the map

$$\Phi(x) = \frac{1}{n} \sum_{i\in I} v_i^* x v_i \qquad x \in M.$$

We claim that $\Phi(xy) = \Phi(yx)$ for $x, y \in M$. Indeed,

$$\begin{aligned}\Phi(xy) &= \frac{1}{n}\sum_{i\in I} v_i^* xy v_i = \frac{1}{n}\sum_{i,k\in I} v_i^* x v_k v_k^* y v_i \\ &= \frac{1}{n}\sum_{i,k\in I} (p_{i_0} v_i^* x v_k p_{i_0})(p_{i_0} v_k^* y v_i p_{i_0}) \\ &= \frac{1}{n}\sum_{i,k\in I} (p_{i_0} v_i^* y v_k p_{i_0})(p_{i_0} v_k^* x v_i p_{i_0}) = \Phi(yx).\end{aligned}$$

Consequently $\Phi(p_i) = \Phi(q_j) = \Phi(p_{i_0})$ for all $i \in I$ and $j \in J$ by equivalence, and it is easy to see that their common image is p_{i_0}. Thus we have

$$np_{i_0} = \sum_{i\in I} \Phi(p_i) = \Phi\left(\sum_{i\in I} p_i\right) = \Phi(1) = \Phi\left(\sum_{j\in J} q_j\right) = \sum_{j\in J} \Phi(q_j) = mp_{i_0}.$$

So $n = m$. 3/21/2017

Next, suppose n is infinite. We must have m infinite, otherwise the above argument would imply n is finite. Let $\varphi \in \mathcal{Z}(M)_*$ be a normal state with support $z \in \mathcal{Z}(M)$; that is, $w \in \ker(\varphi)$ if and only if $zwz = 0$. For each $i \in I$, define $\sigma_i \colon \mathcal{Z}(M) \to p_i M p_i$ by

$$\sigma_i(z) = zp_i.$$

By Lemma 5.3.11, this is an isomorphism since $p_iMp_i = \mathcal{Z}(p_iMp_i) = \mathcal{Z}(M)p_i$. **Fact:** σ_i and σ_i^{-1} are automatically σ-WOT continuous. Define $\varphi_i \in M_*$ by

$$\varphi_i(x) = \varphi(\sigma_i^{-1}(p_ixp_i)).$$

Note that φ_i is positive and in particular a state. Also, φ_i has support zp_i. Now, by σ-WOT continuity we have

$$1 = \varphi_i(1) = \varphi_i\left(\sum_{j\in J} q_j\right) = \sum_{j\in J} \varphi_i(q_j).$$

Since $\varphi_i(q_j) \geq 0$ for each $j \in J$, the set

$$J_i := \{j \in J \colon \varphi_i(q_j) > 0\}$$

is countable. Observe that if $j \notin J_i$, then $\varphi_i(q_j) = 0$ which means $(zp_i)q_j(zp_i) = 0$, or $|q_jp_iz|^2 = zp_iq_jp_iz = 0$. Thus $q_jp_iz = 0$. However, for any $j \in J$

$$\sum_{i\in I} q_jp_iz = q_jz,$$

which is non-zero since $z(q_j) = 1$. Thus $q_jp_iz \neq 0$ for some $i \in I$, which means $j \in J_i$ for some $i \in I$. That is,

$$J = \bigcup_{i\in I} J_i.$$

Thus $m \leq n \cdot \aleph_0 = n$. By symmetry, we have $n = m$. □

Proposition 5.3.15. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a type* I *von Neumann algebra. Then M can be uniquely decomposed as a direct sum of type* I_n *von Neumann algebras.*

Proof. Fix a cardinal n. We say $z \in \mathcal{P}(\mathcal{Z}(M))$ is *n-homogeneous* if z is a sum of n pairwise orthogonal abelian projections each with central support equal to z. Observe that if $\{z_i\}_{i\in I} \subset \mathcal{P}(\mathcal{Z}(M))$ is a pairwise orthogonal family of n-homogeneous, then $z := \sum_i z_i$ is also n-homogeneous. Indeed, let J be an index set of cardinality n, then for each $i \in I$ let $\{p_j^{(i)}\}_{j\in J}$ be a family of pairwise orthogonal abelian projections that sum to z_i and each with central support z_i. For each $j \in J$, define

$$p_j := \sum_{i\in I} p_j^{(i)}.$$

Since the family $\{p_j^{(i)}\}_{i\in I}$ is centrally orthogonal, we have that p_j is abelian by Lemma 5.2.3 and

$$z(p_j) = \sum_{i\in I} z(p_j^{(i)}) = \sum_{i\in I} z_i = z.$$

Since $z = \sum_j p_j$, we see that z is indeed n-homogeneous.

For each cardinal n, let $\{z_i\}_{i\in I}$ be a *maximal* family of pairwise orthogonal n-homogeneous projections, and denote $Z_n := \sum_i z_i$. Then MZ_n is a type I_n von Neumann algebra since Z_n is n-homogeneous. For distinct ardinals n and m it is easy to see that Z_nZ_m is both n-homogeneous and m-homogeneous. So by Lemma 5.3.14, $Z_nZ_m = 0$. Thus $\sum_n MZ_n$ is a direct sum decomposition. To see that this is all of M, it suffices to show that for any non-zero $z \in \mathcal{P}(\mathcal{Z}(M))$, there exists a non-zero n-homogeneous subprojection for some cardinal n.

Let $p \leq z$ be an abelian subprojection such that $z(p) = z$ (which exists by a straightfoward maximality argument). Let $\{p_i\}_{i\in I}$ be a maximal family of pairwise orthogonal abelian projections such that $z(p_i) = z(p) = z$ for each $i \in I$. Consequently, $p_i \sim p$ for each $i \in I$ by Lemma 5.3.13. Denote $q := \sum_i p_i \leq z$. By the comparison theorem, there is $z_1 \in \mathcal{P}(\mathcal{Z}(M))$ such that

$$(z - q)z_1 \preceq pz_1 \qquad \text{and} \qquad p(1 - z_1) \preceq (z - q)(1 - z_1).$$

Observe that we cannot have $p \preceq z - q$ since this would contradict the maximality of $\{p_i\}_{i\in I}$. Thus z_1 above is non-zero. Let $(z - q)z_1 \sim q_1 \leq pz_1$. If $q_1 = 0$, then $(z - q)z_1 = 0$ and hence

$$\sum_{i\in I} p_i z_1 = zz_1.$$

Also, $z(pz_1) = z(p)z_1 = zz_1$, so zz_1 is an $|I|$-homogeneous subprojection of z. If $q_1 \neq 0$, then $q_1 \in pMp = \mathcal{Z}(pMp) = \mathcal{Z}(Mz)p$. In particular, $q_1 = z(q_1)p$. It follows that

$$z(q_1) = z(q_1)(q + z - q) = \sum_{i\in I} p_i z(q_1) + (z - q)z(q_1).$$

We also have $z(p_i z(q_1)) = z(q_1)$, and $z((z - q)z(q_1)) = z((z - q)z_1)z(q_1) = z(q_1)z(q_1) = z(q_1)$. Thus, $z(q_1)$ is an $(|I| + 1)$-homogeneous subprojection of z.

Finally, the uniqueness of the decomposition follows from Lemma 5.3.14. □ 3/22/2017

Corollary 5.3.16. *A type* I *factor is type* I_n *for exactly one cardinal* n.

