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Banach Algebras and Spectral Theory

Lecture Notes by Srivatsav.K.E. *Math 206, Fall 2016*

Lecture 1 August 24th. A Broad Survey of Banach Algebras

Definition 1 *A Normed Algebra is a normed vector space over a field, preferably $\mathbb{R}$ or $\mathbb{C}$, equipped with another sub-multiplicative, associative binary operation (multiplication) (i.e, $\|ab\| \leq \|a\| \|b\|$)*

From a norm, we get the standard metric, $d(a, b) = \|a - b\|$. This metric determines a topology on the space for which addition and multiplication are uniformly continuous. A Banach Algebra is a normed algebra which is complete for this metric. Given a normed algebra $\mathcal{A}$, form it's completion, and from uniform continuity, the addition and multiplication extend to the completion to produce a Banach Algebra.

Example: Let X be a compact topological space. Let $\mathcal{A} = C(X)$ the space of continuous $\mathbb{R}$ valued functions with the usual structure. With the supremum norm $\|f\|_\infty = sup\{|f(x) : x \in X\}$ this is a Banach Algebra. If X is locally compact, then $C_b(X)$, the space of bounded continuous functions under the same norm is also a Banach Space. Another important space is $C_\infty(X)$, the subalgebra of functions that vanish at infinity. This means that for all f and every $\epsilon > 0$, there is a compact $K \subset X$ with $|f(x)| < \epsilon$ for $x \notin K$.

Example: Let $\mathcal{O} \in \mathbb{C}^n$ be a bounded open subset. Let $\mathcal{H}^\infty(\mathcal{O})$ be the functions that are holomorphic and bounded on $\mathcal{O}$, under the supermum norm. Let $\mathcal{A}^\infty(\mathcal{O})$ be the functions that are continuous on the completion of $\mathcal{O}$ and holomorphic on the original set. These two are interesting Banach Algebras.

Example: Let (X, d) be a compact metric space. For $f \in C(X)$, define $L(f) = sup\{\frac{|f(x)-f(y)|}{d(x,y)}, x \neq y\}$. Note that this quantity may be infinity (when?). Define the set of Lipshitz functions $\mathcal{L}^d(X) = \{f \in C(X) : L(f) < \infty\}$, with a norm $\|f\|_d = \|f\|_\infty + L(f)$, this forms a Banach algebra. Notice that L gives a seminorm on the space, and in fact, the metric can be recovered from L. This is discussed later in this class.

If a Topological Vector space has a topology determined by a countable number of seminorms and is complete, it is called a Frechet space. If the product is continuous, it's a Frechet Algebra. We shall discuss about these algebras in greater detail later. We now see a non commutative example of a Banach Algebra.

Example: If X is a Banach Space, the space of all bounded/continuous operators on X, denoted by $\mathfrak{B}(X)$, is a Banach Algebra with the operator norm. Any closed subalgebra of $B(X)$ is also Banach. Notably, if X is a Hilbert Space, we also have the operation of taking adjoints, with $\|T\| = \|T^*\|$.

Definition 2 *A C^* Algebra is a closed subalgebra of $\mathfrak{B}(\mathcal{H})$, the algebra of operators on a hilbert space, that is stable under taking adjoints.*

Lecture 2 August 26th. More Surveying, and Introductions

We stopped with the definition of C^*Algebras in the previous lecture. Gelfand and Neimark in 1943 gave an abstract characterization of these algebras. A consequence of this, also called "little" G-N Theorem, is as follows.

Corollary 3 *Let $\mathcal{A}$ be a commutative C^*Algebra of $\mathfrak{B}(\mathcal{H})$. Then $\mathcal{A}$ is isometrically and *-algebraically isomorphic to some $C(X)$ for a locally compact space X.*

This shall be discussed in the upcoming lectures. An easier example where this corollary works is when you consider some $T \in \mathfrak{B}(\mathcal{H})$ which is normal (i.e, commutes with it's adjoint). If A is the C^* algebra generated by T, then $A \cong C(X)$ for some compact space X. It is interesting to also discuss when $\mathcal{H}$ is a finite dimensional space.

This class shall also discuss to a certain extent, Von Neumann Algebras, *-subalgebras of $\mathfrak{B}(\mathcal{H})$ that is norm closed, closed under the weak topology on the space. Before we begin a more careful study of the objects at hand, we shall look at one final example of a banach algebra.

Example: Let G be a discrete group, let $\pi : G \to Aut(X) \subset \mathfrak{B}(X)$ be a representation of G on the banach space X. π is bounded if all the images of π are norm bounded by a K. $L^1(G)$ with convolution, i.e, $(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x)$, is a fine banach algebra.

For a while, all algebras have an identity. Let $\mathcal{A}$ be an algebra over $\mathbb{F}$. Suppose $f \in C(X)$ for some locally compact X. How do we look at the range of this function? We ask if $f - \lambda 1_{\mathcal{A}}$ is invertible, in which case λ is in the range. This is the motivation of the concept of spectrum.

Definition 4 *The spectrum of $a \in \mathcal{A}$ is $\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}}$ is not invertible$\}$. Note that this is a purely algebraic object. The spectrum is sometime denoted by $\sigma(a)$.*

Lemma 5 *Let $\mathcal{A}$ be a unital Banach Algebra and let $a \in \mathcal{A}$, $\|a\| < 1$. Then $1 - a$ is invertible and $\left\|(1-a)^{-1}\right\| \leq \frac{1}{1-\|A\|}$.*

Proof: We clearly want this: $\frac{1}{1-a} = \sum a^n$ with $a^0 = 1$. Let $S_n = \sum_{k=0}^{n} a^k$. We want the S_n to form a cauchy sequence. For $n > m$, we have

$$\|S_n - S_m\| = \left\|\sum_{k=m+1}^{n} a^k\right\| \leq \sum_{k=m+1}^{n} \left\|a^k\right\| \leq \sum_{k=m+1}^{n} \|a\|^k$$

Since $\|a\| < 1$ this is exactly the situation of the ordinary geometric series, and it follows from here that the sequence is a cauchy sequence. Because $\mathcal{A}$ is complete, $\exists b \in \mathcal{A}$ such that $s_n \to b$. We need to now show that $(1-a)b = 1 = b(1-a)$. $(1-a)b = \lim_{n\to\infty}(1-a).S_n = \lim_{n\to\infty}(1 - a^{n+1}) = 1$. ∎

Corollary 6 *Let $a \in \mathcal{A}$ and suppose that $\|1 - a\| < 1$. Then a is invertible and $\left\|a^{-1}\right\| \leq \frac{1}{1-\|1-a\|}$*

The proof is trivial from the previous corollary. Note that this means that the open ball around the identity element consists of invertible elements.

For any $a \in \mathcal{A}$, define L_a, R_a as operators on $\mathcal{A}$ by $L_a b = ab$ and $R_a b = ba$. Have $L_a L_b = L_{ab}$. L is called the left regular representation of $\mathcal{A}$ on itself. $L : \mathcal{A} \to End(\mathcal{A})$ is an algebra homomorphism. Let

$GL(\mathcal{A})$ be the set of invertible elements in $\mathcal{A}$. If $a \in GL(\mathcal{A})$, then L_a is a homeomorphism of $\mathcal{A}$ onto itself. Then for any $a \in GL(\mathcal{A})$ let $\theta_1 = \{b \in \mathcal{A} : \|1 - b\| < 1\}$. This is open. Hence, $L_a(\theta_1) \subset GL(a)$ is open and contains a. This easily leads us to an important result:

Proposition 7 *$GL(\mathcal{A})$ is an open subset of $\mathcal{A}$.*

Lecture 3 August 29th. On the Non-Emptiness of Spectra

Let $\mathcal{A}$ be a unital Banach Algebra. Let $a \in \mathcal{A}$. Then $\phi : \lambda_a \to a + \lambda$, $\lambda \in C$ is continuous and thus $\phi^{-1}(GL(\mathcal{A}))$ is an open set. But, this is nothing but $\{\lambda : a - \lambda \text{ is invertible}\}$, which is the complement of $\sigma(a)$. Hence, $\sigma(a)$ is a closed subset of $\mathbb{C}$. The complement of $\sigma(a)$ is hereby called the resolvent set $\rho(a)$.

Proposition 8 *$\forall a \in \mathcal{A}$, $\lambda \in \sigma(a)$ we have $|\lambda| \leq \|a\|$.*

Proof: Suppose we have the otherwise, i.e, $\lambda > \|a\|$, then $(a - \lambda) = -\lambda(1 - \frac{a}{\lambda})$. This is invertible since λ is invertible and $1 - \frac{a}{\lambda}$ is invertible from the lemma from last lecture. So $\lambda \notin \sigma(a)$. which is a contradiction. ■

If we are working over $\mathbb{R}$, we can have $\sigma(A) = \phi$, and a quick example is when taking the rotation matrix on $\mathbb{R}^2$. Another good set of examples can be found in $C([0, 1])$. This is a complete space, however, the space of polynomials, a subalgebra (supremum norm) of this is dense but certainly not closed or complete. Here, only the constant functions are invertible.

Definition 9 *Define the resolvent of a as the function $R(a, \lambda) = (\lambda - a)^{-1}$, defined on $\rho(a)$.*

Proposition 10 *R is an analytic function, and analytic and bounded about ∞.*

Proof: By definition, $f^{-1}(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ if limit exists. For $a, b \in GL(\mathcal{A})$ we have $a^{-1} - b^{-1} = a^{-1}(b - a)(b^{-1})$. A sidenote here is the fact that on $GL(\mathcal{A})$, the mapping $a \to a^{-1}$ is norm continuous (the proof is using the ball of radius 1/2 trick), which leads to the fact that $GL(\mathcal{A})$ is a topological group. In any case, we proceed with showing the limit exists.

$$\frac{(z - a)^{-1} - (z_0 - a)^{-1}}{z - z_0} = \frac{(z - a)^{-1}((z_0 - a) - (z - a))}{z - z_0} = -(z - a)^{-1}(z_0 - a)^{-1}$$

Hence, it follows that the limit is indeed defined, and as predicted by freshman calculus. Look at $R(a, z)$ at ∞. $R(a, 1/z) = (1/z - a)^{-1} = \frac{1}{1/z-a} = z(1 - za)^{-1}$ which exists for all $|z| < \frac{1}{\|A\|}$. For $z = 0$, $R(a, \frac{1}{z}) = 0$. This ends the proof. ■

We are now ready to prove the main theorem of this lecture.

Theorem 11 *Let $\mathcal{A}$ be a Banach Algebra over $\mathbb{C}$. Then $\forall a \in \mathcal{A}$, $\sigma(a) \neq \phi$.*

Proof: Suppose the spectrum were empty. Then the resolvent set is the whole of $\mathbb{C}$. Then the resolvent function is defined on the entire complex plane, and is analytic and bounded. We can't directly apply Liouville's theorem from complex analysis here, simply because the resolvent is not a complex valued function. For this we take a slight detour. Viewing $\mathcal{A}$ as a Banach Space, we know that its dual space $\mathcal{A}'$ is a large space. If f is an $\mathcal{A}$ valued analytic function and if $\phi \in \mathcal{A}'$, then the map $z \to \phi(f(z))$ is now a complex valued function that is entire and bounded. But this is clearly impossible since the function has to be 0 from Liouville. ■

We end the lecture with a theorem (the quite obviously follows from the above theorem) that will be used next lecture,

Theorem 12 (Gelfand-Mazur) *Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$. If every non zero element is invertible, then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$. (However, this is not correct over $\mathbb{R}$)*

The proof uses the fact that the spectrum of an element is non empty. This gives a cannonical equality between elements of $\mathcal{A}$ and $\lambda \in \mathbb{C}$.

Lecture 4 August 31st. On Maximal Ideals

For an algebra, we can consider left, right or two sided ideals, with the usual definition. If $\mathcal{A}$ is a normed or a topological algebra and if I is an ideal, then it's closure is again an ideal (since multiplication is continuous).

Proposition 13 *Let $\mathcal{A}$ be a unital Banach Algebra and let I be an ideal in $\mathcal{A}$. If I is a proper ideal, then its closure is a proper ideal.*

Proof: We know that $1 \in GL(\mathcal{A})$ which is an open set. And since the ideal is proper, it cannot intersect with invertible elements. Hence, 1 cannot meet the closure of I. ■

However, this property is not shared by more general algebras. Good examples include $C_c(\mathbb{R})$ the subalgebra of $C_\infty(X)$ with functions of compact support; The subalgebra of polynomials.

Corollary 14 *If $\mathcal{A}$ is a Banach algebra with 1, then every maximal 2-sided ideal is closed.*

Proof: Since the closure of every proper ideal is also proper, we have that the closure of a maximal ideal is itself. ■

Let X be a normed vector space and Y a subspace. We can consider X/Y and have $\pi : X \to X/Y$. For $z \in X/Y$ set $\|z\| = inf\{\|x\| : \pi(x) = z\}$. This can also be seen as the distance of the point from the subspace Y. This is a seminorm, except when Y is closed, in which case it becomes a norm. If Y is closed and if X is a Banach space, then X/Y is also a Banach space.

We now look at the natural extensions of these ideas to Algebras. If $\mathcal{A}$ is a normed algebra and if I is a closed 2 sided ideal in $\mathcal{A}$, then $\mathcal{A}/I$ is a normed algebra. If $\mathcal{A}$ is a Banach algebra, so is the quotient. Given $a, b \in \mathcal{A}$ and $c, d \in I$, we have $(a-c)(b-d) = ab - \{(cb - ad - cd) \in I\}$. Hence, $\|\pi(ab)\| = \|\pi(a)\pi(b)\| \leq \|(a-c)(b-d)\| \leq \|a-c\|\,\|b-d\|$. It follows from taking infemum that $\|\pi(ab)\| \leq \|\pi(a)\|\,\|\pi(b)\|$. This along with other minor details establish the extensions we seek.

Definition 15 *A ring is said to be simple if it contains no proper 2 sided ideal except $\{0\}$.*

A good example of a simple ring is $M_n(\mathbb{F})$ for any field $\mathbb{F}$.

Proposition 16 *If R is a ring and if I is a maximal 2 sided ideal in R, then R/I is simple.*

In fact, if $\mathcal{A}$ is a Banach algebra and if I is a maximal 2 sided ideal, then A/I is a simple Banach algebra. Suppose A is a commutative ring with 1, and $a \in A$ isn't invertible, then $aA = Aa$ is a 2 sided proper ideal.

Corollary 17 *If A is a simple commutative ring with 1, then A is a field.*

The proof of the above is obvious since every element is invertible from the previous reasoning. Now, let $\mathcal{A}$ be a commutative Banach algebra with 1, over $\mathbb{C}$. Let I be a maximal ideal in $\mathcal{A}$. Then A/I is a simple commutative Banach algebra with no non-invertible elements. We know from Gelfand Mazur theorem that this means A/I is canonically isomorphic to $\mathbb{C}$. Thus, from a maximal ideal I, you get a homomorphism ϕ not identically 0, from $\mathcal{A}$ onto $\mathbb{C}$ with kernel as I.

Proposition 18 *If $\phi : A \to \mathbb{C}$ is a non trivial homomorphism and if $a \in \mathcal{A}$ then, $\phi(A) \in \sigma(a)$.*

Proof: We just consider $\phi(a - \phi(a))$. This is 0 so $a - \phi(a)$ is not invertible. ■

Corollary 19 *Let A be a Banach algebra with* 1 *over* $\mathbb{C}$. *Then every homomorphism* $\phi : \mathcal{A} \to \mathbb{C}$ *is continuous.*

Proof: Given $a \in \mathcal{A}$, we have $\phi(a) \in \sigma(a)$. This implies from a previous lemma, $|\phi(a)| \leq \|a\|$, so we have $\|\phi\| = 1$. Since this is norm bounded, it is continuous. This is one of our first results following a common theme called Automatic Continuity. ■

Note that if $\mathcal{A}$ is a normed algebra with 1, and if $\phi : A \to \mathbb{C}$ is continuous, then $Ker(\phi)$ is a closed maximal ideal in $\mathcal{A}$.