Theorem 5.3.17. *Let* $M \subset \mathcal{B}(\mathcal{H})$ *be a type* I_n *von Neumann algebra. Then* $M \cong \mathcal{Z}(M)\bar{\otimes}\mathcal{B}(\mathcal{K})$ *for a Hilbert space* $\mathcal{K}$ *with* $\dim(\mathcal{K}) = n$. *In particular, all abelian von Neumann algebras are type* I_1, *and if* M *is a type* I_n *factor, then* $M \cong \mathcal{B}(\mathcal{K})$.

Proof. We will only consider $n < \infty$, in which case $\mathcal{Z}(M)\bar{\otimes}\mathcal{B}(\mathcal{K}) = \mathcal{Z}(M) \otimes \mathcal{B}(\mathcal{K})$ by Lemma 2.5.4.

Let $\{p_i\}_{i\in I}$ be a family of pairwise orthogonal, equivalent, abelian projections such that

$$\sum_{i\in I} p_i = 1$$

and $|I| = n$. Recall that in the proof of Lemma 5.3.14 it was shown that $z(p_i) = 1$ for all $i \in I$.

Fix $i_0 \in I$. For each $i \in I$, let $v_i \in M$ be such that

$$v_i^* v_i = p_{i_0}$$
$$v_i v_i^* = p_i$$

Then for each pair $i, j \in I$, define $e_{ij} = v_i v_j^*$. Note that $e_{ii} = p_i$ and $e_{ij} = p_i e_{ij} p_j$ for each $i, j \in I$. So it follows that for $i, j, k, l \in I$:

$$e_{ij}^* = e_{ji}$$
$$e_{ij}e_{kl} = \delta_{j=k} e_{il}$$
$$\sum_{i\in I} e_{ii} = 1$$

A collection of operators satisfying these properties is called a *system of matrix units (of size* n*)*. From these relations, it is clear that $W^*(e_{ij} : i, j \in I) \cong M_n(\mathbb{C}) = \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ with $\dim(\mathcal{K}) = n$.

For $x \in M$ and $i, j \in I$, observe that $v_i^* x v_j = p_{i_0} v_i^* x v_j p_{i_0} \in p_{i_0} M p_{i_0}$. Since p_{i_0} is abelian with $z(p_{i_0}) = 1$, Lemma 5.3.11 implies

$$p_{i_0} M p_{i_0} = \mathcal{Z}(p_{i_0} M p_{i_0}) = \mathcal{Z}(M) p_{i_0}.$$

Let $x_{ij} \in \mathcal{Z}(M)$ be such that $x_{ij} p_{i_0} = v_i^* x v_j$. Then we have

$$x = \left(\sum_{i \in I} p_i\right) x \left(\sum_{j \in J} p_j\right) = \sum_{i,j \in I} v_i v_i^* x v_j^* v_j = \sum_{i,j \in I} v_i x_{ij} p_{i_0} v_j^* = \sum_{i,j \in I} x_{ij} e_{ij}.$$

Thus the map

$$\mathcal{Z}(M) \otimes \mathcal{B}(\mathcal{K}) \ni \sum_{i,j \in I} z_{ij} \otimes e_{ij} \mapsto \sum_{i,j \in I} z_{ij} e_{ij} \in M$$

is a surjective $*$-homomorphism. To see that it is injective, suppose

$$x := \sum_{i,j \in I} z_{ij} \otimes e_{ij} = \sum_{i,j \in I} w_{ij} \otimes e_{ij}$$

for some $z_{ij}, w_{ij} \in \mathcal{Z}(M)$, $i, j \in I$. Then $z_{ij} p_{i_0} = v_i^* x v_j = w_{ij} p_{i_0}$ for each $i, j \in I$. But then Lemma 5.3.11 implies $z_{ij} = w_{ij}$ for each $i, j \in I$. Thus we have the claimed isomorphism. $\square$

5.3.2 Type II von Neumann Algebras

Definition 5.3.18. A type II von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is said to be **type** II_1 if it is finite, and is said to be **type** II_∞ if M is properly infinite.

Theorem 5.3.19. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a type* II *von Neumann algebra. Then there exists unique, central, pairwise orthogonal projections $Z_{\mathrm{II}_1}, Z_{\mathrm{II}_\infty} \in \mathcal{P}(\mathcal{Z}(M))$ satisfying $Z_{\mathrm{II}_1} + Z_{\mathrm{II}_\infty} = 1$, and such that MZ_{II_1} is type II_1 and MZ_{II_∞} is type II_∞.*

Proof. Let $\{z_i\}_{i \in I}$ be a maximal family of pairwise orthogonal central finite projections. By Lemma 5.2.3, $Z_{\mathrm{II}_1} := \sum_i z_i$ is finite. Thus MZ_{II_1} is type II_1.

Define $Z_{\mathrm{II}_\infty} := 1 - Z_{\mathrm{II}_1}$. Then by maximality of $\{z_i\}_{i \in I}$, Z_{II_∞} has no finite central subprojections. That is, Z_{II_∞} is properly infinite. Thus MZ_{II_∞} is type II_∞. $\square$

5.3.3 Examples

We revisit some of the examples from Section 2.4 and determine their type.

Example 5.3.20. By Theorem 5.3.17, $\mathcal{B}(\mathcal{H})$ is type $\mathrm{I}_{\dim(\mathcal{H})}$, $M_n(\mathbb{C})$ is type I_n, and $L^\infty(X, \mu)$ is type I_1.

Example 5.3.21. Let Γ be a discrete group. If Γ is abelian, then $L(\Gamma)$ is abelian and hence is type I_1.

Suppose Γ is an i.c.c. group, so that $L(\Gamma)$ is a factor. We claim that $L(\Gamma)$ is in in fact a type II_1 factor. Indeed, $L(\Gamma)$ cannot be a type I factor as this would imply $L(\Gamma) \cong \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and consequently $L(\Gamma)'$ would be trivial. But we know its commutant contains $R(\Gamma)$ (actually this is an equality, but we have yet to establish this). To see that that $L(\Gamma)$ is neither type III nor type II_∞, it suffices to show 1 is finite. Suppose $v \in L(\Gamma)$ is such that $v^*v = 1$ and $vv^* \leq 1$. Recall that we have a faithful tracial state τ. Then

$$\tau(1 - vv^*) = \tau(1) - \tau(vv^*) = \tau(1) - \tau(v^*v) = 0.$$

Since τ is faithful, $vv^* = 1$ and so 1 is finite.

Example 5.3.22. A similar argument as above shows that the hyperfinite II_1 factor $\mathcal{R}$ is actually type II_1. Indeed, we showed that is a factor with a faithful tracial state and a large commutant.

Example 5.3.23. Let m be the Lebesgue measure on $\mathbb{R}$. Consider the group of rational affine transformations on $\mathbb{R}$:

$$G := \left\{ s = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Q},\ a > 0 \right\}.$$

Then G acts on $\mathbb{R}$ via $s \cdot t := at + b$. This action fails to be measure preserving for $a \neq 1$, but it is still free and ergodic. Indeed, then $s \cdot t = t$ if and only if either $a = 1$ and $b = 0$ or $t = -\frac{b}{a-1}$. Thus the fixed points of s are an m-null set. To see that G acts ergodically, consider the subgroup of translations:

$$H := \left\{ s = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Q} \right\}.$$

Clearly H acts ergodically on $\mathbb{R}$, and so G acts ergodically as well. Consequently, $M := L^\infty(\mathbb{R}, m) \rtimes G$ is a factor.