Proposition 20 *For a commutative Banach algebra with* 1 *over* $\mathbb{C}$, *there is a natural bijection between the set of maximal ideals of* $\mathcal{A}$ *and the set of non zero homomorphism from* $\mathcal{A} \to \mathbb{C}$.

For a commutative Banach Algebra, $\hat{\mathcal{A}}$ is defined to be the space of non zero $\mathbb{C}$ homomorphisms, or the Maximal Ideal space of $\mathcal{A}$. Observe that $\hat{\mathcal{A}} \subset \mathcal{A}'$ (the dual of $\mathcal{A}$ as a Banach Space), and as a matter of fact, $\hat{\mathcal{A}} \subseteq Ball_1(\mathcal{A}')$ (closed unit ball) under the weak* topology.

Lecture 5 September 2nd. The Gelfand Transform

Let $\mathcal{A}$ be a commutative Banach algebra with 1, over $\mathbb{C}$. We have $\hat{A} = \{\phi : A \to \mathbb{C}$:one to one unital algebra homomorphisms$\}$. Since all of these have norm 1, they lie in the closed unit ball of $\mathcal{A}'$.

Proposition 21 *$\hat{A}$ is closed in $\mathcal{A}'$ for the weak* topology.*

Proof: Let $\psi \in Ball_1(\mathcal{A}')$, and let $\{\phi_\alpha\}$ be a net in $\hat{A}$ that converges to ψ for the weak* topology. This implies for every $a \in \mathcal{A}$, $\phi_\alpha(a) \to \psi(a)$. The linearity of ψ is obvious. Now, for $a, b \in \mathcal{A}$, $\phi(ab) = \lim \phi_\alpha(ab) = \lim \phi_\alpha(a)\phi_\alpha(b) = \lim \phi_\alpha(b) \lim \phi_\alpha(b)$ from the boundedness and continuity of the functionals in the net. It is easy to show that $\psi(1) = 1$. Hence, we have shown the closure of $\hat{\mathcal{A}}$ for the weak* topology. ■

From Alaoglu's theorem, we have that $\hat{\mathcal{A}}$ is now compact, and by itself it's a nice compact Hausdorff space. Hence, the space $C(\hat{\mathcal{A}})$ is a unital Banach Algebra. For any $a \in \mathcal{A}$, define $\hat{a} \in C(\hat{\mathcal{A}})$ by $\hat{a}(\phi) = \phi(a)$.

Claim 22 *$\hat{a}$ is continuous.*

Proof: Suppose that $\{\phi_\alpha\}$ is a net in $\hat{\mathcal{A}}$ and $\phi_\alpha \to \phi_0 \in \hat{\mathcal{A}}$. Then $\hat{a}(\phi_0) = \phi_0(a) = \lim \phi_\alpha(a) = \lim \hat{a}(\phi_\alpha)$. This is synonymous to continuity. ■

Claim 23 *The map $a \to \hat{a}$ is a unital algebra homomorphism of A into $C(\hat{\mathcal{A}})$.*

Proof: Let $a, b \in \mathcal{A}$. $\hat{ab}(\phi) = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi)\hat{b}(\phi)$, which is consistent with the definition of multiplication. Hence $\hat{ab} = \hat{a}\hat{b}$. $\hat{1_{\mathcal{A}}}(\phi) = \phi(1) = 1_{\mathbb{C}}$. Furthermore, $\|\hat{a}\|_\infty \leq \|a\|$ since $\phi(a) \in \sigma(a)$. ■

The homomorphism $\Lambda : \mathcal{A} \to C(\hat{\mathcal{A}})$ is called the Gelfand transform. This work was carried out by Gelfand during the mid 1930's and 40's. $\hat{\mathcal{A}}$ is often called the Gelfand spectrum. Observe that for $a \in \mathcal{A}$, the range of the gelfand transform is contained in $\sigma(a)$, as consistent with our very first motivations, in lecture 2, of the concept of a spectrum. When does equality occur is a pertinent question. Let us look at a special case.

Let $\mathcal{A}$ be a unital Banach algebra and let a_0 be an element that generates $\mathcal{A}$, i.e, the space of polynomials of a_0 is a dense subalgebra of $\mathcal{A}$. Define $\Phi : \hat{\mathcal{A}} \to \sigma(a_0)$ given by $\Phi(\phi) = \phi(a_0)$ which is clearly in the spectrum.

Theorem 24 *Φ is continuous and injective, and onto.*

Proof: If $\{\phi_\alpha\}$ is a net with $\phi_\alpha \to \phi$, then $\Phi(\phi_\alpha) = \phi_\alpha(a_0) \to \phi(a_0) = \Phi(\phi)$. This is synonymous with continuity. Suppose $\Phi(\phi_1) = \Phi(\phi_2)$, we have $\phi_1(a_0) = \phi_2(a_0)$. Remember that these are algebra homomorphisms. This means for any polynomial p, $\phi_1(p(a_0)) = \phi_2(p(a_0))$. Since the polynomial space is dense in $\mathcal{A}$, we have $\phi_1 = \phi_2$. For ontoness, let $\lambda \in \sigma(a)$. This means $a - \lambda$ is not invertible. Consider an ideal $I \subset$ polynomials in z, precisely, the ideal $(z - \lambda)Poly(z)$. $I(\mathcal{A})$ consists of non invertible elements, so it is proper, and thus it's closure is proper too. (FINISH THE PROOF HERE) ■

Lecture 6. September 8th. On Semisimplicity

We start off by treating an important example. Let us prove that for a compact Hausdorff space X, the maximal ideal space in the algebra of continuous functions is homeomorphic to X. Let X be a compact Hausdorff space. Let $\mathcal{A} = C(X)$ with the supremum norm. Each $x \in X$ gives a $\phi_x \in \hat{\mathcal{A}}$ by $\phi_x(f) = f(x)$.

Proposition 25 *Let I be an ideal in $\mathcal{A}$. If $\forall x \in X$ there is $f_x \in I$ with $f_x(x) \neq 0$ then $I = \mathcal{A}$.*

Proof: For each x, $f_x\overline{f_x}$ (which is in the ideal I) is a non negative function with non-zero value at x. Thus, we can assume that f_x's are non negative. Then for each x, set $\theta_x = \{y \in X : f_x(y) > 0\}$ is an open set containing x. Thus, these sets form an open cover of X. By compactness, we have a finite subcover. There are $x_1, x_2, \ldots x_n$ such that $\bigcup_{j=1}^n \theta_{x_j} = X$. Let $f = f_{x_1} + f_{x_2} + \ldots + f_{x_n}$. This function is in the ideal I since it is a sum of functions in the ideal. We know that this is a non negative function, and in fact it is also non zero since for every $x \in X$, $f_x(x) \neq 0$. Thus, since $f^{-1} \in C(X)$, ff^{-1} is in I, but this means I contains 1, and thus equals to the whole space $\mathcal{A}$. ∎

It follows that $x \to \phi_x$ is a bijection from X onto $\hat{\mathcal{A}}$.

Claim 26 *This map is continuous.*

Proof: Let $\{x_\alpha\}$ be a a net of elements in X that converge to x_0, then for any $f \in C(X)$, $\phi_{x_\alpha}(f) = f(x_\alpha) \to f(x_0) = \phi_{x_0}(f)$. Thus $\phi_{x_\alpha} \to \phi_{x_0}$ for the weak* topology. ∎

Since X and $\hat{A}$ are compact, this map is a homeomorphism.

Now, we move on to a more general result concerning the range of the Gelfand Transform of an element.

Theorem 27 *Let $\mathcal{A}$ be a commutative Banach Algebra with 1 over $\mathbb{C}$. For $a \in \mathcal{A}$, the range of the Gelfand Transform ($\hat{a}$) is equal to $\sigma(a)$.*

Proof: Let $\lambda \in \sigma(a)$, so that $a - \lambda$ is not invertible. $(a - \lambda)\mathcal{A}$ is an ideal, and is proper in A. By zorn's lemma, this is contained in a maximal ideal. So, there exists $\phi \in \hat{\mathcal{A}}$ with $\phi((a - \lambda)\mathcal{A}) = 0$, and hence we get $\phi(a) = \lambda$. Thus, λ is in the range of the gelfand transform. ∎

It is important to note here that this proof can be completed without using the Zorn's Lemma, for finitely or even countably generated Banach Algebras. But, once you have infinitely generated Banach Algebras, we require Zorn's lemma. In fact, constructing the maximal ideal is almost impossible in most cases. Here is an example of such a situation.

Example: Let F be a finite field. For each n, let $F_n = F$. Let $\mathcal{A} = \prod_{n \in \mathbb{N}} F_n$. Let I be the ideal, $\oplus_{n \in \mathbb{N}} F_n$. This certainly lives in a maximal ideal, and it is impossible to construct this maximal ideal

The Gelfand transform for $\mathcal{A}$ is injective if the intersection of the maximal ideals is $\{0\}$. We then say that $\mathcal{A}$ is semisimple.

Lecture 7. September 10th. Spectral Radius

As usual, let $\mathcal{A}$ be our Banach space with 1 over $\mathbb{C}$ and commutative, and $\hat{\mathcal{A}}$ be it's maximal ideal space. We have seen that the range of the Gelfand transform of $a \in \mathcal{A}$ is nothing but the the spectrum $\sigma(a)$. Define the usual norm for elements, $\|\hat{a}\|_\infty = \sup|\lambda|, \lambda \in \sigma(a)$. Call this, rather conveniently, the spectral radius of a, $\mathfrak{r}(a)$.

We quickly see the following: If $a, b \in \mathcal{A}$, then

$$\mathfrak{r}(ab) = \left\|\hat{ab}\right\|_\infty = \left\|\hat{a}\hat{b}\right\| \leq \|\hat{a}\|_\infty \left\|\hat{b}\right\|_\infty = \mathfrak{r}(a)\mathfrak{r}(b)$$

Similarly sub-linearity follows. So $\mathfrak{r}$ is a fine algebra seminorm on $\mathcal{A}$. Note, if $\mathcal{A}$ is not commutative, then submultiplicity of this spectral radius fails.

Now consider $a \in \mathcal{A}$. We have that $\sigma_{\mathcal{A}}(a) \subseteq \sigma_{(a,1)}(a)$, where $(a, 1)$ is the algebra generated by $a, 1$. This is just because a bigger algebra has more potential to create more invertible elements. Let f be a holomorphic function, $\mathbb{C}$-valued defined in an open subset of $\mathbb{C}$ that contains $\{z : |z| \leq \|a\|\}$. Thus, there exists a power series that converges absolutely and uniformly on the $\|a\|$-ball, $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$. Then define $f(a) = \sum_{n=0}^{\infty} \alpha_n a^n$, where $a^0 := 1_{\mathcal{A}}$.

Proposition 28 (First version of spectral mapping theorem) *If $\lambda \in \sigma_{(a,1)}(a)$, (then we know $|\lambda| \leq \|a\|$) then $f(\lambda) \in \sigma(f(a))$*

Proof: Look at

$$f(a) - f(\lambda) = \sum_{n=0}^{\infty} \alpha_n(a^n - \lambda^n) = \sum_{n=1}^{\infty} \alpha_n(a^n - \lambda^n) = \sum_{n=1}^{\infty} \alpha_n(a - \lambda)(a^{n-1} + a^{n-2}\lambda + \dots \lambda^{n-1}) \leq n\, \|a\|^{n-1}$$

Now, we have that this is equal to $(a - \lambda)b_1$ for some b_1, and this b_1 exists from the above convergence argument. Thus, the invertibility of $f(a) - f(\lambda)$ implies the invertibility of $a - \lambda$. This means λ is not in the spectrum, which is a contradiction. ■

So let $f(z) = z^n$. See that if $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$, and thus it is contained in the $\|a^n\|$ ball. Hence, $|\lambda| \leq \|a^n\|^{1/n}$, and thus, $\mathfrak{r}(a) \leq liminf\, \|a^n\|^{1/n}$.

Definition 29 *$a \in \mathcal{A}$ is quasi-nilpotent if $\|a^n\|^{1/n} \to 0$ as $n \to \infty$.*

Note, that a is quasi nilpotent implies $\sigma(a) = 0$. We now look at an important theorem which we shall prove shortly afterwards.

Theorem 30 (Gelfand Spectral Radius Formula) *For $\mathcal{A}$ commutative Banach Algebra with 1, and for $a \in \mathcal{A}$, we have $\lim \|a^n\|^{1/n}$ exists, and is equal to the spectral radius of a.*

Here is an important consequence.

Proposition 31 *Let $\mathcal{A}$ be commutative Banach Algebra with 1. Then, $\hat{}: \mathcal{A} \to C(\hat{\mathcal{A}})$ is isometric exactly if $\|a^2\| = \|a\|^2$ for all $a \in \mathcal{A}$.*

Proof: If this holds, then we have $\left\|a^{2n}\right\| = \|a\|^{2n}$ for all n. Thus, we have $\mathfrak{r}(a) = \|a\| = \|\hat{a}\|_\infty$. Conversely, if $\left\|a^2\right\| = S^2$, then $\left\|a^{2k}\right\| \leq S^{2k}$ and $\left\|a^{2^k}\right\|^{\frac{1}{2^k}} \leq S$. So, $\mathfrak{r}(a) = \|\hat{a}\| \leq S$. Hence, if $S \neq \|a\|$, the Gelfand transform is not isometric. ■

We now define an important algebraic object that we shall discuss in great depth in the next few lectures.

Definition 32 *A $*$ Algebra over $\mathbb{C}$ is an algebra $\mathcal{A}$ equipped with an involution $a \to a*$ that satisfies $a^{**} = a$, is additive and $(\alpha a)^* = \overline{\alpha} a*$, and $(ab)^* = b^* a^*$. A normed $*$ algebra is one that preserves involution under the norm.*

Lecture 8. September 12th. Gelfand's Spectral Radius Formula

Our strategy for proving the Gelfand Spectral Radius Formula we saw last time, is to prove that the liminf is greater than or equal to limsup, thereby establishing the existence of the limit.

Theorem 33 (Gelfand Spectral Radius)

$$\liminf \|a^n\|^{1/n} \geq \|\hat{a}\|_\infty \geq \limsup \|a^n\|^{1/n}$$

Proof: We know that the Resolvant function is analytic at infinity. $R(a,z) = (z-a)^{-1}$ is holomorphic outside of the $\|a\|$ ball. We look at $z \to \infty$, i.e, consider $z \to 0$ and look at $R(a,z^{-1}) = \frac{1}{z^{-1}-a} = \frac{z}{1-za} = z\sum_{n=0}^{\infty}(za)^n$, and this exists (the convergence) as long as $|z| < \|a\|^{-1}$. But $R(a,z)$ is holomorphic for $|z| > \mathfrak{r}(a)$, that is, for $|z^{-1}| < \mathfrak{r}(a)^{-1}$. Note: $\|a\| \geq \mathfrak{r}(a)$ and $\|a^{-1}\| \leq \mathfrak{r}(a)^{-1}$.

Let $\phi \in \mathcal{A}'$ be given. Then $\phi(R(a,z^{-1}))$ is a $\mathbb{C}$ valued function that is holomorphic for $|z| < \mathfrak{r}(a)^{-1}$. The power series about 0 for this function is (which converges for $|z| < \mathfrak{r}(a)^{-1}$) $z\sum_{n=0}^{\infty}\phi(a^n)z^n$. Let $r > \mathfrak{r}(a)$ be given so that $r^{-1} < \mathfrak{r}(a)^{-1}$. The power series now converges absolutely and uniformly for $|z| \leq r^{-1}$. This gives us that $\sum \phi(a^n r^{-n})$ converges absolutely, meaning each individual term is norm bounded. Thus, there exists a constant M_α such that $|\phi(a^n r^{-1})| \leq M_\alpha$ for all n.

Now, for each n, let F_n be the element of A'' defined by $F_n(\psi) = \psi(a^n r^{-n})$. Given ψ there is a constant M_ψ such that $\forall n$, $F_n(\psi) \leq M_\psi$. By the principle of uniform boundedness (which is a result in classical functional analysis, following from the Baire Category Theorem), we have that this collection is totally bounded. Hence, $\exists M$ such that $\|F_n\| \leq M$ for all n. But, it is clear that $\|F_n\| = \|a^n r^{-n}\|$. So, $\|a^n\|^{1/n} \leq M^{1/n} r$ for all n. As n goes to infinity, we have *limsup* $\|a^n\|^{1/n} \leq r$, and this is for all $r > \mathfrak{r}(a)$. Hence, we have the right hand side inequality. The left hand side inequality follows from last lecture. Hence proved. ∎

Lecture 9. September 14th. Group Algebras and Symmetry

We start off our discussion with an interesting class of abstract *-algebras, which we shall define later in this lecture.

Let S be a semigroup with e. Let $\pi : S \to End(V)$ for a vector space V, with the discrete topology (This is called a representation of S on V, forgetting the addition in V) . Let $f \in C_c(S)$, finite $\mathbb{C}$ linear combinations of elements in S, compact support. Set $\pi_f \in End(V)$, with the following natural map.

$$\pi_f(v) = \sum_{x \in S} f(x).\pi_x(v)$$

We also have defined the natural convolution product of elements in $C_c(S)$ by $(f * g)(x) = \sum_{yz=x} f(y)g(z)$, and one should note that this convolution product is defined precisely in a way that makes $\pi_{f*g} = \pi_f \pi_g$. So, $f \to \pi_f$ is an algebra homomorphism of $C_c(S)$ into $End(V)$.

We take a slight detour. Suppose V is a Banach space. Let $\pi : S \to \mathfrak{B}(V)$ bounded operators on V. Then, set $w(x) = \|\pi(x)\|$ for $x \in S$. We have $w(xy) \leq w(x)w(y)$ with $w(e) = 1$. This is a 'weight function' on S. A way to see this would be by considering $f \in C_c(S)$, $\|\pi_f\| = \|\sum f(x)\pi(x)\| \leq \sum |f(x)| \, \|\pi(x)\|$. So, on $C_c(S)$, put a norm defined by $\|f\|_{1,w} = \sum |f(x)| w(x)$ which is nothing but the l^1 norm for weight w. Thus, $\|\pi_f\| \leq \|f\|_{1,w}$. We also have $\|f * g\|_{1,w} \leq \|f\|_{w,1} \|g\|_{w,1}$. Thus $C_c(S)$ with $\|\|_{1,w}$ is a normed algebra with identity, and it's completion is $l^1(S, w)$.

So, π_f is well defined for $f \in l^1(S, w)$. In our studies, we shall assume $w = 1$, and mostly spend time on S being a group G. Assume that G now acts on isometries of V. Let V be a hilbert space $\mathcal{H}$ and π is a unitary representation on $\mathcal{H}$. Define $\pi_{x^{-1}} = \pi_x^*$. Then, $(\pi_f)^* = (\sum f(x)\pi(x))^* = \sum \overline{f(x)} \pi_{x^{-1}} = \sum \overline{f(x^{-1})} \pi(x)$. So on $C_c(G)$, set f^* by $f^*(x) = \overline{f(x^{-1})}$. It is easy to see that this is an involution, and preserves norm. Thus $l^1(G)$ is a * Banach Algebra with 1. If G is commutative, $l^1(G)$ is commutative.

We have encountered so far, 3 kinds of * Banach Algebras. $l^1(G)$, C(X) and concrete C*Algebras $\subseteq \mathfrak{B}(\mathcal{H})$. We shall return to this subject soon, but now we carry on our study of spectra, leading to the little Gelfand Naimark Theorem.

Definition 34 *Let $\mathcal{A}$ be a * normed Algebra. Then $a \in \mathcal{A}$ is said to be self adjoint if $a^* = a$.*

Definition 35 *A *-Banach Algebra is symmetric if whenever $a \in \mathcal{A}$ and a is self adjoint implies $\sigma(a) \in \mathbb{R}$.*

Example: Let $\mathcal{A} = \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$, with sup norm. Define $(z, w)^* = (\overline{w}, z)$. This is not symmetric! (check)

Let $\mathcal{A}$ be a *-Banach Algebra that is symmetric. Then if $a \in \mathcal{A}$, $a^* = a$, then for any $\phi \in \hat{\mathcal{A}}$, $\phi(a) \in \sigma(a)$, so $\hat{(a)}$ is $\mathbb{R}$ valued.

In any *-Algebra, given $a \in \mathcal{A}$,

$$a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i}$$

Define the real and imaginary parts of a from the above equation. Observe that $Re(a)$ and $Im(a)$ are both self adjoint. For any $a \in \mathcal{A}$, $\hat{a^*} = \overline{\hat{a}}$. We now propose a very important claim.

Proposition 36 *For a commutative *-Banach Algebra that is symmetric, the range of the Gelfand Transform is dense in $C(\hat{\mathcal{A}})$*

Proof: Follows from Stone Weierstrass. Exercise. ■

Let $\mathcal{A} \subseteq \mathfrak{B}(\mathcal{H})$ be a $C*$ Algebra. We have the key property defining this, $\|T^*T\| = \|T\|^2$. The proof of this is as follows. Given $\psi \in \mathcal{H}$, $\|T\psi\|^2 =< T\psi, T\psi >=< T * T\psi, \psi >\leq \|T^*T\| \|\psi\|^2$. Thus, $\|T\| \leq \|T^*T\|^{1/2}$. Finally, $\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2$. The result follows. We end with an important definition.

Definition 37 *An Abstract C-* Algebra is a *Banach Algebra with an involution such that the key property* $\|a^*a\| = \|a\|^2$.

Lecture 10. September 16th. Little Gelfand Naimark Theorem, Continuous Functional Calculus

Let $\mathcal{A}$ be a *-Algebra. $a \in \mathcal{A}$ is said to be normal if it commutes with it's adjoint.

Proposition 38 *Let $\mathcal{A}$ be a C* Algebra with 1, and let a be a normal element of $\mathcal{A}$. Then $\mathfrak{r}(a) = \|a\|$.*

Proof: First let's do for $a^* = a$. From the characteristic property of C* Algebras, we have $\|a\|^2 = \|a^2\|$, which is a precise condition for the gelfand transform to be an isometry. Hence proved. Now, for a normal, $\|a\|^2 = \|a^*a\| = \mathfrak{r}(a^*a) \leq \mathfrak{r}(a^*)\mathfrak{r}(a) \leq \|a\| \mathfrak{r}(a)$. Now, since a sits inside a commutative Banach Algebra, we have $\|a\| = \mathfrak{r}(a)$. ■

Corollary 39 *Let $\mathcal{A}$ be a commutative C* Algebra with 1. The Gelfand Transform is isometric.*

From the previous lecture, we have that the Gelfand Transform is onto if $\mathcal{A}$ is symmetric. Here's is an important proposition.

Proposition 40 *Let $\mathcal{A}$ be a C* Algebra with 1. Then $\mathcal{A}$ is symmetric.*

Proof:[Richard Arens]
Let $a \in \mathcal{A}$ be self adjoint. We need to show that $\sigma(a) \in \mathbb{R}$. Let $\lambda \in \sigma(a)$, and $\lambda = r + is$. For any $t \in \mathbb{R}$, let $b = b_t = a + it.Id$. So, $\sigma(b) = \sigma(a) + it$. Now, we have $\|b^*b\| = \|b\|^2$. Hence, $b^*b = (a - it)(a + it) = a^2 + t^2$. Hence, $\|b^*b\| \leq \|a^2\| + t^2$. But, we know that $\|b\|^2 \leq \|r + i(s + t)\|^2 = r^2 + (s + t)^2$. It follows that $\|a\|^2 \geq r^2 + s^2 + 2st$. But, this is true for all $t \in \mathbb{R}$. Hence, $s = 0$ as desired. ■

This culminates in the theorem we are interested in.

Theorem 41 (Little Gelfand Naimark Theorem) *Let $\mathcal{A}$ be a commutative $C*$ Algebra with 1. Then $A \cong C(\hat{\mathcal{A}})$, isometrically *isomorphic.*

Proposition 42 *For any $C*$ Algebras with q, its norm is determined by the Algebraic and * structure.*

Proof: $\|a\|^2 = \|a^*a\| = \mathfrak{r}(a^*a)$. Let V be a finite dimensional vector space over $\mathbb{C}$. Let $\mathcal{A} = L(V)$. If you choose an inner product on V, it gives a norm and a star structure in fact. ■

We now look at one form of the spectral theorem for self adjoint operators. Let $T \in \mathfrak{B}(\mathcal{H})$ be self adjoint. Then, from the Gelfand Naimark theorem and a previous result, we have $C^*(T, 1_{\mathcal{H}}) \cong C(\sigma(T))$, with the canonical map $T \to \hat{T}(r) = r$, and for a polynomial p, we have $p(T) \to p(r)$. A quick application is the following result.

Proposition 43 *Let $\mathcal{A}$ be a C* Algebra with 1 and let B be a sub C* Algebra containing $1_{\mathcal{A}}$. Then for any $b \in B$, $\sigma_B(b) = \sigma_{\mathcal{A}}(b)$.*

The proof of this involves a standard trick, and is a good beginning to the ideas of continuous functional calculus. We leave it to the reader as an exercise. We first prove it for self adjoint operators and then prove it for the general case.

Lecture 11. September 19th. Group Algebras and some Special Examples

Let S be a discrete semigroup with e. $l^1(S)$ with convolution is a Banach Algebra with 1 (δ_e) assume S is commutative so that $\mathcal{A} = l^1(S)$ is a commutative Banach Algebra with 1. We are concerned about finding what $\hat{\mathcal{A}}$ looks like.

Elements of $\hat{\mathcal{A}}$ are homomorphism $\phi : S \to \mathbb{C}$ with $\phi(1) = 1$. Furthermore, for $\phi \in \hat{\mathcal{A}}$, $\phi(a) \in \sigma(a)$ so $\|\phi(a)\| \leq \|a\|$. So ϕ sits inside the unit ball of A'. From previous knowledge, $A' = l^\infty(S)$. So ϕ is associated with a bounded function on S with $\|\phi\|_\infty \leq 1$.

ϕ is stable under the structure of a semigroup homomorphism. Hence, under the l^∞ norm, ϕ is a semigroup homomorphism of S into the unit disk. That is,

Proposition 44 $\hat{\mathcal{A}} \cong Hom(S, D)$

where D is the unit disk in $\mathbb{C}$. The topology on $\hat{\mathcal{A}}$ is the weak * topology restricted to A'.

For a discrete set Y, the weak * topology of l^∞ from l^1 coincides with the topology of uniform convergence on finite subsets of Y. We now look at an interesting special case where the semigroup we have at hand is $\mathbb{Z}^+$.

Let $S = \mathbb{Z}^+$, it's generated by 1. So, $l^1(S)$ is generated by δ_1, so any ϕ belonging to $\hat{\mathcal{A}}$ is determined by $\phi(\delta_1)$. So $\hat{A} = D$, is identified with the unit disk.

Given $w \in D$ define $\phi_2(n) = w^n$, $\phi_w \in Hom(\mathbb{Z}^+, D)$. So, $\phi_w \in \hat{A}$. For $f \in l^1(Z^+)$, define the gelfand transform in the usual way $\hat{f} = \sum_{i=0}^{\infty} f(n)w^n$. This gives a function on D that is holomorphic in the interior and has an absolutely convergent power series on D.

Suppose $f \in l^1(\mathbb{Z}^+)$ is such that $\hat{f}(w) \neq 0$ for all $w \in D$. So, q/f is a continuous function on D that is holomorphic inside D. So it has a power series. We claim in fact that $1/\hat{f}$ has a power series whose coefficients form an element of $l^1(\mathbb{Z}^+)$.

Now we consider G is a discrete group and is commutative. $\mathcal{A} = l^1(G)$, and $\hat{\mathcal{A}} =$ Unital Homomorphisms from $G \to D$. But for $x \in G$, $\phi(1) = \phi(x)\phi(x^{-1}) = 1$. Since $\|x\| \leq 1$, we have $\|x\| = 1$ for all $x \in G$. Thus we have the theorem:

Theorem 45 $\hat{\mathcal{A}} = Hom(G \to T)$ *where* $T = \{z \in \mathbb{C} : |z| = 1\}$

On $\hat{\mathcal{A}}$, the weak * topology coincides with the topology of uniform convergence on compact sets. We now look at another specific and rather illustrative example.

Example: Let $G = \mathbb{Z}$, where $l^1(\mathbb{Z})$ is generated by δ_1 and δ_{-1}. So, any $\phi \in \hat{\mathcal{A}}$ is determined by $\phi(\delta_1)$. So $\hat{\mathcal{A}} = T$. We can visualize elements as rotations on the circle. So, for $f \in l^1(\mathbb{Z})$,

$$\hat{f}(e^{it}) = \sum_{n=-\infty}^{\infty} f(n)e^{int}$$

And voila, we have the Fourier Series. Now perhaps the reader can confirm his guess that the Gelfand transform is really what's going on inside the fourier transform.

Lecture 12. September 21st. Fourier Transform is Injective?

Let G be a discrete commutative group. We have from the previous lecture that $l^1\hat{(G)} = Hom(G,T)$. Specially, for $\mathbb{Z}$, we have $l^1\hat{(\mathbb{Z})} = T$. For $f \in l^1(\mathbb{Z})$, we have it's gelfand transform, $\hat{f}(e^{it}) = \sum f(n)e^{int}$. Similarly, for $\mathbb{Z}^n$, we have $l^1\hat{(\mathbb{Z}^n)} = T^n$, where we define the Gelfand transform accordingly. To simplify notation, we say $\hat{G} := Hom(G,T) = l^1\hat{(G)}$.

We are interested in the following question. Is the Gelfand Transform, in this case, the Fourier Transform injective? First, we have the following theorem.

Theorem 46 *If $f \in l^1(G)$ and if $\hat{f}$ never takes value 0 on $\hat{G}$ so that $\frac{1}{\hat{f}} \in C(\hat{G})$. Then there exists $g \in l^1(G)$ with $\hat{g} = \frac{1}{\hat{f}}$.*

Now let's deal with the injectivity. Let G be any discrete group not necessarily commutative. We have for $C_c(G)$- continuous functions of finite support, defined the left regular representation λ as follows: $\lambda_x f(y) = f(x^{-1}y)$. We have $\lambda_{x_1}\lambda_{x_2} = \lambda_{x_1x_2}$.

For $1 \leq p \leq \infty$, we have that the λ_x is an isometry on $l^p(G)$. The integrated form (as we have defined in the previous couple of lectures) of λ is a representation of $l^1(G)$ algebra as operators on l^p. We then have:

Theorem 47 *The integrated form $f \to \lambda_f$ is injective. i.e, if $\lambda_f = 0$ then $f = 0$.*

Proof: Consider $\delta_e \in l^p(G)$. We have $\lambda_f(\delta_e) = f$. Thus, every f can be viewed as a function in $l^p(G)$. The result follows. ■

Consider the case of λ on $l^2(G)$. We have that $\lambda : l^1(G) \to \mathfrak{B}(l^2(G))$ is injective. Furthermore, $\lambda_f^* = \lambda_{f^*}$ where $f^*(x) = \overline{f(x^{-1})}$. Therefore, the image of $l^1(G)$ in $\mathfrak{B}(l^2(G))$ is a $*$ algebra. Then the norm closure $C_r(G)$ of it is a $C*$ algebra with 1. From Gelfand Naimark, we have that $C(C_r\hat{(G)}) = \hat{G}$, wherein the gelfand transform is an injective isomorphism. Hence, we have proved our original question.

Lecture 13. September 23rd. Topological Groups, Duality, Spectral Synthesis

Let G be a locally compact abelian group. Set $\hat{G} = \{\phi : G \to T\} \subset C_b(G)$. Put on $\hat{G}$ the topology of uniform convergence on compact sets. If K is a compact subset of G, we imbibe the infinity-norm on K.

Proposition 48 *Then for this topology, $\hat{G}$ is a topological group. That is, product is jointly continuous and taking inverse is continuous.*

Proof: Given $\phi, \psi, \phi_0, \psi_0 \in \hat{G}$ we have

$$\|\phi\psi - \phi_0\psi_0\| \leq \|\phi\psi - \phi\psi_0\| + \|\phi\psi_0\phi_0\psi_0\| \leq \|\phi\| \|\phi - \psi_0\| + \|\phi - \phi_0\| \|\phi_0\|$$

The result follows quickly. For inverse, we just take conjugates, which preserves norm. ■

Thus for G being a discrete abelian group, $\hat{G}$ with these norms, is a topological group. This topology also agrees with the weak * topology from $l^1\hat{(}G)$ which is compact. Thus, $\hat{G}$ is a compact topological group. This is called the Dual group of G.