We also claim that m is not equivalent to any measure that is invariant under the action of G. Indeed, if μ is a measure on $\mathbb{R}$ which is invariant under G, then it is invariant under H. That is, μ is a translation invariant measure on $\mathbb{R}$. By the uniqueness of the Haar measure, it must be that $\mu = cm$ for some $c \geq 0$. If μ is equivalent to m then $c > 0$, but cm is not invariant under G since it is not invariant under scalings.

We further claim that M is purely infinite. Let $z \in M$ be the supremum of all finite projections in M. Since conjugating a finite projection gives a finite projection, we have $z = uzu^*$ for all unitaries $u \in M$. That is, $z \in \mathcal{Z}(M)$. Since M is a factor, we have either $z = 0$ or $z = 1$. If $z = 0$, we are done. So suppose, towards a contradiction, that $z = 1$. Note that then 1 is semi-finite as the supremum of finite projections. We then have the following *fact*: there exists a *semi-finite trace* τ on M such that the restriction of τ to $L^\infty(\mathbb{R}, m)$ is given by integration against a measure equivalent to m. Since a trace is invariant under unitary conjugation, this measure is necessarily invariant under the action of G. But by the above we know this is a contradiction.

In certain cases, it is easy to show that projections are not finite. Let $u \in M$ be the unitary operator corresponding to

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in G.$$

Then for $f \in L^\infty(\mathbb{R}, m)$ we have $(ufu^*)(t) = f(2t)$. In particular, $u\chi_S u^* = \chi_{\frac{1}{2}S}$ for any $S \subset \mathbb{R}$ measurable. Thus $\chi_S \sim \chi_{\frac{1}{2}S}$ via the partial isometry $u\chi_S$. For $S = [-\epsilon, \epsilon]$ for some $\epsilon > 0$, $\chi_{\frac{1}{2}S} < \chi_S$ and hence χ_S is not finite.

Chapter 6

The Trace

In this chapter we show for finite factors the existence of a unique, faithful, normal, tracial state . More generally, for a finite non-factor M we show the existence of a unique, faithful, normal "center-valued trace," and for semi-finite factors we show the existence of a unique, faithful, normal "tracial weight." The material in this chapter has been adapted from [3, Sections 4.7, 4.8, and 4.9].

6.1 Center-Valued Traces

Lemma 6.1.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra and $p \in \mathcal{P}(M)$ non-zero. If $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ is a family of pairwise orthogonal projections satisfying $p_i \sim p$ for all $i \in I$, then $|I| < \infty$.*

Proof. If I is infinite, then there exists a proper subset $J \subset I$ with $|J| = |I|$. But then

$$\sum_{i\in I} p_i \sim \sum_{j\in J} p_j < \sum_{i\in I} p_i,$$

contradicting M being finite. □

Lemma 6.1.2. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a type II_1 von Neumann algebra. Then there exists a projection $p_{1/2} \in \mathcal{P}(M)$ so that $p_{1/2} \sim 1-p_{1/2}$. Moreover, there exists a family of projections $\{p_r\}_r$ indexed by dyadic rationals $r \in [0, 1]$ such that:*

(i) $p_r \leq p_s$ if $r \leq s$

(ii) $p_s - p_r \sim p_{s'} - p_{r'}$ whenever $0 \leq r \leq s \leq 1$ and $0 \leq r' \leq s' \leq 1$ satisfy $s - r = s' - r'$.

Proof. Let $\{p_i, q_i\}_{i\in I}$ be a maximal family of pairwise orthogonal projections such that $p_i \sim q_i$ for all $i \in I$. Define $p_{1/2} := \sum_i p_i$ and $q = \sum_i q_i$. Then $p_{1/2} \sim q$, and we further claim $q = 1 - p_{1/2}$. If not, then $1 - (p_{1/2} + q) \neq 0$. Since M is type II, $1 - (p_{1/2} + q)$ is not abelian and consequently there exists $p_0 \in \mathcal{P}([1 - (p_{1/2} + q)]M[1 - (p_{1/2} + q)])$ which is strictly less than its central support (in this corner), which we will denote by z. Therefore, if $q_0 = z - p_0$, then p_0 and q_0 are not centrally orthogonal, and consequently by Proposition 5.1.12 they have equivalent subprojections. However, this contradicts the maximality of $\{p_i, q_i\}_{i\in I}$. Thus $q = 1 - p_{1/2}$.

Now, we construct the family of projections indexed by dyadic radicals $r \in [0, 1]$ inductively. We let $p_{1/2}$ be as above, and set $p_1 := 1$ and $p_0 := 0$. Let $v \in M$ be such that $v^*v = p_{1/2}$ and $vv^* = 1 - p_{1/2}$. Since $p_{1/2}Mp_{1/2}$ is type II_1 by Theorem 5.3.4, the above argument yields $p_{1/4} \leq p_{1/2}$ such that $p_{1/4} \sim p_{1/2} - p_{1/4}$. Set $p_{3/4} := p_{1/2} + vp_{1/4}v^*$. It is easily observed that $p_0 \leq p_{1/4} \leq p_{1/2} \leq p_{3/4} \leq p_1$ and $p_{1/4} \sim p_{(k+1)/4} - p_{k/4}$ for each $k = 0, 1, 2, 3$. Induction then yields the desired family. □

Lemma 6.1.3. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a type II_1 von Neumann algebra, and let $\{p_r\}_r \subset \mathcal{P}(M)$ be the family of projections indexed by dyadic rationals $r \in [0, 1]$ as in the previous lemma. If $p \in \mathcal{P}(M)$ is non-zero, then there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ so that $pz \neq 0$ and $p_rz \preceq pz$ for some positive dyadic rational $r \in (0, 1]$.*

Proof. By considering the compression $Mz(p)$, we may assume $z(p) = 1$. We proceed by contradiction and suppose we cannot find such a $z \in \mathcal{P}(\mathcal{Z}(M))$. Then by the comparison theorem $p \preceq p_r$ for all positive dyadic rationals $r \in (0, 1]$. In particular, we have for each $k \in \mathbb{N}$

$$p \preceq p_{2^{-(k+1)}} \sim p_{2^{-k}} - p_{2^{-(k+1)}}.$$

For each $k \in \mathbb{N}$, let $q_k \leq p_{2^{-k}} - p_{2^{-(k+1)}}$ be such that $p \sim q_k$. But then $\{q_k\}_{k\in\mathbb{N}}$ is an infinite family or pairwise orthogonal projections that contradicts Lemma 6.1.1. $\square$

Definition 6.1.4. For $M \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra, a projection $p \in \mathcal{P}(M)$ is said to be **monic** if there exists a finite collection of pairwise orthogonal projections $\{p_1, \ldots, p_n\} \subset \mathcal{P}(M)$ such that $p_i \sim p$ for each $i = 1, \ldots, n$, and $\sum_{i=1}^n p_i \in \mathcal{Z}(M)$.

For any $k \in \mathbb{N}$, the projections $p_{1/2^k}$ from Lemma 6.1.2 are monic since

$$p_{1/2^k} \sim p_{(i+1)/2^k} - p_{i/2^k} \qquad i = 1, \ldots, 2^k - 1$$

and

$$p_{1/2^k} + \sum_{i=1}^{2^k-1} p_{(i+1)/2^k} - p_{i/2^k} = 1 \in \mathcal{Z}(M).$$

If $z \in \mathcal{Z}(M)$ is an n-homogeneous projection for a finite cardinal n, then $z = \sum_{i=1}^n p_i$ for abelian projections p_i with $z(p_i) = z$, $i = 1, \ldots, n$. In particular, the p_i are all equivalent and thus each p_i is monic.

Proposition 6.1.5. *If $M \subset \mathcal{B}(\mathcal{H})$ is a finite von Neumann algebra, then every projection is the sum of pairwise orthogonal monic projections.*

Proof. Via a maximality argument, it suffices to show that any non-zero projection has a monic subprojection. By taking compressions with central projections, we can consider separately the type I and type II cases. More precisely, we need only consider the type I_n with $n < \infty$ and the type II_1 cases.

If M is type I_n, then we showed in Proposition 5.3.15 that any non-zero projection contains a non-zero n-homogeneous subprojection. By the discussion proceeding this proposition, there exists a monic subprojection.

If M is type II_1 and $p \in \mathcal{P}(M)$ is non-zero, then Lemma 6.1.3 implies there is a central projection z and a positive dyadic rational r such that $p_r z \preceq pz$. Thus pz has a monic subprojection. $\square$

Definition 6.1.6. For $M \subset \mathcal{B}(\mathcal{H})$ a von Neumann algebra, a **center-valued state** is a linear map $\varphi\colon M \to \mathcal{Z}(M)$ such that:

(i) $\varphi(x^*x) \geq 0$ for all $x \in M$

(ii) $\varphi(z) = z$ for all $z \in \mathcal{Z}(M)$

(iii) $\varphi(zx) = z\varphi(x)$ for all $x \in M$ and $z \in \mathcal{Z}(M)$.

We say that φ is **faithful** if $\varphi(x^*x) = 0$ for $x \in M$ implies $x = 0$. We say that φ is **normal** if it is σ-WOT continuous. We say that τ is a **center-valued trace** if it is a center-valued state and $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

Recall the map Φ from the proof of Lemma 5.3.14. Up to the isomorphism $p_{i_0}Mp_{i_0} \cong \mathcal{Z}(M)$, Φ is a faithful center-valued trace.

Theorem 6.1.7. *Every von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ has a normal, center-valued state.*

Proof. Since $\mathcal{Z}(M)$ is abelian, it is a type I von Neumann algebra. By Theorem 5.3.9, $N := \mathcal{Z}(M)'$ is also type I. By a maximality argument, we find an abelian projection $q \in \mathcal{P}(N)$ whose central support in N is 1. Observe that by Corollary 5.3.12

$$qMq \subset qNq = \mathcal{Z}(qNq) = \mathcal{Z}(N)q,$$

and by Lemma 5.3.11, $\theta : z \mapsto zq$ defines an isomorphism from $\mathcal{Z}(M)$ to $\mathcal{Z}(M)q$. **Fact:** θ and θ^{-1} are automatically normal. Define $\varphi(x) := \theta^{-1}(qxq)$ for $x \in M$. Then φ is a normal, center-valued state. $\square$

Recall that $\mathcal{Z}(M)$, as abelian von Neumann algebra, is isomorphic to the $*$-algebra of essentially bounded, measurable functions over some measure space. Thus the following lemma is proved by the same argument that shows a positive linear functional attains its norm at the identity.

Lemma 6.1.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a center-valued state $\varphi: M \to \mathcal{Z}(M)$. The φ is bounded with $\|\varphi\| = 1$.*

We know give some alternate characterizations of center-valued traces.

Lemma 6.1.9. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\varphi: M \to \mathcal{Z}(M)$ a center-valued state. The following are equivalent:*

(i) τ is a center-valued trace;

(ii) $\tau(x^*x) = \tau(xx^*)$ for all $x \in M$;

(iii) $\tau(p) = \tau(q)$ whenever $p, q \in \mathcal{P}(M)$ are equivalent.

Proof. Clearly (i)⇒(ii)⇒(iii). Suppose (iii) holds. Then for any $p \in \mathcal{P}(M)$ and any unitary $u \in M$ we have $\tau(upu^*) = \tau(p)$. Since τ is bounded, $\tau(uxu^*) = \tau(x)$ for all $x = x^* \in M$ and unitaries $u \in M$ by applying the Borel functional calculus to the real and imaginary parts of x. Replacing x with xu yields $\tau(ux) = \tau(xu)$ for all $x \in M$ and unitaries $u \in M$. Since $y \in M$ can be written as a linear combination of four unitaries, we have $\tau(yx) = \tau(xy)$ for all $x, y \in M$. □

Our next goal is to show that any finite von Neumann algebra has a faithful, normal center-valued trace. First we require a few lemmas.

Lemma 6.1.10. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra. If $\varphi: M \to \mathcal{Z}(M)$ is a normal center-valued state, then for each $\epsilon > 0$ there exists $p \in \mathcal{P}(M)$ such that $\varphi(p) \neq 0$ and*

$$\varphi(xx^*) \leq (1+\epsilon)\varphi(x^*x).$$

for all $x \in pMp$.

Proof. Let $\{q_i\}_{i\in I} \subset \mathcal{P}(M)$ be a maximal family of pairwise orthogonal projections with $\varphi(q_i) = 0$. Define $q_0 := 1 - \sum_i q_i$. By normality we have $\varphi(q_0) = 1$, and by maximality of $\{q_i\}_{i\in I}$ we have that φ is faithful on $q_0 M q_0$.

Let $\{e_j, f_j\}_{j\in J} \subset \mathcal{P}(q_0 M q_0)$ be a maximal family of projections such that: (i) $\{e_j\}_{j\in J}$ and $\{f_j\}_{j\in J}$ are both pairwise orthogonal families of projections; (ii) $e_j \sim f_j$ for all $j \in J$; and (iii) $\varphi(e_j) > \varphi(f_j)$ for each $j \in J$. If $J = \emptyset$, then $\varphi(p) = \varphi(q)$ for all equivalent $p, q \in \mathcal{P}(q_0 M q_0)$, and so φ is a center-valued trace on $q_0 M q_0$ by Lemma 6.1.9, and we are done. Otherwise, letting $e := q_0 - \sum_j e_j$ and $f := q_0 - \sum_j f_j$ we have

$$\varphi(f) > \varphi(e) \geq 0.$$

Thus $f \neq 0$. Since $\sum_j e_j \sim \sum_j f_j$ are equivalent finite projections, Proposition 5.2.11 implies $e \sim f$. Consequently $e \neq 0$, and $\varphi(e) > 0$ since φ is faithful on $q_0 M q_0$.

We claim that whenever $\bar{e} \leq e$ and $\bar{f} \leq f$ satisfy $\bar{e} \sim \bar{f}$ then $\varphi(\bar{e}) \leq \varphi(\bar{f})$. Since $\varphi(\bar{f}) - \varphi(\bar{e}) \in \mathcal{Z}(M)$ is self-adjoint, we can consider

$$z := \chi_{(-\infty,0)}(\varphi(\bar{f}) - \varphi(\bar{e})) \in \mathcal{P}(\mathcal{Z}(M)).$$

If the desired inequality does not hold, then $z \neq 0$ and we have

$$\varphi(z\bar{f}) = z\varphi(\bar{f}) < z\varphi(\bar{e}) = \varphi(z\bar{e}).$$

Moreover, $z\bar{f} \sim z\bar{e}$ since $\bar{f} \sim \bar{e}$, but this contradicts the maximality of $\{e_i, f_i\}_{i\in I}$. 4/10/2017

Now, define

$$\mu = \inf\{t \in [0,1] : \varphi(\bar{e}) \leq t\varphi(\bar{f})\ \forall \bar{e} \leq e,\ \bar{f} \leq f,\ \bar{e} \sim \bar{f}\}.$$

Note that $\mu_0 > 0$ since $\varphi(e) > 0$. Letting $\epsilon > 0$ be as in the statement of the lemma, there exists $\bar{e} \leq e$ and $\bar{f} \leq f$ satisfying $\bar{e} \sim \bar{f}$ and such that $(1+\epsilon)\varphi(\bar{e}) \not\leq \varphi(\bar{f})$. Thus the central projection

$$\chi_{(-\infty,0)}(\mu\varphi(\bar{f}) - (1+\epsilon)\varphi(\bar{e}))$$

is non-zero, and cutting down $\bar{e}$ and $\bar{f}$ by it (which maintains equivalence) we may assume $(1+\epsilon)\varphi(\bar{e}) > \mu\varphi(\bar{f})$).