If G, H are discrete commutative groups and if $F : G \to H$ is a group homomorphism, then for $\psi \in \hat{H}$ we have $\psi(F) \in \hat{G}$. Define as usual, $\hat{F} : \hat{H} \to \hat{G}$ by $\hat{F}(\psi) = \psi(F)$.

Proposition 49 *Then $\hat{F}$ is a continuous group homomorphism from $\hat{H} \to \hat{G}$.*

More generally, we can see that the category of compact commutative groups is in concrete realizations of the dual of the category of discrete commutative groups. This is a result due to Pontryagin.

We saw last time that for a discrete group G, $l^1(G)$ can be embedded densely in a $C*$ Algebra.

Let X be a compact Hausdorff space, and let $\mathcal{A} = C(X)$. For a closed ideal I, define $Ker(I) = \{x \in X : f(x) = 0 \forall f \in I\}$. The kernel is a closed subset of X. For any closed subset K of X, define $Hull(K) = \{f : f(K) = 0\}$. This is a closed ideal.

Theorem 50 *For $\mathcal{A} = C(X)$, $Hull(Ker(I)) = I$ for all closed ideals I.*

Definition 51 *Let $\mathcal{A}$ be a commutative Banach Algebra with 1. Say that $\mathcal{A}$ satisfies spectral synthesis if $Hull(Ker(I)) = I$ for all closed ideals.*

Does spectral synthesis hold for $l^1(\mathbb{Z})$? Laurent Schwartz in 1950 showed that spectral synthesis failed for $L^1(\mathbb{R}^n)$. In 1958, it was shown that spectral synthesis failed for all $L^1(G)$ G not compact.

Lecture 14. September 26th. Strong Continuity of Representations of Locally Compact Groups

Let G be a topological group and π be a representation of G on V. Then π is said to be strongly continuous if for every $v \in V$ the function $x \to \pi_x(v)$ is continuous.

Proposition 52 *If π is strongly continuous at e, then it is strongly continuous.*

Proof: Assume strong continuity at e. Then for any $y \in G$, we have the following. If $\{x\} \to y$ then for $v \in V$ we have $\pi_x v = \pi_y v = \pi_y(\pi_{y^{-1}x} v - \pi_e v)$ since $x^{-1}y \to e$. Since π_y is fixed, we have the result. ∎

Proposition 53 *If V is a Banach Space and if $\exists M$ so that $\|\pi_x\| \leq M$ for all x, then if π is strongly continuous on some dense subset S of V, then π is strongly continuous.*

Proof: Given V, $\epsilon > 0$ choose $w \in S$ such that $\|v - w\| \leq \epsilon/3$. Then there is a neighborhood O of e such that for $x \in O$, $\|\pi_x(w) - w\| \leq \epsilon/3$.

Then, we have
$$\|\pi_x(v) - v\| \leq \|\pi_x(v) - \pi_x(w)\| + \|\pi_x(w) - w\| + \|w - v\| \leq M\epsilon$$
∎

Theorem 54 *Let G be a locally compact group and let M be a locally compact space. Let α be an action of G on M, i.e, α maps G into the $Homeo(M)$. If α is jointly continuous, i.e, $(x, m) \to \alpha_x(m) \in M$ is continuous. Then representation of G on $C_\infty(M)$ by $\alpha_x(f) = f(\alpha_{x^{-1}})$ is strongly continuous.*

Proof: It suffices to show strong continuity on $C_c(M)$ (Since that is a dense subspace of C_∞). It suffices to show strong continuity at e here.

Let $f \in C_c(M)$. To show that $x \to \alpha_x(f)$ is continuous at e. Let K be the support of f. It is compact. Let $C \subset G$ be a compact neighborhood of e, with $C^{-1} = C$. Since α is jointly continuous, we have $\alpha_C(K)$ is compact. We call this CK.

Let $\epsilon > 0$ be given. Since α is jointly continuous, for every $m \in CK$ there is a neighborhood $\emptyset_m X U_m$ of (e, m) with $O \subset C$, such that for $(y, n) \in O_m U_m$, we have $\left\|f(\alpha_{y^{-1}}) - f(m)\right\| < \epsilon/3$. The U_m's form an open cover of CK. So there is a finite subcover, $\{U_{m_j}\}$. Let $O = \cap O_{m_j}$, open neighborhood of e.

We claim that if $x \in O$, then $\|\alpha_x(f) - f\| < \epsilon$. The proof is as follows. Let $m \in M$. If $\|f(\alpha_{x^{-1}}(m)) - f(m)\| \neq 0$, then either $\alpha_{x^{-1}}(m)$ or m belongs to K. So, $m \in CK$ or $m \in CK$. Thus m is in some U_{m_j} then $\|f(\alpha_{x^{-1}}(m)) - f(m_j)\| < \epsilon/3$. But also, $\|f(m) - f(m_j)\| \leq \epsilon/3$. Thus the result follows for any m. ∎
Having concluded the important theorem of this lecture, we move on to some important consequences.

Let M be a locally compact space. By a Radon Measure on M we mean a linear functional on $C_c(M)$ that is continuous for the Inductive Limit Topology on $C_c(M)$. This comes from: If $O \subset M$ is open, $\overline{O}$ is compact, $C_\infty(O)$ embeds into $C_c(M)$ operationally. A net $\{f_\alpha\} \subset C_c(M)$ converges to $f \in C_c(M)$ for inductive limit topology for exactly if there is a compact $K \subset M$ such that for some α_0 have support of $f_\alpha \subseteq K$ if $\alpha \geq \alpha_0$ and $f_\alpha \to f$ for the sup norm. We end with a proposition and leave the proof to the reader.

Proposition 55 *If μ is a positive linear functional on $C_c(M)$, i.e, if $f \geq 0$ then $\mu(f) \geq 0$, then μ is a Radon Measure.*

Lecture 15. September 28th. Radon Measures and Measure Preserving Actions

Let X be a locally compact space. We have $C_c(X)$ and the space of positive radon measures. Given a positive radon measure μ. For $f \in C_c(X)$ set $\|f\|_p = \mu(|f|^p)^{1/p}$. This is a seminorm. Let $L^p(X, \mu)$ be the completion of $C_c(X)$ for this seminorm. Let $\mathcal{A}$ be any *algebra. We say that μ is a positive linear functional if $\mu(a^*a) \geq 0 \; \forall a \in \mathcal{A}$. Then define $< a, b >_\mu = \mu(a^*b)$ is a pre inner product.

Now let G be a locally compact group and α an action of G on M a locally compact space. We say that a Radon Measure on M is α-invariant if $\forall x \in G$, $f \in C_c(M)$, $\mu(\alpha_x(f)) = \mu(f)$.

Theorem 56 *Then this action extends to a strongly continuous action on $L^p(M, \mu)$ for each p, $1 \leq p < \infty$.*

Proof: Must show that α as an action on $C_c(M)$ is strongly continuous. We showed last time that α as an acton of $C_c(M)$ is strongly continuous for inductive limit topology. i.e, given f $\exists$ neighborhood O of e, $\overline{O}$ is compact, such that for $x \in O$, $\|f - \alpha_x(f)\|_\infty < \epsilon$.

Then we can choose $h \in C_c(M)$ with $h \geq 0$ and $h = 1$ on K, then for $x \in O$, $|\alpha_x(f) - f|^p \leq \epsilon^p$ and $|\alpha_x(f) - f|^p = h|\alpha_x(f) - f|^p \leq \epsilon h$. So, $\mu(|\alpha_x(f) - f|^p) \leq \epsilon^p \mu(h)$. ■

Let G be a locally compact group, and let λ be the left action of G on itself by translation. In 1933, Haar showed that every locally compact group has a left invariant Borel Measure (subsequently improved to Radon Measure by Cartan). We shall call these Haar Measures, and study some of their consequences.

Lecture 16. September 30th. Locally Compact Group Representations on Banach Spaces

Let G be a locally compact group, μ a left invariant Haar Measure. Let $y \in G$ let ρ_y be the right translation of the function $\rho_y(f)(x) = f(xy)$. Define ν on $C_c(G)$ by $\nu(f) = \int \rho_y(f)(x)d\mu(x)$. Then ν is left invariant,

$$\nu(\lambda_z f) = \int \rho_y(x_z f)d\mu = \int \lambda_z(\rho_y f)d\mu = \int \rho_y(f)d\mu$$

So ν is a left invariant Haar Measure. So $\exists\Delta(y) \in \mathbb{R}^+$ with $\nu_g(f) = \Delta(y)\mu(f)$ (uniqueness of Haar measure). Δ is called the modular function of G.

Proposition 57 *$\Delta : G \to \mathbb{R}^+$ is a continuous group homomorphism.*

Definition 58 *A group is unimodular if $\Delta = 1$ i.e, the left Haar measure is right translation invariant.*

$f \to \int \Delta(x)f(x)d\mu = \int f(x^{-1})d\mu$ is a right haar measure. δ is the Radon Nikodym derivative of the right haar measure with respect to the left haar measure.

Let G be a locally compact group, H is a closed subgroup. $M = G/H$ is locally compact, on which G acts, jointly continuously.

Proposition 59 *There is a G invariant measure on G/H exactly if $\Delta_H = \Delta_G$restricted to H. We get $L^p(G/H)$ with strongly continuous action of G.*

Let G be a locally compact group with Haar Measure μ, let π be a strongly continuous action of G by isometries on a Banach Space V. We want the integrated form: For $f \in C_c(G)$, $v \in V$, set $\pi_f(v) = \int f(x)\pi_x(v)d\mu \in C_c(G, V)$. The author urges the reader to make peace in some form or the other with the concept of integration on Banach Spaces.

Lecture 17. October 3rd. On Approximate Identities

Let G be a locally compact group, with a left Haar Measure. Let π be a strongly continuous representation of G on a Banach Space V, by isometries. Then for $f \in C_c(G)$ and $v \in V$, we want to set the integrated form:

$$\pi_f = \int_G f(x)\pi_x v dx$$

Note that $f(x)\pi_x v \in C_c(G, V)$, and hence, we assume the reader is familiar with integration with values on a Banach Space. Let μ be a positive radon measure on M.

For $f, g \in C_c(G)$ we have

$$\pi_f(\pi_g v) = \int f(x)\pi_x(\int g(y)\pi_y v dy)dx = \int f(x)(\int g(y)\pi_{xy} v dy)dx = \int f(x)(\int g(x^{-1}y)\pi_y v dy)dx$$

We assume Fubini's results here. We thus have

$$\pi_f(\pi_g v) = \int (\int f(x)g(x^{-1}y)dx)\pi_y v dy$$

Defining $\int f(x)g(x^{-1}y)dx = f * g(y)$, we have $\pi_f\pi_g = \pi_{f*g}$. We thus have $C_c(G)$ is an algebra for convolution.

$$\|\pi_f v\| = \left\|\int f(x)\pi_x v dx\right\| \leq \int |f(x)| \, \|\pi_x v\| \, dx = \|f\|_{l^1} \, \|v\|$$

Thus, $\|\pi_f\| \leq \|f\|_{l^1}$. We then have that π extends to the completion of $C_c(G)$, namely, $L^1(G, Haar)$. $L^1(G, Haar)$ is a Banach Algebra. And $\pi : L^1(G) \to \mathfrak{B}(V)$ is an algebra homomorphism. However, if G is not a discrete group, the Haar measure gives individual points measure 0. Then $L^1(G)$ does not have an identity.

Definition 60 *Let $\mathcal{A}$ be a normed algebra without 1. By a left approximate identity element for $\mathcal{A}$, we mean a net $\{e_\alpha\}$ of elements in $\mathcal{A}$ such that $\|a - e_\alpha a\| \to 0$ By a right identity, we mean the same thing, except $\|a - ae_\alpha\| \to 0$ and similarly, a 2 sided identity.*

By a bounded left or right or 2-sided approximate $\{e_\alpha\}$, we need K so that $\|e_\alpha\| \leq K$ for all α. We end with an important proposition:

Proposition 61 *For G a locally compact group, $C_c(G)$ has a 2-sided approximate identity of norm 1.*

Proof: Let N be a neighborhood system at e. For $O \in N$ choose $f \in C_c(G)$ $f \geq 0$, $supp(f) \subseteq O$, $\int_G f(x)dx = 1$. In the case of $C_\infty(M)$, with a locally compact M. We have the directed set N of compact subsets, directed by inclusion. For $K \in N$, choose $f_K \in C_c(M)$, $1 \geq f_K \geq 0$ and $f_K = 1$ on K. Then for any $g \in C_\infty(M)$, we have $\|g - f_k g\| \to 0$ as K goes to infinity. ∎

Lecture 18. October 5th. The Multiplier Algebra

Let G be a locally compact group. Let N be the directed set of open neighborhoods of e. For each $O \in N$ choose $f_O \in C_c(G)$, $\mathrm{supp}(f_O) \subset O$, $f_O \geq 0$, $\int_G f_O = 1$. This is called an approximate delta function.

Proposition 62 *Let $\pi : G \to Aut(V)$ be a strongly continuous representation on isometries of V, a Banach Space. Let $f \to \pi_f$ be it's integrated form. Then for any $v \in V$ $\pi_{f_O} v \to v$.*

Proof: We have the following norm equalities.

$$\|\pi_{f_O} v - v\| = \left\| \int f_O \pi_x v dx - (\int f_O(x) dx) v \right\| = \left\| \int f_O(x)(\pi_x v - v) dx \right\| \leq \epsilon$$

■

In $L^1(G)$, we have $(f * g)(x) = \int f(y) g(y^{-1}x) dy = (\pi_f g)(x)$ wherein π is the left translation action on $L^1(G)$. We have that, $e_O * g \to g$ for all $g \in L^1(G)$. So, $\{e_O\}$ is a left approximate identity. It is also true that this a right approximate identity (we leave the details to the reader). We thus have that $L^1(G)$ has a two sided approximate identity of norm 1.

Let $\mathcal{A}$ be a normed algebra and let $\pi : \mathcal{A} \to End(V)$ so that $\|\pi_a\| \leq \|a\|$.

Definition 63 *The representation π is non degenerate if the linear span $\{\{\pi_a v\} : a \in \mathcal{A}, v \in V\}$ is dense.*

Proposition 64 *if $\mathcal{A}$ has a 2 sided approximate identity $\{e_\lambda\}$ then π is non degenerate if and only if $\pi_{e_\lambda} v \to v$ for all $v \in V$.*

Proof: For any $\pi_a v$, have $\pi_{e_\lambda(\pi_a)v} = \pi_{e_\lambda a} v \to \pi_a v$. So for $w \in span(\{\pi_a v\})$ we have our result. If span is dense, we get the desired result from a simple $\epsilon/3$ argument. ■

Proposition 65 *For G locally compact, and any strongly continuous representation of π of G on a Banach Space V on isometries, the integrated form of π is a non degenerate representation of $L^1(G)$. The author wonders about the converse.*

Let S be a locally compact semi group with e. Define the usual $C_c(S)$. Let $M(S)$ be all the finite $\mathbb{C}$ valued Radon Measures on S, continuous for the infinity norm. $M(S)$ is the dual Banach Space of $C_\infty(S)$. For $\mu, \nu \in M(S)$ define $\mu * \nu$ by, for any $h \in C_\infty(S)$, set

$$(\mu * v)(h) = \int (\int h(xy) d\mu(x)) d\nu(y)$$

$*$ is an associative product on $M(S)$ and $\|\mu * \nu\|' \leq \|\mu\|' \|\nu\|'$. This is indeed a fine banach algebra with identity element: δ_e.