Let $\{\bar{e}_k, \bar{f}_k\}_{k\in K}$ be a maximal family such that: (i) $\{\bar{e}_k\}_{k\in K}$ and $\{\bar{f}_k\}_{k\in K}$ are both pairwise orthogonal families of projections; (ii) $\bar{e}_k \leq \bar{e}$, $\bar{f}_k \leq \bar{f}$, and $\bar{e}_k \sim \bar{f}_k$ for each $k \in K$; and (iii) $(1+\epsilon)\varphi(\bar{e}_k) \leq \mu\varphi(\bar{f}_k)$ for each $k \in K$. Define $p := \bar{e} - \sum_k \bar{e}_k$ and $q := \bar{f} - \sum_k \bar{f}_k$. Proposition 5.2.11 implies $p \sim q$. We have $p \neq 0$ since otherwise by normality we have

$$(1+\epsilon)\varphi(\bar{e}) = \sum_{k\in K}(1+\epsilon)\varphi(\bar{e}_k) \leq \sum_{k\in K}\mu\varphi(\bar{f}_k) \leq \mu\varphi(\bar{k}),$$

contradicting $(1+\epsilon)\varphi(\bar{e}) > \mu\varphi(\bar{f})$.

Just as was argued above with the maximality of the family $\{e_j, f_j\}_{j\in J}$, the maximality of $\{\bar{e}_k, \bar{f}_k\}_{k\in K}$ implies that we have $\mu\varphi(\bar{q}) \leq (1+\epsilon)\varphi(\bar{p})$ whenever $\bar{p} \leq p$ and $\bar{q} \leq q$ satisfy $\bar{p} \sim \bar{q}$. So, suppose $\bar{p}_1, \bar{p}_2 \leq p$ satisfy $\bar{p}_1 \sim \bar{p}_2$. If $v \in q_0 M q_0$ is such that $v^*v = p$ and $vv^* = q$, then $\bar{p}_1 \sim v\bar{p}_1 v^* =: \bar{q} \leq q$. Then the previous argument and the definition of μ imply

$$\varphi(\bar{p}_1) \leq \mu\varphi(\bar{q}) \leq (1+\epsilon)\varphi(\bar{p}_2).$$

Let $x \in pMp$ and let u be a unitary in pMp. By Corollary 3.1.13, x^*x can be uniformly approximated by linear combinations of projections in pMp. Thus the previous argument and the normality of φ implies

$$\varphi(u^*x^*xu) \leq (1+\epsilon)\varphi(x^*x)$$

since $\bar{p} \sim u^*\bar{p}u$ for any $\bar{p} \in \mathcal{P}(pMp)$. Now, if $x^* = v|x^*|$ is the polar decomposition, the finiteness of pMp implies (via the proof of Proposition 5.2.11) that v can be extended to a unitary $u \in pMp$. Then

$$u^*x^*xu = v^*x^*xv = v^*v|x^*|^2v^*v = |x^*|^2 = xx^*,$$

and consequently $\varphi(xx^*) \leq (1+\epsilon)\varphi(x^*x)$. □

Lemma 6.1.11. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra and $\epsilon > 0$. Then there exists a normal center-valued state φ such that*

$$\varphi(xx^*) \leq (1+\epsilon)\varphi(x^*x)$$

for all $x \in M$.

Proof. Let ψ be the normal center-valued state guaranteed by Theorem 6.1.7. By Lemma 6.1.10, there exists $p \in \mathcal{P}(M)$ with $\psi(p) \neq 0$ and

$$\psi(xx^*) \leq (1+\epsilon)\psi(x^*x)$$

for all $x \in pMp$. By Proposition 6.1.5, $p = \sum_i p_i$ for p_i monic projections. At least one p_i satisfies $\varphi(p_i) \neq 0$, and so replacing p with this p_i we may assume p is monic.

Thus there exists a finite family $\{p_1, \ldots, p_n\}$ of pairwise orthogonal projections such that $p_i \sim p$, $i = 1, \ldots, n$ and $z := \sum_{i=1}^n p_i \in \mathcal{P}(\mathcal{Z}(M))$. Take $v_i \in M$ such that $v_i^*v_i = p_i$ and $v_iv_i^* = p$ for each $i = 1, \ldots, n$. For $x \in Mz$, define $\varphi_0(x) := \sum_{i=1}^n \psi(v_ixv_i^*)$. For $x \in Mz$ we have

$$\begin{aligned}0 \leq \varphi_0(xx^*) = \varphi_0(xzx^*) &= \sum_{j=1}^n \varphi_0(xp_jx^*) = \sum_{i,j=1}^n \psi(v_ixv_j^*v_jx^*v_i^*)\\ &\leq (1+\epsilon)\sum_{i,j=1}^n \psi(v_jx^*v_i^*v_ixv_j^*) = (1+\epsilon)\sum_{i=1}^n \varphi_0(x^*p_ix) = (1+\epsilon)\varphi_0(x^*x)\end{aligned}$$

Observe that φ_0 is valued in $\mathcal{Z}(Mz)$, is normal, and satisfies properties (i) and (iii) in Definition 6.1.6 of a center-valued state on Mz. Also, $\bar{z} \in \mathcal{Z}(Mz)$, we have

$$\varphi_0(\bar{z}) = \sum_{i=1}^n \psi(v_i\bar{z}v_i^*) = \sum_{i=1}^n \bar{z}\psi(v_izv_i^*) = \bar{z}\varphi_0(z).$$

Thus, in order for φ_0 be a center-valued state on Mz, we need $\varphi_0(z) = z$. This need not be the case; however, since $\varphi_0(z) > 0$ there exists $\delta > 0$ such that 4/14/2017

$$z_0 := \chi_{[\delta,\infty)}(\varphi_0(z)) \neq 0.$$

Since $\varphi_0(z) = z\varphi_0(1)$, $z_0 \leq z$. If we set $y := f(\varphi_0(z))$ for $f(t) := \frac{1}{t}\chi_{[\delta,\infty)}(t)$, then $z_0 = y\varphi_0(z)$. For $x \in Mz_0$ define $\varphi(x) := y\varphi_0(x)$. Then

$$\varphi(xx^*) = y\varphi_0(xx^*) \leq y(1+\epsilon)\varphi_0(x^*x) = (1+\epsilon)\varphi(x^*x).$$

Also, φ is valued in $\mathcal{Z}(Mz_0)$, is normal, and satisfies properties (i) and (iii) in Definition 6.1.6 of a center-valued state on Mz_0. Lastly, we have

$$\varphi(z_0) = y\varphi_0(z_0) = y\varphi_0(z_0 z) = yz_0\varphi_0(z) = y\varphi_0(z) = z_0.$$

Consequently, $\varphi(\bar{z}) = \bar{z}$ for any $\bar{z} \in \mathcal{Z}(Mz_0)$. Thus φ is a normal center-valued state on Mz_0.