If G is locally compact, and choose a Haar measure dx, then for each $f \in L^1(G)$ can be viewed as a measure $f(x)dx$, whence $L^1(G) \to M(G)$ is an isometry. We have the following interesting fact:

Proposition 66 *$L^1(G)$ is a 2 sided ideal in $M(G)$. And in fact, $M(G)$ is the stone cech compactification of $L^1(G)$.*

For any algebra $\mathcal{A}$ without 1, what is it's β compactification? It is the "biggest" algebra $\mathfrak{B}$ with 1 in which $\mathcal{A}$ sits as a 2sided essential ideal. i.e, if $b \in \mathfrak{B}$ and if $ba = 0$ for all a then $b = 0$. We lead on to the construction of the Multiplier Algebra.

If $\mathcal{A}$ is a two sided ideal in $\mathfrak{B}$, then for all $b \in B$ have operators on $\mathcal{A}$, L_b, R_b of left and right multiplication on $\mathcal{A}$, with:

1. $L_b(aa') = L_b(a)a'$
2. $R_b(aa') = aR_b(a')$
3. $R_b(a)a' = aL_ba'$

Definition 67 *A multiplier or a double centralizer of $\mathcal{A}$ is any pair (S,T) of operators on $\mathcal{A}$ that satisfy the above conditions. The collection of double centralizers forms an algebra $M(\mathcal{A})$ in which $\mathcal{A}$ sits as a 2 sided ideal. This is called the Multiplier Algebra.*

Lecture 19. October 7th. Strongly Continuous Representations and Non-Degenerate Representations

In the spirit of last lecture, we propose that the multiplier algebra of $C_\infty(M)$ where M is locally compact, is nothing but $C_b(M)$. We leave it as an exercise. For G locally compact group (not discrete), we have the multiplier algebra of $L^1(G)$ is nothing but the space of finite Radon Measures on G.

Let G be a locally compact, not discrete group. We have that points have Haar Measure 0. Let G_d be the group with discrete topology, and $l^1(G_d)$ is the algebra with convolution. Let, $B = L^1(G) \oplus l^1(G_d)$. Define $\|f \oplus F\| := \|f\| + \|F\|$. Given f, δ_z, we have after some computation that $(\delta_z * f)(h \in C_c(G)) \in L^1(G)$. We have that $L^1(G)$ sits as a 2 sided ideal in G.

So, B is a Banach Algebra with 1, containing $L^1(G)$ as a 2-sided ideal. $L^1(G)$ has an approximate identity of norm 1 for itself, from last time. We prove an important theorem here.

Theorem 68 *Let B be a banach algebra, and let I be a closed 2-sided ideal in B. Assume I has an approximate identity of norm 1. If π is a representation of I on a Banach Space V, with $\|\pi(d)\| \le \|d\|$, and if π is non degenerate, then π extends uniquely to a representation of B on V.*

Proof: We first prove uniqueness, as it gives us a clue for the other part. Suppose $\overline{\pi}$ exists, then for $b \in B$, $d \in I$, $v \in V$, we have

$$\overline{\pi}(b)(\pi_d v) = \overline{\pi_b}(\overline{\pi_d} v) = \overline{\pi_{bd}}(v) = \pi_{bd} v$$

Hence, what $\overline{pi}$ does on the range of π is completely determined by π. Also, the finite linear combinations of $\pi_d v$ is dense in V, thus by continuity, it's uniquely determined.

Now we discuss existence. We try defining $\overline{\pi}$ on finite sums by

$$\overline{\pi_b}(\sum \pi_{d_j} v_j) = \sum \pi_{bd_j}(v)$$

Is this well defined? We need if $\sum \pi_{d_j} v_j = 0$ then $\sum \pi_{bd_j}(v_j) = 0$ for all $b \in B$. Let $\{e_\lambda\}$ be an approximate identity. Suppose $\sum \pi_{d_j} v_j = 0$. We then have

$$\sum \pi_{bd_j} v_j = \lim_\lambda \sum \pi_{b(e_\lambda d_j)} v_j = \lim_\lambda \sum \pi_{(be_\lambda) d_j} v_j = \lim_\lambda \pi_{be_\lambda} \sum \pi_{d_j} v_j$$

Hence we have that it's well defined. We just need to check one more detail.

$$\left\|\overline{\pi_b}(\sum \pi_{d_j}(v_j))\right\| = \left\|\sum \pi_{bd_j} v_j\right\| = \lim \left\|\sum \pi_{be_\lambda d_j} v_j\right\| = \left\|\sum \pi_{be_\lambda}(\sum \pi_{d_j}(v_j))\right\| \le \|\pi_{be_\lambda}\| \left\|\sum \pi_{d_j} v_j\right\|$$
$$\le \|be_\lambda\| \le \|b\|$$

Thus $\|\overline{\pi_b}\| \le \|b\|$ on the dense subspace. Hence we have the extension. ∎

Let G be locally compact, not discrete. If π is a representation of $L^1(G)$ on V. Then π extends uniquely to a representation $\overline{\pi}$ on $L^1(G) \oplus l^1(G_d)$, and we have G is embedded inside $l^1(G_d)$. So, $\overline{\pi}|_{G \subset l^1(G)}$ is a representation of G on V by isometries. $\|\overline{\pi}_{\delta_x}\| \le \|\delta_x\| = 1$.

We wonder why $\overline{\pi}|_G$ is strongly continuous. We only need to check this on a dense subspace $\{\sum \pi_{f_i}(v_j)\}$. We are aiming to prove the important fact that Strongly continuous representations and non degenerate representations are in bijection!

Lecture 21. October 12th. Duals of Locally Compact Groups

Let G be a commutative locally compact group. Choose a Haar measure and form the $\mathcal{A} = L^1(G)$ algebra. Consider $\overline{\mathcal{A}} = L^1(G) \oplus \mathbb{C}\delta_e$. We are interested in finding the maximal ideal space of the above commutative Banach Algebra with 1. It is going to consist of homomorphisms of $\overline{\mathcal{A}}$ into $\mathbb{C}$. If $\mathcal{A}$ is commutative, we have one homomorphism with $\mathcal{A}$ as the kernel and every other homomorphism restricts to a unique non zero homomorphism of $\mathcal{A}$ into $\mathbb{C}$. We thus symbolically have:

$$\hat{\overline{\mathcal{A}}} = \hat{\mathcal{A}} \cup \{\text{point at infinity}\}$$

Note, $\hat{\overline{\mathcal{A}}}$ is compact for the weak-* topology. This gives that $\hat{\mathcal{A}}$ is the complement of one point in a compact space, and thus is locally compact. We know that the non zero homomorphisms of $L^1(G) \to \mathbb{C}$ are non-degenerate representations of $L^1(G)$ on $\mathbb{C}$. From previous lectures, these homomorphisms correspond to strongly continuous representations of G on $\mathbb{C}$, non zero and bounded. These are basically continuous homomorphisms from G into the circle group T.

$$L^1\hat{(G)} = \hat{G} = ContHom(G,T)$$

We also have that for the topology of uniform convergence on compact sets of G, saw that $\hat{G}$ is a topological group, and this coincides with the weak* topology.

Let V be a finite dimensional vector space over $\mathbb{R}$ with usual topology. Let $G = V$, forgetting scalar multiplication. Taking $G = \mathbb{R}^n$, we then have $\hat{G} = (\mathbb{R}^n)' \cong \mathbb{R}^n$. This is the main theorem we shall prove next lecture. The isomorphism is not cannonical though.

Lecture 22. October 14th. Inspirations towards Pontryagin Duality

We are interested in showing that $\hat{\mathbb{R}} = \mathbb{R}$. I.e, every $\phi : \mathbb{R} \to \pi$ continuous group homomorphism is of the form $r \to e^{isr}$ for some real s.

Theorem 69 $\hat{\mathbb{R}} \cong \mathbb{R}$

Proof: Let ϕ be given and $\phi(0) = 1$. ϕ is continuous so find an a such that on $(0, a)$, we have $|\phi(r)| \geq 1/2$. So $\int_0^a \phi(r)dr \neq 0$. Then for any $t \in \mathbb{R}$, we have

$$\phi(t)K := \phi(t) \int_0^a \phi(r)dr = \int_0^a \phi(r+t)dr = \int_t^{a+t} \phi(r)dr$$

So, $\phi(t)$ is differentiable, and we have $\phi'(t)K = \phi(a+t) - \phi(t) = (\phi(a) - 1)\phi(t)$. So, weh ave $\phi'(t) = K^{-1}(\phi(a) - 1)(\phi(t))$. So we have the differential equation $\phi'(t) = C\phi(t)$ which gives us that $\phi(t) = e^{Ct} + C_0$. Substituting the constraints, and the fact that the homomorphism is into the circle T, we have that $\phi(t) = e^{ist}$ for some real s. ■

This concludes the proof. For a general G commutative, let $f \in L^1(G)$. We have the Gelfand Transform given by $\phi \in \hat{G}$, $\hat{f}(\phi) = \int_G f(x)\phi(x)dx$ with the appropriate connotations of ϕ, respecting the correspondence between non degenerate representations of the L^1 group on V and strongly continuous representations of G on V. For $G = \mathbb{R}^n$, we have $\phi_s(x) = e^{ix.s}$, $s \in \mathbb{R}^n$. Thus, we have

$$\hat{f}(s) = \int_{\mathbb{R}^n} f(x)e^{ix.s}$$

is the fourier transform.

We have that the fourier transform $\hat{f} \in C_\infty(\hat{G})$ (with the weak* topology). For $G = \mathbb{R}^n$. Want to show that the weak * topology on $\hat{\mathbb{R}^n} = \mathbb{R}^n$ agrees with the usual topology. We start at that with some bookkeeping. We have $\hat{f}(s) = \int f(x)e^{ix.s}$, $f \in L^1(\mathbb{R}^n)$. By Lebesgue Dominated Convergence theorem, $\hat{f}$ is continuous. We also have the Riemann Lebesgue Lemma: $\hat{f}$ vanishes at ∞. We prove this (for $\mathbb{R}$) first for $f(x) = \chi_{[a,b]}$, and then for finite linear combinations, which forms a dense subspace.

We then have that every $\hat{f}$ for $f \in L^1(\mathbb{R}^n)$ extends continuously to the 1-pt compactification of $\mathbb{R}^n$, with $\hat{f}(\infty) = 0$. So, $\overline{\mathbb{R}^n}$(Usual Topology) $\to \overline{\mathbb{R}^n}$(Weak* Topology). By definition, $\hat{f}$ is continuous for the weak* topology, this concludes the proof.

Let G be a locally compact commutative group. $\hat{G} = \{\phi : G \to T\}$. On $L^1(G)$, have $*$, $f^*(x) = \overline{f(x^{-1})}$. After some computation, we have as before, $\hat{f^*} = \overline{\hat{f}}$. We have the range of the Fourier Transform on $L^1(G)$ is dense in $C_\infty(\hat{G})$, by Stone Weierstrass. Yet again, we want that fourier transform is injective.

$L^1(G)$ is represented faithfully on $L^2(G)$. Let $C^*(G) = C^*$-Algebra generated by the range of this representation. We have

$$L^1(G) \to C^*(G) \to Closure(C^*(G)) \cong C(X)$$

for some compact hausdorff X. If $f \neq 0$, then $\lambda_f \neq 0$, so $\hat{\lambda_f} \in C(X)$, which means there exists some $\phi \in X$ so that $\overline{\lambda_f}(\phi) \neq 0$. This completes the proof.

Let G be compact commutative. Let $\phi \in \hat{G}$. $\phi \in L^1(G)$. Consider from the invariance of Haar measure, $\int_G \phi(x)dx = \int \phi(xy)dx = \int \phi(x)\phi(y)dx = \phi(y)\int \phi(x)dx$. Hence, if $\exists y$ with $\phi(y) \neq 1$, then we have $\int \phi(x)dx = 0$. If $\phi, \xi \in \hat{G}.$, we have $\phi, \xi \in L^2(G)$. $< \phi, \xi >= \int \phi(x)\xi(x)dx = \int(\phi\bar{\xi})(x)dx = 0$ if $\phi\xi \neq 1$, i.e, $\phi \neq \xi$. View $\phi \in L^1(G)$, $\hat{\phi}(\xi) = \int \phi(x)\xi(x)dx = \int \phi\xi(x)dx = 0$ if $\overline{\phi} \neq \xi$. Thus $\hat{\phi}(\bar{(\phi)}) = \int \phi(x)\overline{\phi}(x)dx = \|1\|_1 = \|1\|_2$. Thus, if G is compact, then $\hat{G}$ is discrete, and if G is discrete, $\hat{G}$ is compact. This builds to a much harder result, the Pontryagin duality: $\hat{\hat{G}} = G$.

Lecture 23. October 16th. Detour to Spectral Mapping Theorem

Let $\mathcal{A}$ be a unital Banach Algebra over $\mathbb{C}$. Let $a \in \mathcal{A}$, and let f be a $\mathbb{C}$ valued function defined and holomorphic on an open set O of $\mathbb{C}$ with $\sigma(a) \subset O$. We want to define f in the following way.

Suppose $\sigma(a) = C_1 \sqcup C_2$. Let O_1 and O_2 be disjoint open sets with O_i containing C_i. Define f on O by $f(z) = 1$ if $z \in O_1$ and 0 if $z \in O_2$. (we want something like $f(a) = p$ where $p^2 = p$, i.e, projection).

Let γ be a finite collection of closed piecewise smooth paths in O that surround the $\sigma(a)$ so that $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}dz = 1$ if $w \in \sigma a$ and 0 outside γ. Define $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$.

Let $\mathcal{H}(O)$ be the algebra of holomorphic functions on O. We have the following theorem:

Theorem 70 *$f \to f(a)$ is a unital algebra homomorphism from $\mathcal{H}(O)$ into $\mathcal{A}$.*

We also have,

$$\|f(a)\| \leq \frac{1}{2\pi}\int_\gamma |f(z)| \left\|(z-a)^{-1}\right\| dz \leq \|f\|_{in\gamma} \sup_{z\in\gamma} \left\|(z-a)^{-1}\right\| length(\gamma)$$

If $f_n \in \mathcal{H}(O)$, and $f_n \to f$ uniformly on γ, then $f_n(a) \to f(a)$ in norm. ($f(a)$ does not depend on choice of γ).

Proof: Independence from path: Let γ_1 and γ_2 be two paths. Can find γ_3 that is inside both γ_1 and γ_2. Thus it suffices to do for γ_2 inside γ_1, $\int_{\gamma_1} - \int_{\gamma_2} = 0$?

But this is true if and only if $\int_{\gamma_1-\gamma_2} = 0$. But, $z \to f(z)(z-a)^{-1}$ between the two curves is holomorphic between γ_1 and γ_2. Thus we have the result from Cauchy's theorem (after sufficiently applying to an element of the dual Banach Space, to avoid issues with domain of cauchy's theorem). This map is obviously linear. But we wonder why it is an algebra homomorphism. This is slightly non trivial.

We have the following:

$$\begin{aligned}
f(a)f(b) &= \left(\frac{1}{2\pi i}\int_{\gamma_1} f(z_1)(z_1-a)^{-1}dz_1\right)\left(\frac{1}{2\pi i}\int_{\gamma_1} g(z_2)(z_2-a)^{-1}dz_2\right) \\
&= \frac{1}{2\pi i}^2 \int_{\gamma_1}\int_{\gamma_2} f(z_1)g(z_2)(z_1-a)^{-1}(z_2-a)^{-1}dz_1dz_2
\end{aligned}$$

But we know that $(z_1-a)^{-1} - (z_2-a)^{-1} = (z_1-a)^{-1}(z_2-z_1)(z_2-a)^{-1}$. Assuming that γ_2 is inside γ_1, we continue:

$$\begin{aligned}
&\frac{1}{2\pi i}^2 \int_{\gamma_1}\int_{\gamma_2} f(z_1)g(z_2)\frac{(z_1-a)^{-1}}{z_2-z_1}dz_1dz_2 - \frac{1}{2\pi i}^2 \int_{\gamma_1}\int_{\gamma_2} f(z_1)g(z_2)\frac{(z_2-a)^{-1}}{z_2-z_1}dz_1dz_2 \\
&= \int_{\gamma_1}\left(\int_{\gamma_2}\frac{g(z_2)}{z_2-z_1}dz_1\right)(z_1-a)^{-1}f(z_1)dz_1 + \int_{\gamma_2}\left(\int_{\gamma_1}\frac{f(z_1)}{z_2-z_1}dz_1\right)(z_2-a)^{-1}g(z_2)dz_2 \\
&= \frac{1}{2\pi i}\left(0 + \frac{1}{2\pi i}\int_{\gamma_2} f(z_2)g(z_2)(z_2-a)^{-1}dz_2\right) = (fg)(a)
\end{aligned}$$

■

Once we have finished the proof, the following proposition (spectral mapping) follows:

Proposition 71 $\sigma(f(a)) = f(\sigma(a))$

Proof: Let $\lambda \in \sigma(a)$. Then, $f(z) - f(\lambda)$ is holomorphic (0 at $z = \lambda$). Thus, this is $(z - \lambda)g(z)$ for some holomorphic g. Thus, $f(a) - f(\lambda) = (a - \lambda)g(a)$. If the LHS were invertible, then $(a - \lambda)$ is invertible which is a contradiction. Thus, $f(\sigma(a)) \subseteq \sigma(f(a))$. For the other inclusion, we have the following:

If $\lambda \notin f(\sigma(a))$, then, $f(z) - \lambda$ is not 0 on $\sigma(a)$. So, $(f(z) - \lambda)^{-1}$ is holomorphic on $\sigma(a)$. So $(f(a) - \lambda)^{-1}$ so $\lambda \notin \sigma(f(a))$ ∎

Lecture 24. October 19th. On Approximate Eigenvectors

We start with a remark. We consider G is locally compact commutative, $L^1(G)$, $\hat{G}$ and $f \to \hat{f} \in C_\infty(\hat{G})$. Consider $C_c(G) \subset L^1(G) \cap L^2(G)$ is a subalgebra of $L^1(G)$ for convolution. We have the following theorem.

Theorem 72 (Plancheral) *For a suitable choice of Haar measure on $\hat{G}$, find that for $f \in L^1(G) \cap L^2(G)$, have $\hat{f} \in L^2(\hat{G}) \cap C_\infty(\hat{G})$ and $\left\|\hat{f}\right\|_2 = \|f\|_2$. Thus, $f \to \hat{f}$ extends to a unitary operator from $L^2(G) \to L^2(\hat{G})$.*

Now, we look more closely at operators on Hilbert Space, to get a different treatment of the theory that we have built.

Let V be a vector space over $\mathbb{C}$. Define a Sesquilinear form, to be one that is linear in the first variable and conjugate linear in the second (Not required to be positive). We have the following polarization identity, inspired from physics, as one would think.

Theorem 73 (Polarization) *For ξ, $\psi \in V$, $<\xi, \psi>= \frac{1}{4}\sum_{k=0}^{3} i^k <\xi + i^k\psi, \xi + i^k\psi>$*

Let $T \in \mathfrak{B}(\mathcal{H})$, a bounded operator on a Hilbert Space over $\mathbb{C}$.

Proposition 74 *If $<T\xi, \xi>= 0$ for all $\xi \in \mathcal{H}$, then $T = 0$.*

Proof: Define a sesquilinear form given by $<\xi, \psi>' = <T\xi, \psi>$. From the polarization identity, we have $<\xi, \psi>' = 0$, and thus, $<T\xi, \psi>= 0$ for all ξ and ψ, as desired. ■

Proposition 75 *If $<T\xi, \xi> \in \mathbb{R}$ for all ξ, then T is self adjoint.*

Proof: $<T\xi, \xi>=<\xi, T^*\xi>=<T^*\xi, \xi>$. Thus, we have $T - T^* = 0$ from the previous proposition. ■

Proposition 76 *For $T \in \mathfrak{B}(\mathcal{H})$ we have $Ker(T) = range(T^*)^\perp$*

Proof: If $\xi \in ker(T)$, we have $T(\xi) = 0 \Leftrightarrow <T\xi, \psi>= 0$ for all ψ, $\Leftrightarrow <\xi, T^*\psi>= 0$
$\Leftrightarrow \psi \in \perp (range(T^*))$ ■

Proposition 77 *Let $T \in \mathfrak{B}(\mathcal{H})$. If there are $a, b \in \mathbb{R}$, $a > 0$, $b > 0$ such that $\|T\xi\| \geq a\|\xi\|$ for all ξ, and $\|T^*\xi\| \geq b\|\xi\|$ for all ξ, then T is invertible in $\mathfrak{B}(\mathcal{H})$.*

Proof: If $T\xi = 0$, then from the first inequality, we have $\xi = 0$. Hence, T is injective. Similarly, T^* is injective. But $Ker(T) = (range(T))^\perp \Rightarrow range(T)$ is dense in $\mathcal{H}$. Also, if $\xi \in range(T)$, $\|\xi\| = \|TT^*\xi\| \geq a\left\|TT^{-1}\xi\right\| \geq a\left\|TT^{-1}\xi\right\| \geq a\left\|T^{-1}\xi\right\|$. Since T^{-1} is bounded on the range of T, we have that T^{-1} extends, which concludes the proof. ■

We have the following immediate consequence:

Proposition 78 *Let $T \in \mathfrak{B}(\mathcal{H})$. If T is not invertible, then either*

1. *There is a sequence $\{\xi_n\}$, $\|\xi_n\| = 1$ for all n, and $T\xi_n \to 0$*
2. *There is a sequence $\{\xi_n\}$, $\|\xi_n\| = 1$ for all n, and $T^*\xi_n \to 0$*

Corollary 79 *Let $T \in \mathfrak{B}(\mathcal{H})$. If $\lambda \in \sigma(T)$, then, $\exists\{\xi_n\}$, $\|\xi_n\| = 1$ with $T(\xi_n) - \lambda\xi_n \to 0$, or $\exists\{\xi_n\}$, $\|\xi_n\| = 1$ with $T^*(\xi_n) - \overline{\lambda}\xi_n \to 0$. We then say (in the first case), ξ_n is an approximate λ eigenvector for T.*

Let $T \in \mathfrak{B}(\mathcal{H})$ with T normal. Then for any $\xi \in \mathcal{H}$, $\|T^*\xi\| = \|T\xi\|$. This is an immediate consequence. This gives us the following punchline of the lecture:

Theorem 80 *Let $T \in \mathfrak{B}(\mathcal{H})$, T is normal, then if $\lambda \in \sigma(T)$, then there exists a λ approximate eigenvector for T.*

Lecture 25. October 21st. Non Degenerate Representations, and Cyclic Vectors

We start with the definition of "positivity".

Definition 81 *For $T \in \mathfrak{B}(\mathcal{H})$, we say that T is positive ($T \geq 0$) if $< T\xi, \xi > \geq 0$ for all ξ (note that this automatically implies T is self adjoint).*

Proposition 82 *For $T \in \mathfrak{B}(\mathcal{H})$, the following are equivalent:*

1. $T \geq 0$
2. *T is self adjoint and $\sigma(T) \in \mathbb{R}^+$*
3. $\exists S \in \mathfrak{B}(\mathcal{H})$, $S^* = S$, $T = S^2$

Proof: For $1 \Rightarrow 2$: If $\lambda \in \sigma(T)$, then $\exists \{\xi_n\}$ with norm 1 such that $(T-\lambda)\xi_n \to 0$. Then, $< (T-\lambda)\xi_n, \xi_n > = < T\xi_n, \xi_n > - < \lambda\xi_x, \xi_n > \to 0$, it immediately follows that $\lambda \geq 0$

For $3 \Rightarrow 1$: $< T\xi, \xi > = < S^*S\xi, \xi > = < S\xi, S\xi > \geq 0$.

For $2 \Rightarrow 3$: Let $\mathcal{A} = C^*$ Subalgebra generated by T and I. Then $\mathcal{A} \cong C(\sigma(T))$, given by the map $T \to (t \to t)$. By continuous functional calculus, we have $t \to \sqrt{t}$ corresponds to an operator S in $\mathcal{A}$, and S is non negative and satisfies our requirement. ■

Given an abstract C^* Algebra, $\mathcal{A}$, given a, say $a \geq 0$ if $a^* = a$ and $\sigma(a) \in \mathbb{R}^+$, equivalently $a = b^2$ and $b^* = b$. We remark here that showing that sums of positive elements is positive, is really a non trivial statement.

Let $\mathcal{H}$ be a Hilbert Space, S is a subset of $\mathfrak{B}(\mathcal{H})$, self adjoint i.e, $T \in S$ then $T^* \in S$.

Definition 83 *A subspace K of $\mathcal{H}$ is $S-$invariant if $\xi \in K \Rightarrow T\xi \in K$ for all $T \in S$.*

We have an immediate consequence:

Proposition 84 *If K is $S-$ invariant, then so is $K^\perp$.*

Definition 85 *A closed subspace K is S-irreducible if K contains no proper S invariant subspaces.*

Let $\mathcal{A}$ be a *Algebra and π a *-representation of $\mathcal{A}$ on $\mathcal{H}$, i.e, $\pi : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$, non-degenerate, then choose $\xi \in \mathcal{H}$, $\xi \neq 0$, let $K = \overline{\{\pi(a)\xi\}}$. We say that K is a π-cyclic subspace of $\mathcal{H}$, and ξ is a cyclic vector for this subspace.

We end the lecture with an important proposition:

Proposition 86 *For any non-degenerate *=representation of $\mathcal{A}$ on a Hilbert $\mathcal{H}$, can find a family of mutually orthogonal cyclic- $\mathcal{A}$-invariant subspaces, $\{K_\lambda\}$ such that $\mathcal{H} = \oplus_\lambda K_\lambda$.*

Proof: Mimic the existence of basis proof, using Zorn's lemma. ■

Lecture 26. October 24th. Pre Inner Product Derived from Positive Linear Functionals

We start first by discussing direct sums of Hilbert Spaces.

Let Λ be an indexing set. For each $\lambda \in \Lambda$, let $\mathcal{H}_\lambda$ be a Hilbert Space. Then, $\oplus_{\lambda\in\Lambda}\mathcal{H}_\lambda = \{(\xi_\lambda)_{\lambda\in\Lambda} : \sum \|\xi_\lambda\|^2 < \infty\}$. Define $< (\xi_\lambda), (\phi_\lambda) >= \sum < \xi_\lambda, \phi_\lambda >_{\mathcal{H}_\lambda}$. Why is $\oplus\mathcal{H}_\lambda$ complete? We look at the space of Complex valued functions on Λ, $l^2(\Lambda)$, this is complete, and the same steps follow for the completeness of our direct sum.

Now, consider $\{T_\lambda\}$, $T_\lambda \in \mathfrak{B}(\mathcal{H}_\lambda)$. Define $\oplus T_\lambda$ on $\oplus\mathcal{H}_\lambda$ by $(\oplus T_\lambda)(\xi_\lambda) = (T_\lambda \xi_\lambda)$. This is well defined only if there exists k such that $\|T_\lambda\| \leq k$ for all λ. (Then, $\|\oplus T_\lambda\| \leq k$)

If $\mathcal{A}$ is a *-Algebra, and if π_λ is a *-representation of $\mathcal{A}$ on $\mathcal{H}_\lambda$, then define $\oplus\pi_\lambda$ by $(\oplus\pi_\lambda)(a) = \oplus\pi_\lambda(a) \in \mathfrak{B}(\oplus\mathcal{H}_\lambda)$ if there is a constant k such that $\|\pi_\lambda(a)\| \leq k_a$ for all λ, $\|\oplus\pi_\lambda(a)\| \leq k_a$. We now get more specific:

If $\mathcal{A}$ is a normed *-algebra, then if there exists k so that $\|\pi_\lambda\| \leq k$ for all λ, then $\oplus\pi_\lambda(a) \in \mathfrak{B}(\oplus\mathcal{H}_\lambda)$, and $\|\oplus\pi_\lambda(a)\| \leq k\|a\|$, furthermore, $a \to \oplus\pi_\lambda(a)$ is a *-representation of $\mathcal{A}$ on $\oplus\mathcal{H}_\lambda$.

For a while, assume that $\mathcal{A}$ has $1_\mathcal{A}$, and that for any representation π, $\pi(1_\mathcal{A}) = I_\mathcal{H}$. Now let π be a representation of $\mathcal{A}$ on a Hilbert Space, $\mathcal{A}$ is *-normed with 1 and $\|\pi\| \leq k$. Let $\mathcal{H} = \oplus\mathcal{H}_\lambda$ be a cyclic decomposition of $\mathcal{H}$ for $\pi(\mathcal{A})$, with cyclic vectors ξ_λ. Let π_λ be the restriction of π to $\mathcal{H}_\lambda$, so that $\|\pi_\lambda\| \leq \|\pi\|$. We then have $\pi' = \oplus\pi_\lambda$ is a *-representation of $\mathcal{A}$.

Definition 87 *Let $(\pi, \mathcal{H})$, $(\pi', \mathcal{H}')$ be *-representations of S, say that they are unitarily equivalent if there is a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that, $U(\pi(a)\xi) = \pi'(a)(U\xi)$.*

We have in our case, $(\mathcal{H}, \pi)$ is unitarily equivalent to $(\oplus\mathcal{H}_\lambda, \pi_\lambda)$. Let $\mathcal{A}$ be a unital *-Algebra and let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$, and let $\xi \in \mathcal{H}, \xi \neq 0$. Define $\mu : \mathcal{A} \to \mathbb{C}$ by $\mu(a) =< \pi(a)\xi, \xi >$. Then, $\mu(a^*a) =< \pi(a^*a)\xi, \xi >=< \pi(a)\xi, \pi(a)\xi >\geq 0$. This leads us to make the following definition:

Definition 88 *A linear functional μ on $\mathcal{A}$ is positive if $\mu(a^*a) \geq 0$ for all $a \in \mathcal{A}$. If $\mu(1_\mathcal{A}) = 1$, then μ is a state (analagous to a probability measure).*

Let μ be some positive linear functional on $\mathcal{A}$. For any $a, b \in \mathcal{A}$, set $< a, b >_\mu= \mu(b^*a)$. Then, $<>$ is a pre inner product. We just check here the proof of $< b, a >_\mu= \overline{< a, b >_\mu}$. For some $z \in \mathbb{C}$, we have $< a+zb, a+zb >_\mu= \mu((a+zb)^*(a+zb)) \geq 0$. But, this is nothing but $\mu(a^*a)+\overline{z}\mu(b^*a)+z\mu(a^*b)+\|z\|^2\mu(b^*b)$ which belongs to $\mathbb{R}$. This, $\overline{z}\mu(b^*a) + z\mu(a^*b) \in \mathbb{R}$. Taking $z = 1, i$ gives us our result.

Factor $\mathcal{A}$ by $\mathcal{N}_\mu = \{a :< a, a >_\mu= 0\}$ so that we get an inner product on $\mathcal{A}/\mathcal{N}_\mu$. Taking the completion, we get a Hilbert space $L^2(\mathcal{A}, \mu)$. Let L be the left regular representation. We have $< L_a b, c >_\mu=< ab, c >= \mu(c^*(ab)) = \mu((a*c)^*b) =< b, a^*c >_\mu=< b, L_{a^*}c >$. So, $a \to L_a$ is a *-representation.

Lecture 27. October 26th. Gelfand-Naimark-Segal Construction

Let $\mathcal{A}$ be a *-Algebra with 1. We have defined earlier what a positive linear functional is. Let μ be positive. We have also defined $< a, b >_\mu = \mu(b^* a)$. Look at $< L_a(b), c >_\mu = < b, L_a^*(c) >_\mu$. Let $\mathcal{N}_\mu = \{a \in \mathcal{A} : < a, a >_\mu = 0\}$. Then we have from the Cauchy Schwartz inequality (works for pre-inner products too), $\mathcal{N}_\mu = \{a \in \mathcal{A} : < a, b >_\mu = 0 \forall b \in \mathcal{A}\}$. Thus, $\mathcal{N}_\mu$ is a linear subspace of $\mathcal{A}$. In fact, $\mathcal{N}_\mu$ is carried into itself by the Left regular representation. Given $a \in \mathcal{N}_\mu$, $b \in \mathcal{A}$, $< L_b a, c >_\mu = < a, b^* c >_\mu = 0$. In other words, $\mathcal{N}_\mu$ is a left ideal in $\mathcal{A}$. Thus, $< ., . >_\mu$ drops to an inner product on $\mathcal{A}/\mu$. Let $L^2(\mathcal{A}, \mu)$ be the completion of this quotient space. This is a Hilbert Space.