Now, let $\{z_i\}_{i\in I} \subset \mathcal{P}(\mathcal{Z}(M))$ be a maximal family of pairwise orthogonal central projections such that for each $i \in I$ there exists a normal center-valued state φ_i on Mz_i such that

$$\varphi_i(xx^*) \leq (1+\epsilon)\varphi_i(x^*x)$$

for all $x \in Mz_i$. We must have $\sum_i z_i = 1$, otherwise we could apply the above argument to $M(1 - \sum_i z_i)$ to contradict maximality. Thus if we define for $x \in M$

$$\varphi(x) := \sum_{i\in I} \varphi_i(xz_i),$$

then φ is a normal center-valued state with the desired property. □

Theorem 6.1.12. *A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is finite if and only if there exists a normal center-valued trace. Moreover, any such center-valued trace is faithful and unique.*

Proof. We first show that any such center-valued trace τ is automatically faithful. Indeed, if $p \in \mathcal{P}(M)$ is monic and $\{p_1, \ldots, p_n\} \subset \mathcal{P}(M)$ are pairwise orthogonal such that $p_i \sim p$ for each $i = 1, \ldots, n$ and $z := \sum_{i=1}^n p_i \in \mathcal{Z}(M)$, then

$$z = \tau(z) = \tau\left(\sum_{i=1}^n p_i\right) = \sum_{i=1}^n \tau(p_i) = n\tau(p).$$

Consequently, $\tau(p) = \frac{1}{n}z \neq 0$. By Proposition 6.1.5, any projection $p \in \mathcal{P}(M)$ the sum of pairwise orthogonal monic projections. and consequently $\tau(p) \neq 0$. Finally, approximating positive $x \in M$ by positive linear combinations of its spectral projections, we have $\tau(x) > 0$. Thus τ is faithful.

Now, suppose there exists a normal center-valued trace τ. Then if $v \in M$ satisfies $v^*v = 1$ and $vv^* \leq 1$, then

$$1 = \tau(v^*v) = \tau(vv^*).$$

Since τ is faithful by the above argument, we have $vv^* = v^*v = 1$. Thus M is finite.

Conversely, suppose M is finite. Let $(a_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of real numbers converging to 1. By Lemma 6.1.11, there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of normal center-valued states such that

$$\varphi_n(xx^*) \leq a_n\varphi_n(x^*x)$$

for $x \in M$ and each $n \in \mathbb{N}$.

For each $1 \leq m < n$, we claim that $a_m^2\varphi_m - \varphi_n$ is a positive linear map. It suffices to check positivity on projections, and since the map is normal, it suffices, by Proposition 6.1.5, to check it on monic projections. Let $\{p_1, \ldots, p_k\} \subset \mathcal{P}(M)$ be pairwise orthogonal, equivalent projections with $z := \sum_{i=1}^k p_i \in \mathcal{Z}(M)$. Then $\varphi_n(p_1) \leq a_n\varphi_n(p_i)$ and $\varphi_m(p_i) \leq a_m\varphi_m(p_1)$ for each $i = 1, \ldots, k$. Hence

$$k\varphi_n(p_1) \leq a_n\sum_{i=1}^k \varphi_n(p_i) = a_n\varphi_n(z) = a_n z = a_n\varphi_m(z) = a_n\sum_{i=1}^k \varphi_m(p_i) \leq ka_n a_m\varphi_m(p_1) \leq ka_m^2\varphi_m(p_1).$$

Thus $[a_m^2\varphi_m - \varphi_n](p_1) \geq 0$ and $a_m^2\varphi_m - \varphi_n$ is positive. It follows from Lemma 6.1.8 that

$$\|a_m^2\varphi_m - \varphi_n\| \leq \|[a_m^2\varphi_m - \varphi_n](1)\| = a_m^2 - 1.$$

Since $a_m \searrow 1$, we see that $\|\varphi_m - \varphi_n\| \to 0$ and thus we can define $\tau\colon M \to \mathcal{Z}(M)$ by

$$\tau(x) = \lim_{n\to\infty} \varphi_n(x).$$

One immediately has that τ is a center-valued state. Moreover, $\tau(xx^*) \leq \tau(x^*x)$ for all $x \in M$. By reversing the roles of x and x^*, we obtain $\tau(xx^*) = \tau(x^*x)$ for all $x \in M$. Thus by Lemma 6.1.9, τ is a center-valued trace. To see that τ is normal, let $\phi \in \mathcal{Z}(M)_*$, then

$$\|\phi\circ\tau - \phi\circ\varphi_n\| \leq \|\varphi\|\|\tau - \varphi_n\| \to 0.$$

We have $\phi\circ\varphi_n \in M_*$ by normality of φ_n. Since M_* is closed, the above implies $\phi\circ\tau \in M_*$, and hence τ is normal.

To see that τ is unique, suppose $\psi\colon M \to \mathcal{Z}(M)$ is another normal center-valued trace. Then for p a monic projection as above, we have

$$\tau(p) = \frac{1}{k}\sum_{i=1}^{k} \tau(p_k) = \frac{1}{n}\tau(z) = \frac{1}{n}z = \frac{1}{n}\psi(z) = \psi(p).$$

Thus τ and ψ agree on monic projections, and so by Proposition 6.1.5 they agree on all projections. Since M is the norm closure of the span of $\mathcal{P}(M)$, we see that τ and ψ agree everywhere. Hence τ is unique. ☐

Proposition 6.1.13. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra with a normal center-valued trace τ. Then for $p, q \in \mathcal{P}(M)$, $p \preceq q$ if and only if $\tau(p) \leq \tau(q)$.*

Proof. If $p \preceq q$, let $v \in M$ be such that $v^*v = p$ and $vv^* \leq q$. Then $\tau(p) = \tau(v^*v) = \tau(vv^*) \leq \tau(q)$. Conversely, suppose $\tau(p) \leq \tau(q)$. By the comparison theorem there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$. To show $p \preceq q$, it suffices to show $q(1-z) \sim p(1-z)$. Let $v \in M$ be such that $v^*v = q(1-z)$ and $vv^* \leq p(1-z)$. Then

$$\tau(vv^*) \leq \tau(p(1-z)) = \tau(p)(1-z) \leq \tau(q)(1-z) = \tau(q(1-z)) = \tau(v^*v) = \tau(vv^*).$$

Since τ is faithful by Theorem 6.1.12, this implies $vv^* = p(1-z)$ and hence $q(1-z) \sim p(1-z)$. ☐

6.1.1 Dixmier's Property

We now show that in a general von Neumann algebra, the operator norm closure of the convex hull of the unitary orbit of any element intersects the center, which is known as *Dixmier's property.* This property also yields an additional proof of the uniqueness of the center-valued trace on a finite von Neumann algebra.