Here is a key proposition:

Proposition 89 1. *If $\mathcal{A}$ is a Banach *-Algebra with 1, then any positive linear functional μ on $\mathcal{A}$ is continuous with $\|\mu\| = \mu(1)$.*

2. *If $\mathcal{A}$ is a normed *-Algebra with 1 and if μ us a continuous positive linear functional on $\mathcal{A}$, then $\|\mu\| = \mu(1)$.*

And further, as in $1, 2$, left regular representation on $\mathcal{A}/\mu$ satisfies $\|L_a\| \leq \|a\|$. (note, proving 1 is more than sufficient)

Proof: Let first $a \in \mathcal{A}$ with $a^* = a$, $\|a\| \leq 1$. Consider the function f on $\mathbb{C}$ defined by $f(z) = (1-z)^{1/2}$ is holomorphic on $\{\|z\| < 1\}$. So, there's a power series $\sum \alpha_n x^n$ converging absolutely for any z, $|z| < 1$. Since $f(z)$ is to have values in $\mathbb{R}$ for $z \in \mathbb{R} \cap \{|z| < 1\}$, $\alpha_n \in \mathbb{R}$ for all n. Let $b = f(a)$. Since $a^* = a$ and $\alpha_n \in \mathbb{R}$, we get $b^* = b$. Also, $b^2 = 1 - a$ and $\mu(1-a) = \mu(b * b) \geq 0$, so $\mu(1) \geq \mu(a)$. We do the same thing to $-a$ to get $\mu(1) \geq \|\mu(a)\|$ as desired. For any $a \in \mathcal{A}$ with $a^* = a$, we just consider $\frac{a}{\|a\|+\epsilon}$ and apply the above arguments. We get $|\mu(q)| \geq \left\|\mu(\frac{a}{\|a\|+\epsilon})\right\|$, so $|\mu(a)| \leq \|a\| \mu(1)$. Now, finally, for any $b \in \mathcal{A}$, we have $|\mu(b)|^2 = | < b, 1 >_\mu | \leq < b, b >_\mu < 1, 1 >_\mu \leq \|b^* b\| (\mu(1)^2) \leq (\|b\| \mu(1))^2$. ■

Now, to complete the last part of the proposition, i.e, the left regular representation is by bounded operators on $\mathcal{A}/\mathcal{N}_\mu$, $< ., . >_\mu$.

Let $a, b \in \mathcal{A}$. Consider $L_a b$. $< L_a b, L_a b > \mu = < ab, ab >_\mu = \mu(b^* a^* ab)$. Now define ν by $\nu(c) = \nu(b^* cb)$. We then have that $< L_a b, L_a b > \mu = \nu(a^* a)$. Then, $\|\nu\| \leq \nu(1)$. Hence, $\nu(a^* a) \leq \|a^* a\| \nu(1) = \|a^* a\| \mu(b^* b) = \|a^* a\| < b, b >_\mu = \|a\|^2 < b, b >_\mu$.

Lecture 28. October 28th. Unitary Equivalence Classes for Positive Cyclic Representations

Let $\mathcal{A}$ be a *-normed algebra, unital. And μ is a continuous positive positive linear functional. As in GNS, we have a pre inner product $< a, b >_\mu = \mu(b^* a)$ and the respective "null vectors" space $\mathcal{N}_\mu$. We consider the norm completion of $\mathcal{A}/\mathcal{N}_\mu$, $L^2(\mathcal{A}, \mu)$, and $a \to L_a$ is a bounded $*$ representation.

Define $\xi_0 \in L^2(\mathcal{A}, \mu)$ to be the image of $1_\mathcal{A}$. Then, $\{L_a \xi_0 : a \in \mathcal{A}\}$ is the image of $\mathcal{A}$ in $L^2(\mathcal{A}, \mu)$, and is therefore dense. Thus, ξ_0 is a cyclic vector.

We say that the positive linear functional associated with ξ_0 to be $< L_a \xi_0, \xi_0 >=< a, 1 >_\mu = \mu(a)$.

Proposition 90 *Let* $(\mathcal{H}, \pi, \xi_0)$ *and* (K, ρ, ϕ) *be two cyclic representations of* $\mathcal{A}$ *with cyclic vectors* ξ_0, ϕ_0. *Let* μ *be the positive linear functional for* ξ_0 *and* ν *for* ϕ_0. *If* $\nu = \mu$, *then there exists a unitary operator* $U : \mathcal{H} \to K$ *such that* $U\xi_0 = \phi_0$, *and* U *is an "intertwining operator" for* π *and* ρ. *i.e,* $U(\pi(a)\xi) = \rho(a)U\xi$ *for all* $\xi \in \mathcal{H}$, *and* $U(a\xi) = a(U\xi)$.

Proof: Try to define U on $\{\pi(a)\xi_0 : a \in \mathcal{A}\} \subseteq \mathcal{H}$ by $U(\pi(a)\xi_0) = \rho(a)\phi_0$. Is this well defined? We need to show that if $\pi(a)\xi_0 = 0$, then $\rho(a)\phi_0 = 0$. Note, $< U\pi(a)\xi_0, U\pi(b)\xi_0 >=< \rho(a)\phi_0, \rho(b)\phi_0 >=< \rho(b^* a)\phi_0, \phi_0 >= \nu(b^* a) = \mu(b^* a) =< \pi(b^* a)\xi_0, \xi_0 >=< \pi(a)\xi_0, \pi(b)\xi_0 >$. ■

Definition 91 *A pointed representation of* $\mathcal{A}$ *is a* $(\mathcal{H}, \pi, \xi_0)$ *with* ξ_0 *a cyclic vector. We also define two pointed representations to be unitarily equivalent if there exists a unitary intertwining operator as seen above.*

We end the lecture with a statement that there is a bijection between continuous positive linear functionals on $\mathcal{A}$ and unitary equivalence classes of positive cyclic representations.

Lecture 29. October 30th. Preview to Borel Functional Calculus

We start by remarking that the literature in physics uses different terminology, i.e, "mixed states" for our usual definition of states, and "states" for our usual definition of extreme points (in the convex compact set of states). We state an important theorem:

Theorem 92 *Let $\mathcal{A}$ be a normed *-algebra, unital, and let $(\mathcal{H}, \pi)$ be a non-degenerate *-representation of $\mathcal{A}$, then there is a collection of $\{\phi_\lambda\}$ of positive linear functionals (continuous) on $\mathcal{A}$ such that $(\mathcal{H}, \pi)$ is unitarily equivalent to $\oplus_\lambda(\mathcal{H}_{\phi_\lambda}, \pi_{\phi_\lambda})$ (GNS representation for ϕ_λ).*

In the case of $\mathcal{A}$ is commutative, we have the following examples:

- Let T be a normal operator on a Hilbert Space $\mathcal{H}$. Let $\mathcal{A} = C^*(T, I)$. We want to know how does $\mathcal{A}$, and in specific, T act on $\mathcal{H}$.
- Let $T_1, T_2, T_3, ..., T_n$ be self adjoint operators on $\mathcal{H}$ and assume they commute with each other.
- G is locally compact abelian group. Let $(\mathcal{H}, U)$. We are interested in $C^*(U_{L^1(G)}, I_\mathcal{H})$.

Let $(\mathcal{H}, \pi, \xi)$ be a cyclic representation of $\mathcal{A}$. Consider $C^*(\pi(a)) = \mathfrak{B}$. $\mathfrak{B}$ is commutative and in fact, $\mathfrak{B} = C(\hat{\mathfrak{B}})$. ξ is still a cyclic vector for the action of $\mathfrak{B}$ (because it's smaller). Let ϕ_ξ be the positive linear functional on $\mathfrak{B}$ associated with ξ. Can view ϕ_ξ as a positive radon measure on $C(\hat{\mathfrak{B}})$. Then representation of $\mathfrak{B}$ on $\mathcal{H}$ is unitarily equivalent to representation of $\mathfrak{B}$ on $L^2(\hat{B}, \phi_\xi)$.

In the case of G is locally compact commutative group, we have $L^1\hat{(}G) = \hat{G} = \{\phi : G \to T\}$. Let $(\mathcal{H}, U, \xi)$ be a cyclic representation of $L^1(G)$, so corresponds to a representation π of G on $\mathcal{H}$. Let ϕ_ξ be a Radon Measure on $\hat{G}$. We have $(\mathcal{H}, \pi)$ is unitarily equivalent to $L^2(\hat{G}, \mu_\xi)$, where $\pi(f)$ is pointwise multiplication by $\hat{f}$. For $x \in G$, π_x acts by pointwise multiplication by the function on $\hat{G}$ given by $\hat{x}(\phi) = \phi(x)$.

Lecture 30. November 4th. Borel Functional Calculus and Projection Valued Measures

Starting off with a preview: Let $T \in \mathfrak{B}(\mathcal{H})$, T self adjoint and $\sigma(T) = \mathbb{R}$. Then, we have from G-N, $C^*(T, I) \cong C(\sigma(T))$. Let $\mathcal{H} = \oplus \mathcal{H}_\lambda$, cyclic subspaces for T, i.e, for $C^*(T, I) = \mathcal{A}$. For each λ, $\mathcal{H}_\lambda \cong L^2(\sigma(T), \mu_\lambda)$, with cyclic vector $\xi_0 = 1$, with $C(\sigma(T))$ acts by pointwise multiplication: T acts by $f(t) = t$.

$\mathcal{H}$ as a representation of $C^*(T, I)$ is isomorphic to $\oplus_\lambda L^2(\sigma(T), \mu_\lambda)$. Let $\mathfrak{B}$ be the algebra of bounded $\mathbb{C}$ valued Borel Functions on $\sigma(T)$. For $F \in \mathfrak{B}$, let it act by pointwise multiplication on each $L^2(\sigma(T), \mu_\lambda)$. So, $\mathfrak{B}$ is realized as an algebra of operators on $\mathcal{H}$, and $C(\sigma(T)) \subseteq \mathfrak{B}$. For f in $C(\sigma(T))$, we had the continuous functional calculus, such that f acts on $\mathcal{H}$. The Borel functional calculus extends the continuous functional calculus.

For each $t \in \mathbb{R}$, define $\chi_{(-\infty,t]}|_{\sigma(T)} \in \mathfrak{B}$. Let $E_t = \chi_{(-\infty,t]}(T)$. E_t is a projection operator, i.e, $E_t^2 = E_t$ and $E_t^* = E_T$. Note that $t \to E_t$ is non decreasing, i.e, if $t_1 > t_0$ then $E_{t_1} \geq E_{t_0}$, i.e, $E_{t_1} E_{t_0} = E_{t_0}$.

Lemma 93 *If f_n is a sequence of elements of $\mathfrak{B}$ such that $\exists k$, $\|f_n\|_\infty \leq k$, and F_n converges to $F \in \mathfrak{B}$ pointwise, then $F_n(T)$ converges to $F(T)$ for the strong operator topology, i.e, $F_n(T)\xi \to F(T)\xi$ for the norm of $\mathcal{H}$.*

Proof: Suffices to treat the case when $F = 0$. Suffices to check for $\xi \in H_\lambda$ for some λ. We have

$$\|F(T)\xi\|^2 = \int_{\sigma(T)} |F_n \xi|^2 d\mu_\lambda = \int |F_n|^2 |\xi|^2 (\in L^1) \to 0$$

from Lebesgue Dominated Convergence Theorem. ∎

Let $t_n \in \sigma(T)$, t_n decreases to t_0. Then, E_{t_n} decreases to E_{t_0} for strong operator topology. We "can" integrate continuous $\mathbb{R}$ valued functions, then $T = \int_{\sigma(T)} t dE(t)$. For any Borel set S of $\sigma(T)$, $\chi_S(T)$ is a projection. Such projections are called "Spectral Projections" for T.

The function $S \to \chi_S(T)$ is a "Projection Valued Measure" (for $S \in Borel - sets$).

If $T_1, ..., T_n$ are commuting self adjoint (or normal) operators on $\mathcal{H}$, then $C^*(T_1, ..., T_n, I) \cong C(X)$. X is the joint spectrum of $T_1, T_2, ...T_n$. For any Borel set S, view ξ_S as an operator on $\mathcal{H}$, $S \to \chi_S$ as an operator is a projection valued measure.

Lecture 31. November 7th. On Compact Operators

We conclude our discussion on the Spectral Theorem with a few remarks. Let $\mathcal{A}$ be a commutative *-normed algebra with 1, and if $(\mathcal{H}, \pi)$ is a non-degenerate continuous *-representation of $\mathcal{A}$. We saw that $\mathcal{H} \cong \oplus_j L^2(\hat{\mathcal{A}}_{sa}, \mu_j)$ where "sa" denotes self adjoint. For any bounded Borel Function F on X, it gives an operator on $\mathcal{H}$ by pointwise multiplication on each $L^2(X, \mu_j)$. For each Borel set S, χ_S gives a projection operator.

For $\mathcal{A}$ a commutative C*-Algebra in $\mathfrak{B}(\mathcal{H})$, $\mathcal{A} \cong C(X)$ for some compact hausdorff space X. Suppose $\mathcal{A}$ has 1. Let $(\mathcal{H}, \pi)$ be non degenerate *-representations of $\mathcal{A}$. Assume $\mathcal{H}$ is seperable, then there are positive finite Borel measures on $\hat{\mathcal{A}}$, $\mu_1, ..., \mu_\infty$ mutually disjoint, such that

$$(\mathcal{H}, \mu) = L^2(\hat{A}.\mu_1) \oplus (L^2(\hat{A}.\mu_2) \oplus L^2(\hat{A}.\mu_2)) \oplus (L^2(\hat{A}.\mu_3) \oplus L^2(\hat{A}.\mu_3) \oplus L^2(\hat{A}.\mu_3)) \oplus \ldots$$

With the above statement on multiplicity, we conclude this section of the class on the Spectral Theorem.

Now, we begin our discussion on Compact Operators. Let X and Y be Banach Spaces and let $\mathfrak{B}(X,Y)$ be the Banach Space of bounded operators from X to Y.

Definition 94 *$T \in \mathfrak{B}(X,Y)$ is compact, i.e, belongs to $\mathfrak{B}_0(X,Y)$ if $\overline{T(Ball_1(X))}$ is a compact subset of Y. Equivalently, if $T(Ball_1(X))$ is totally bounded.*

Example: Finite rank operators. Note, the range is finite dimensional, and therefore the closed unit ball is compact.

Example: Many typical integral operators. Let M, N be measurable spaces. Let K be a measurable function on M x N. Define an operator T_K by $T_K(\xi)(n) = \int K(n,m)\xi(m)d\mu(m)$

Example: Let (M,d) be a compact metric space. For $f \in C(M)$, let $L^d(f)$ be continuous Lipschitz constant. Let $L_d(M) = \{f \in C(M) : L(f) < \infty\}$. On $L_d(M)$ put norm $\|f\|_d = \|f\|_\infty + L^d(f)$. Then, $L_d(M)$ is a Banach Algebra for pointwise multiplication and the above norm. Let $T : L_d(M) \to C(M)$ be the inclusion with the sup-norm. Then T is a compact operator. This requires the use of Ascoli Arzela theorem.

We now discuss some basic properties of compact operators. Let $\mathfrak{B}_0(X,Y)$ be the space of compact operators from X to Y.

Proposition 95 *If $T \in \mathfrak{B}_0(X,Y)$ then*

1. *If $S \in \mathfrak{B}(Y,Z)$ then $S(T)$ is a compact operator.*
2. *If $S \in \mathfrak{B}(Z,X)$ then $T(S)$ is a compact operator.*

Let (M,d) be a metric space. A subset E of M is totally bounded if for every $\epsilon > 0$, E can be covered by a finite number of balls of radius ϵ. Any compact subset is totally bounded. Also, if (M,d) is complete, then the closure of a totally bounded set is compact.

Proposition 96 *If $T \in \mathfrak{B}_0(X,Y)$ and α is a scalar then $\alpha T \in \mathfrak{B}_0(X,Y)$.*

Proposition 97 *Let $S,T \in \mathfrak{B}_0(X,Y)$. Then $S+T \in \mathfrak{B}_0(X,Y)$.*

The proof of the above lemma follows from the fact that if E and F are totally bounded subsets of Y, then $E+T$ is totally bounded (translation of the balls and a triangle inequality argument).