Lemma 6.1.14. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For $x = x^* \in M$, there exists a unitary $u \in M$ and $y = y^* \in \mathcal{Z}(M)$ such that*

$$\left\|\frac{1}{2}(x + u^*xu) - y\right\| \leq \frac{3}{4}\|x\|.$$

Proof. We may assume $\|x\| = 1$. Let $p = \chi_{[0,1]}(x)$ and $q = 1 - p = \chi_{[-1,0)}(x)$. By the comparison theorem, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ and $p_1, p_2, q_1, q_2 \in \mathcal{P}(M)$ such that

$$zq \sim p_1 \leq p_1 + p_2 = zp \qquad \text{and} \qquad (1-z)p \sim q_1 \leq q_1 + q_2 = (1-z)q.$$

Let $v, w \in M$ be such that $v^*v = zq$, $vv^* = p_1$, $w^*w = (1-z)p$, and $ww^* = q_1$. Set

$$u := v + v^* + w + w^* + q_2 + p_2.$$

Then, since p and q are orthogonal we have

$$u^*u = uu^* = v^*v + vv^* + w^*w + ww^* + q_2 + p_2 = zq + p_1 + (1-z)p + q_2 + p_2 = p + q = 1.$$

Thus u is a unitary, and moreover we have:

$$\begin{array}{lll} u^*p_1u = zq & u^*q_1u = (1-z)p & u^*p_2u = p_2 \\ u^*zqu = p_1 & u^*(1-z)pu = q_1 & u^*q_2u = q_2. \end{array}$$

Now, by definition of p and q we have $-zq \leq zx \leq zp = p_1 + p_2$. Conjugating by u yields $-p_1 \leq zu^*xu \leq zq + p_2$. Consequently

$$-\frac{1}{2}(zq + p_1) \leq \frac{1}{2}(zx + zu^*xu) \leq \frac{1}{2}(p_1 + zq) + p_2$$

Since $z = zq + zp = zq + p_1 + p_2$, we have

$$-\frac{1}{2}z \leq \frac{1}{2}(zx + zu^*xu) \leq z,$$

or equivalently,

$$-\frac{3}{4}z \leq \frac{1}{2}(zx + zu^*xu) - \frac{1}{4}z \leq \frac{3}{4}z.$$

A similar argument yields

$$-\frac{3}{4}(1-z) \leq \frac{1}{2}((1-z)x + (1-z)u^*xu) + \frac{1}{4}(1-z) \leq \frac{3}{4}(1-z).$$

So summing yields

$$\left\|\frac{1}{2}(x + u^*xu) - y\right\| \leq \frac{3}{4},$$

where $y = \frac{1}{4}(2z-1)$. □

Theorem 6.1.15 (Dixmier's Property)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For $x \in M$, denote by $\overline{K}(x)$ the norm closed convex hull of the unitary orbit of x. Then $\mathcal{Z}(M) \cap \overline{K}(X) \neq \emptyset$.*

Proof. Let $\mathcal{K}$ denote the set of all maps $\alpha \colon M \to M$ of the form $\alpha(x) = \sum_{i=1}^n c_i u_i^* x u_i$, where $u_1, \ldots, u_n \in M$ are unitaries and $c_1, \ldots, c_n \geq 0$ satisfy $\sum_{i=1}^n c_i = 1$. Observe that $\alpha(z) = z$ for any $z \in \mathcal{Z}(M)$ and $\alpha \in \mathcal{K}$.

Denote $a_0 :=$ Re (x) and $b_0 :=$ Im (x). Using Lemma 6.1.14, we can find a unitary $u \in M$ and $y_1 = y_1^* \in \mathcal{Z}(M)$ such that

$$\left\|\frac{1}{2}(a_0 + u^*a_0u) - y_1\right\| \leq \frac{3}{4}\|a_0\|.$$

Define $\alpha_1 \in \mathcal{K}$ by $\alpha_1(x) = \frac{1}{2}(x + u^*xu)$. Also define $a_1 := \alpha_1(a_0)$, which we note is self-adjoint. The above inequality is equivalent to $\|a_1 - y_1\| \leq \frac{3}{4}\|a_0\|$. Next apply Lemma 6.1.14 to $a_1 - y_1$ to obtain $\alpha_2 \in \mathcal{K}$ and $y_2 = y_2^* \in \mathcal{Z}(M)$ such that

$$\|\alpha_2(a_1 - y_1) - y_2\| = \|\alpha_2(a_1) - (y_1 + y_2)\| \leq \frac{3}{4}\|a_1 - y_1\| \leq \left(\frac{3}{4}\right)^2 \|a_0\|.$$

Define $a_2 := \alpha_2(a_1)$. Then the above inequality is equivalent to $\|a_2 - (y_1 + y_2)\| \leq \left(\frac{3}{4}\right)^2 \|a_0\|$. Iterating this construction, we obtain sequences $(\alpha_n)_{n\in\mathbb{N}} \subset \mathcal{K}$, $(a_n)_{n\in\mathbb{N}} \subset M_{s.a.}$, and $(y_n)_{n\in\mathbb{N}} \subset \mathcal{S}(M)_{s.a.}$ such that $a_n = \alpha_n(a_{n-1})$ and

$$\left\|a_n - \sum_{i=1}^n y_n\right\| \leq \left(\frac{3}{4}\right)^n \|a_0\|.$$

Since any finite composition of maps in $\mathcal{K}$ is still an element of $\mathcal{K}$, using the above process for any $\epsilon > 0$ we can find $\alpha \in \mathcal{K}$ and $y \in \mathcal{Z}(M)$ such that $\|\alpha(a_0) - y\| \leq \epsilon$. Similarly, we can find $\beta \in \mathcal{K}$ and $z \in \mathcal{Z}(M)$ such that $\|\beta(\alpha(b_0)) - z\| < \epsilon$. Observe that we also have

$$\|\beta(\alpha(a_0)) - y\| = \|\beta\left(\alpha(a_0) - y\right)\| \leq \|\alpha(a_0) - y\| < \epsilon.$$

Thus we have $\|\beta \circ \alpha(x) - (y + iz)\| < 2\epsilon$.

Iterating this construction yields sequences $(\gamma_n)_{n\in\mathbb{N}} \in \mathcal{K}$ and $(z_n)_{n\in\mathbb{N}} \subset \mathcal{Z}(M)$ such that, if we define $x_0 = x$ and $x_n = \gamma_n(x_{n-1})$ then

$$\|x_n - z_n\| \leq \frac{1}{2^n}.$$

Also

$$\|x_{n+1} - x_n\| = \|\alpha_{n+1}(x_n - z_n) + (x_n - z_n)\| \leq \frac{1}{2^{n-1}}.$$

Thus $(x_n)_{n\in\mathbb{N}}$ converges in norm (to something in $\overline{K}(M)$), which implies $(z_n)_{n\in\mathbb{N}}$ converges to the same limit that is then necessarily in $\mathcal{Z}(M) \cap \overline{K}(M)$. □

Corollary 6.1.16. *A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is finite if and only if $\mathcal{Z}(M)\cap\overline{K}(M)$ contains exactly one element for each $x \in M$. Moreover, if M is finite then it has a unique, faithful, normal center-valued trace τ such that $\mathcal{Z}(M) \cap \overline{K}(M) = \{\tau(x)\}$ for all $x \in M$.*

Proof. Suppose M is finite and let $\tau\colon M \to \mathcal{Z}(M)$ be a center-valued trace. Observe that τ is constantly equal to $\tau(x)$ on $\overline{K}(x)$ for all $x \in M$. Thus $\emptyset \neq \mathcal{Z}(M) \cap \overline{K}(x) \subset \{\tau(x)\}$.

Conversely, if $\mathcal{Z}(M) \cap \overline{K}(M)$ consists of a single element of each $x \in M$, define $\tau\colon M \to \mathcal{Z}(M)$ by letting $\tau(x)$ be the single element. Then one immediately see that τ is a center-valued state. Since $\overline{K}(u^*xu) = \overline{K}(x)$, we have $\tau(u^*xu) = \tau(x)$ for any unitary $u \in M$. Consequently τ is a center-valued trace and M is finite by Theorem 6.1.12.

By the above, we see that for any center-valued trace on a finite von Neumann algebra must have its output in $\mathcal{Z}(M) \cap \overline{K}(M)$, and hence the trace is unique as this set contains exactly one element. □

6.2 Characterizing the Commutant

Let M be a von Neumann algebra with a faithful, normal, tracial state $\tau\colon M \to \mathbb{C}$. We denote by $L^2(M)$ the GNS Hilbert space associated to τ. We will identify M with its GNS representation on $L^2(M)$, so that $M \subset \mathcal{B}(L^2(M))$. Recall that there is a dense subspace of $L^2(M)$ corresponding to M. Thus, we can consider $x \in M$ as both an operator in $\mathcal{B}(L^2(M))$ and an vector in $L^2(M)$. The context will usually make it clear in what way we are thinking of x, but when necessary we might write $\hat{x}$ when we want to think of this element as a vector. We have for $x, y \in M$

$$\langle \hat{x}, \hat{y}\rangle = \tau(y^*x).$$

We may also write $\langle x, y\rangle_2$ for the same quantity.

Now, define for $x \in M$

$$J\hat{x} := \widehat{x^*}.$$

We note that

$$\|J\hat{x}\|^2 = \|\widehat{x^*}\|^2 = \tau(xx^*) = \tau(x^*x) = \|\hat{x}\|^2.$$

Thus J extends to a conjugate linear isometry on $L^2(M)$. Note that since J is conjugate linear, we have $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$ for $\xi, \eta \in L^2(M)$.

Let $x, y, z \in M$. Then

$$\begin{aligned} x(JyJ)\hat{z} &= xJy\widehat{z^*} = xJ\widehat{yz^*} = x\widehat{zy^*} = \widehat{xzy^*} \\ &= J\widehat{yz^*x^*} = Jyz\widehat{^*x^*} = JyJ\widehat{xz} = (JyJ)x\hat{z}. \end{aligned}$$

Thus $x(JyJ) = (JyJ)x$ since $\widehat{M}$ is dense in $L^2(M)$. This implies $JMJ \subset M' \cap \mathcal{B}(L^2(M))$. It turns out that the reverse inclusion also holds:

Theorem 6.2.1. *Let M be a von Neumann algebra with a faithful, normal, tracial state τ. Representing M in $\mathcal{B}(L^2(M))$, we have*

$$M' \cap \mathcal{B}(L^2(M)) = JMJ$$

for J defined as above.

Proof. Define for $\xi \in L^2(M)$ the unbounded operators $L_\xi^0 \colon \widehat{M} \to L^2(M)$ and $R_\xi^0 \colon \widehat{M} \to L^2(M)$ by

$$\begin{aligned} L_\xi^0 \hat{x} &:= (Jx^*J)\xi \\ R_\xi^0 \hat{x} &:= x\xi. \end{aligned}$$

Note that if $\xi = \hat{y}$, then $L_\xi^0 \hat{x} = \widehat{yx}$ and $R_\xi^0 \hat{x} = \widehat{xy}$. Now, we claim that these operators are closable. Suppose $(\hat{x}_n)_{n\in\mathbb{N}} \subset \widehat{M}$ satisfies $\|\hat{x}_n\| \to 0$, and $L_\xi^0 \hat{x}_n \to \eta$ for some $\eta \in L^2(M)$. Then for any $y \in M$ we have

$$\begin{aligned} |\langle \eta, \hat{y}\rangle| &= \lim_{n\to\infty} |\langle L_\xi^0 \hat{x}_n, \hat{y}\rangle| = \lim_{n\to\infty} |\langle (Jx_n^*J)\xi, \hat{y}\rangle| = \lim_{n\to\infty} |\langle \xi, (Jx_nJ)\hat{y}\rangle| \\ &= \lim_{n\to\infty} |\langle \xi, \widehat{yx_n^*}\rangle| = \lim_{n\to\infty} |\langle \xi, y\widehat{x_n^*}\rangle| \leq \lim_{n\to\infty} \|\xi\|\|y\|\|J\hat{x}_n\| = \lim_{n\to\infty} \|\xi\|\|y\|\|\hat{x}_n\| = 0. \end{aligned}$$

Thus $\eta = 0$ and so L_ξ^0 is closable. Similarly for R_ξ^0. We denote the closures of these maps by L_ξ and R_ξ, respectively. We have $L_{\hat{x}} = x$ for all $x \in M$.

For $x, y \in M$ and $\xi \in L^2(M)$, observe that

$$\langle R_{J\xi}\hat{x}, \hat{y}\rangle = \langle xJ\xi, \hat{y}\rangle = \left\langle J\xi, \widehat{x^*y}\right\rangle = \left\langle J\widehat{x^*y}, \xi\right\rangle = \langle y^*\hat{x}, \xi\rangle = \langle \hat{x}, y\xi\rangle = \langle \hat{x}, R_\xi \hat{y}\rangle,$$

therefore $R_{J\xi} = R_\xi^*$. We also note that

$$(JL_\xi J)\hat{x} = JL_\xi\widehat{x^*} = J(JxJ)\xi = xJ\xi = R_{J\xi}\hat{x} = R_\xi^*\hat{x},$$

so that $JL_\xi J = R_\xi^*$. This also implies $L_{J\xi} = L_\xi^*$. Define

$$\begin{aligned} \widetilde{M} &:= \{L_\xi \colon \xi \in L^2(M)\} \cap \mathcal{B}(L^2(M)) \\ \widetilde{N} &:= \{R_\xi \colon \xi \in L^2(M)\} \cap \mathcal{B}(L^2(M)). \end{aligned}$$

The above observations show $J\widetilde{M}J = \widetilde{N}$. Let $L_\xi \in \widetilde{M}$, $R_\eta \in \widetilde{N}$, and $x, y \in M$. In the discussion preceding the theorem, we saw that $JMJ \subset M'$. Using this we have

$$\begin{aligned} \langle L_\xi R_\eta \hat{x}, \hat{y}\rangle &= \langle x\eta, L_{J\xi}\hat{y}\rangle = \langle x\eta, Jy^*J(J\xi)\rangle = \langle (JyJ)x\eta, J\xi\rangle = \langle x(JyJ)\eta, J\xi\rangle \\ &= \langle (Jx^*J)\xi, yJ\eta\rangle = \langle L_\xi\hat{x}, R_{J\eta}\hat{y}\rangle = \langle R_\eta L_\xi \hat{x}, \hat{y}\rangle. \end{aligned}$$

Thus $L_\xi R_\eta = R_\eta L_\xi$ which implies $\widetilde{M} \subset \widetilde{N}'$ and $\widetilde{N} \subset \widetilde{M}'$

It suffices to show $\widetilde{M} = M$ and $\widetilde{N} = M'$. We have already seen that $L_{\hat{x}} = x$ for $x \in M$, thus $M \subset \widetilde{M}$. From the definition of R_ξ, we also immediately have $\widetilde{N} \subset M'$. Let $y \in M'$, then for $x \in M$ we have

$$R_{y\hat{1}}\hat{x} = x(y\hat{1}) = yx\hat{1} = y\hat{x}.$$

Thus $M' \subset \widetilde{N}$, and so $M' = \widetilde{N}$. Consequently, $\widetilde{M} \subset \widetilde{N}' = (M')' = M$, and so $\widetilde{M} = M$. □

Bibliography

[1] J.B. Conway; *A Course in Operator Theory*, American Mathematical Society, 2000.

[2] R.E. Megginson; *An Introduction to Banach Space Theory*, Springer-Verlag, 1998.

[3] J. Peterson; *Notes on von Neumann algebras*, https://math.vanderbilt.edu/peters10/teaching/spring2013/vonNeumannAlgebras.pdf

[4] M. Takesaki; *Theory of Operator Algebras I*, 1st Edition, Springer-Verlag, 1979.