Lecture 32. November 9th. On $\mathfrak{B}_0(V, W)$ and Integral Operators

We define $\mathfrak{B}_0(V, W)$ to be the linear space of compact operators. In fact, $\mathfrak{B}_0(V, W)$ is a $\mathfrak{B}(W) - \mathfrak{B}(V)$ bi-module. In particular, $\mathfrak{B}_0(V)$ is a 2-sided ideal in $\mathfrak{B}(V)$.

Proposition 98 $\mathfrak{B}_0(V, W)$ *is norm closed.*

Proof: Let $\{T_n\} \subset \mathfrak{B}_0(V, W)$, suppose $T_n \to T$ in norm. Let $\epsilon > 0$ be given. Choose N such that for $n \geq N$, we have $\|T - T_n\| < \epsilon/2$. For T_N choose $v_1, v_2, ..., v_k \in V$ such that the open $\epsilon/2$ balls about $T_N v_1, ...T_N v_k$ cover the image of the unit ball. Then the epsilon balls about these vectors will cover $T(V_1)$ as required from triangle inequality. ■

Let $F(V, W)$ be the set of finite rank operators from V to W. These are obviously compact. It is not difficult to see that $F(V)$ is a minimal non trivial 2-sided ideal in $\mathfrak{B}_0(V)$. The question one is interested in is if $\overline{F(V)} = \mathfrak{B}_0(V)$. We now look at an important class of examples of compact operators, called Integral Operators (under some conditions of course).

Suppose $(X, \mu), (Y, \nu)$ are measure spaces. Let K be a measurable function on $X \times Y$ for the product measure. We try to define an operator T_K by $T_K(\xi) = \int K(x, y)\xi(y)d\nu(y)$. We have $\|(T_K\xi)(x)\| \leq \|y \to K(x, y)\|_p \|\xi\|_q$. Following it up with Fubini's theorem, we have $\|T\xi\|_p \leq \|K\|_p \|\xi\|_q$.

Proposition 99 *If* $K \in L^p(X \times Y, \mu \times \nu)$ *then* $T_K \in \mathfrak{B}(L^q(Y), L^p(X))$ *and* $\|T_k\| \leq \|K\|_p$.

Proposition 100 $T_K \in \mathfrak{B}_0(L^q(Y), L^p(X))$

Proof: Let $A \subset X$, $B \subset Y$ be subsets of finite measure. Let $K_{A,B}(x, y) = \chi_{A \times B}(x, y) = \chi_A(x)\chi_B(y)$. Then $(T_{K_{AB}}\xi)(x) = \int K_{AB}(x, y)\xi(y)d\nu(y) = \chi_A(x) < \chi_B, \xi >$, so that T_{AB} is a rank one operator. Finite linear combinations of these give simple integrable functions. So if K is an SIF, then T_K is a finite rank operators. Such functions are norm dense in L^p, and this concludes the proof. ■

Lecture 33. November 11th. Density of Finite Rank Operators

Let G be a compact group. Let λ be the left regular representation on $L^2(G)$, $f \to \lambda_f$ its integrated form as a representation of $L^1(G)$.

Proposition 101 *Then for each f, λ_f is a compact operator.*

Proof: Let $f \in C(G)$, then for each $\xi \in L^2(G)$, $\lambda_f\xi(x) = \int f(y)\xi(y^{-1}x)dy = \int f(xy^{-1})\xi(y)dy$. Let $K_f(x,y) = f(xy^{-1})$. We claim that this kernel is in $L^2(G \times G)$. Note, $\|K_f\|^2 = \int |K_f(x,y)|^2 dxdy = \int \|f\|_2^2 \int 1dy$. Thus, L_f is compact since $C(G)$ is dense in $L^1(G)$. The result follows. ∎

Let V be a banach space.

Proposition 102 *Let $\{S_\alpha\}$ be a net in $\mathfrak{B}(V,W)$ that converges in SOT to $S \in \mathfrak{B}(V,W)$ and assume there is a constant k such that $\|S_\alpha\| \le k$ for all α. Let $T \in \mathfrak{B}_0(U,V)$. Then $S_\alpha T \to ST$ in norm.*

Proof: Given $\epsilon > 0$ find $u_1, ..., u_n \in U$ such that the balls of radius $\epsilon/2K$ around T_{U_j} cover T(Unit ball of U). Since $S_\alpha \to S$ in SOT, can find α_0 such that $\|S_\alpha T u_j - STu_j\| < \epsilon/2$. Then for any u, $\|a\| \le 1$, $\exists j_0$ such that $\|Tu - Tu_j\| \le \epsilon/2K$. Then, $\|STu - S_{\alpha_0}Tu\| \le \|S\|\,\|Tu - Tu_j\| + \|STu_{j_0} - S_\alpha Tu_{j_0}\| + \|S_\alpha Tu_{j_0} - S_\alpha Tu\| \le \epsilon$ as required. ∎

Proposition 103 *Let V be a banach space. If there is a net $\{S_\alpha\} \in \mathfrak{B}(V)$ such that $\exists K$ with $\|S_\alpha\| \le K$ for all α and if each S_α is a finite rank operator, and if $S_\alpha \to I_V$ for SOT, then the finite rank operators are dense in $\mathfrak{B}_0(V)$.*

We now consider an example. Let X be a set with counting measure. Consider $l^p(X)$ for $1 \le p < \infty$. For each finite subset $A \subset X$, let P_A be the projection in $\mathfrak{B}(l^p(X))$ of pointwise multiplication by χ_A. Order finite sets by inclusion. The $\{P_A\}$ is a net of finite rank operators of each norm 1. This net converges to I_{l^p} for the SOT. In particular, any Hilbert Space is isomorphic to an l^2 space.

Let X be a compact set, $V = C(X)$ with the sup norm. Let U be the set of all finite open covers of X ordered by: if $u, v \in U$ say $u \le v$ if every element of v us contained in an element of u. This makes U into a directed set. For each $u \in U$, let $\{\phi_\theta\}_{\theta \in u}$, $(\phi_\theta \in C(X))$ be a partition of unity for U, i.e, for each θ the support of $\phi_\theta \subseteq \theta$ and each $\phi_\theta \ge 0$, and $\sum_{\theta \in U} \phi_\theta = 1$. For each θ, let x_θ^u be some point in θ, given u, define $T_u \in \mathfrak{B}(C(X))$ by $T_u f = \sum_{\theta \in u} f(x_\theta^u)\phi_\theta^u \in C(X)$. Furthermore, T_u is finite rank. We claim that $T_u \to I_{C(X)}$ for SOT.

Lecture 34. November 16th. Schander's Theorem and Spectra of Compact Operators

We start with stating Schander's theorem which is a fundamental result in the theory of compact operators.

Theorem 104 *Let V, W be banach spaces and let $T \in \mathfrak{B}_0(V, W)$. Then T^* is compact (i.e, $\in \mathfrak{B}_0(W^*, V^*)$)*

Proof: Assume $\|t\| \leq 1$ so that T maps the unit ball into the unit ball of W. View the unit ball of W^* as functions on the unit ball of W. As such functions, $ball_1(W^*)$ is uniformly equicontinuous, i.e, if $w_1, w_2 \in ball_1(W)$ and $\phi \in ball_1(W^*)$, then $|\phi(w_1 - w_2)| = |\phi(w_1) - \phi(w_2)| \leq \|w_1 - w_2\|$.

Look at the closure of the image of the unit ball of V under T. It is a norm-compact subset of $ball_1(W)$. So can view $ball_1(W^*)$ as functions on $\overline{T(ball_1(W^*))}$, is uniformly equicontinuous (bounded by 1). So by Ascoli Arzela, this set of functions is totally bounded. Thus, given an $\epsilon > 0$, we can find $\phi_1, ..., \phi_k$ such that the ϵ balls about the ϕ_j's cover $\overline{ball_1(W^*)}$ as functions on $\overline{T(ball_1(V))}$.

Thus given $\phi \in ball_1(W^*)$ there is a j such that $|(\phi - \phi_j)T\nu| \leq \epsilon$ for all $\nu \in ball_1(V)$. So, $\|T^*\phi - T^*\phi_j\| \leq \epsilon$ as required. We leave the proof of the converse as an exercise. ■

Let V be a banach space. $T \in \mathfrak{B}_0(V)$, let λ be an eigenvalue for T, $\lambda \neq 0$. Let V_λ be the λ eigensubspace for λ.

Proposition 105 *Then $T|_{V_\lambda}$ is finite dimensional.*

Proof: $T|_{V_\lambda} = \lambda I_{V_\lambda} \Rightarrow T|_{V_\lambda}$ is compact. Hence V_λ is finite dimensional. ■

Theorem 106 *Let T be a compact operator in the bounded operators on Hilbert Space $\mathcal{H}$, with T normal. Then for any $\lambda \in \sigma(T)$, $\lambda \neq 0$, λ is an eigenvalue, i.e, there exists a corresponding eigenvector.*

Proof: Since $\lambda \in \sigma(T)$ there is a sequence $\{\xi_n\}$ in $\mathcal{H}$ with $\|\xi_n\| = 1$ and $(T - \lambda I)\xi_n \to 0$ in norm. Since T is compact, there is a subsequence $\{\xi_{n_j}\}$ such that, $\{T(\xi_{n_j})\} \to \eta \in \mathcal{H}$. Then, $(T - \lambda I)\xi_{n_j} \to 0$. We ask the reader to fill up the rest of the details in this proof. ■

Lecture 35. November 18th. Polar Decomposition and Traces

Let $T \in \mathfrak{B}(\mathcal{H})$. Set $|T| = (T^*T)^{1/2}$. For any $\xi \in \mathcal{H}$, $\||T|\xi\|^2 =< |T|\xi, |T|\xi >=< T^*T\xi, \xi >= \|T\xi\|^2$. Thus $\||T|\xi\| = \|T\xi\|$.

Let $\eta_T = ker(T) = ker(|T|)$. Define V on η_T to be the 0 operator. Now look at $(ker(T))^\perp = \overline{range(T)}$. For $\xi \in \mathcal{H}$, try setting $V(|T|\xi) = T\xi$. Is this well defined? If $|T|\xi = |T|\eta$, then is $T\xi = T\eta$? This is obvious as the kernels of the two operators are the same.

Also, $\|V(|T|\xi)\| = \|T\xi\| = \||T|\xi\|$, so V is an isometry on the range of $|T|$. So V extends by continuity to $(ker|T|)^\perp = ker(T)^\perp$. So, $\mathcal{H} = (ker(T)) \oplus ker(T)^\perp$.

Definition 107 *By a partial isometry on a Hilbert Space $\mathcal{H}$, we mean an operator V which is an isometry on $(kerV)^\perp$. So, V (as above) is a partial isometry.*

Thus if $T \in \mathfrak{B}_0(\mathcal{H})$, then $T \in \mathfrak{B}_0(\mathcal{H})$ and the converse is also true. We now take a small detour.

Let $\{\xi_\alpha\}$ be an orthonormal basis for $\mathcal{H}$. For any $T \in \mathfrak{B}(\mathcal{H})$, $T \geq 0$ set $tr(T) = \sum < T\xi_\alpha, \xi_\alpha >$. Note, if $S, T \in \mathfrak{B}(\mathcal{H})$, $S \geq 0$, $T \geq 0$, then $tr(S + T) = tr(S) + tr(T)$. If $c \in \mathbb{R}^+, T \geq 0$ then $tr(cT) = ctr(T)$.

Definition 108 *For any C^* Algebra $A \subseteq \mathfrak{B}(\mathcal{H})$, by a weight on A we mean a function $w : A^+ \to [0, \infty]$, such that if $a, b \in A^+$, then $w(a + b) = w(a) + w(b)$ and if $c \in \mathbb{R}^+$, $w(ca) = cw(a)$.*

Proposition 109 *For any $T \in \mathfrak{B}(\mathcal{H})$, $tr(T^*T) = tr(TT^*)$.*

Proof: $| < T\xi_\alpha, \xi_\beta > |^2 = | < \xi_\alpha, T^*\xi_\beta > |^2 = | < T^*\xi_\beta, \xi_\alpha > |^2$. Now consider the sum $\sum_\alpha (\sum_\beta | < T\xi_\alpha, \xi_\beta > |^2)$. From Parseval's law we have this is equal to $\sum_\alpha \|T\xi_\alpha\|^2 = tr(T^*T)$. On the other hand, from Fubini, we have this is equal to $\sum_\beta \|T^*\xi_\beta\|^2 = tr(TT^*)$. ■

Corollary 110 *Let U be a unitary operator, then the $tr(UTU^*) = tr(UT^{1/2}T^{1/2}U^*) = tr((UT^{1/2})^*UT^{1/2}) = Tr(T)$*

Let $\{\eta_\alpha\}$ be any other orthonormal basis for $\mathcal{H}$. Let U be the unitary operator carrying $\{\xi_\alpha\}$ to $\{\eta_\alpha\}$. $\sum < T\eta_\alpha, \eta_\alpha >= \sum < TU\xi_\alpha, U\xi_\alpha >= tr(U^*TU) = tr(T)$. Trace is independent of choice of basis.

If w is a weight on a C^* algebra $A \subseteq \mathfrak{B}(\mathcal{H})$, let $L^2(A, w) = \{a \in A : w(a^*a) < \infty\}$.

Proposition 111 *If $T \in L^2(\mathfrak{B}(\mathcal{H}), tr))$, then $T \in \mathfrak{B}_0(\mathcal{H})$.*

Proposition 112 *If $T \geq 0$, and $tr(T) < \infty$ then $T \in \mathfrak{B}_0(\mathcal{H})$.*

Proposition 113 *For $T \geq 0$, $tr(T) \geq \|T\|$. For any $\epsilon > 0$ can find $\xi, \|\xi\| = 1$, $\|T\xi\| \leq \|T\| - \epsilon$. (use o.n. basis).*

Lecture 36. November 21st. Trace Class Operators

Proposition 114 *Let $T \in \mathfrak{B}(\mathcal{H})$, $T \geq 0$ and $tr(T) < \infty$ then $T \in \mathfrak{B}_0(\mathcal{H})$.*

Proof: Choose orthonormal basis $\{\xi_\alpha\}$ so that $\sum < T\xi_\alpha, \xi_\alpha >< \infty$. Let $\epsilon > 0$ be given. Thus we can find a finite set A of α's such that $\sum_{\alpha \notin A} < R\xi_\alpha, \xi_\alpha > \leq \epsilon$. Let P be the projection on span $\{\xi_\alpha\}_{\alpha \in A}$ - finite rank projection. We have

$\left\|T^{1/2} - T^{1/2}P\right\|^2 = \left\|T^{1/2}(I-P)\right\|^2 = \|(I-P)T(I-P)\| \leq tr((I-P)T(I-P)) < \epsilon$. Since ϵ is arbitrary, $T^{1/2}$ is approximated by a finite rank operator, which means T is compact. ■

Let A be a C^* Algebra which is a subset of $\mathfrak{B}(\mathcal{H})$. Let w be a weight on A. Define $n_w = \{a \in A : w(a^*a) < \infty\}$. We have the parallelogram law for operators: $(a+b)^*(a+b) + (a-b)^*(a-b) = 2(a^*a + b^*b)$. So if $a, b \in n_w$, then $w((a+b)^*(a+b)) < \infty$. Hence $a + b \in n_w$.

Proposition 115 *$a \to w(a^*a)$ is a quadratic form, for a pre inner product on n_w.*

Proof: $b^*a = 1/4 \sum_{k=0}^{3} i^k(a + i^k b)^*(a + i^k b)$. So extend w to n_w by setting $w(b^*a) = 1/4 \sum w(...)$. For tr on $\mathfrak{B}(\mathcal{H})$, $n_tr = \{T \in \mathfrak{B}(\mathcal{H}) : tr(T^*T) < \infty\}$ is called the set of Hilbert Schmidt Operators, and is a 2-sided ideal inside $\mathfrak{B}(\mathcal{H})$. ■

Proposition 116 *n_w is a left ideal in A and for action of A on the left on n_w, $a \to L_a$, $\|L_a\| \leq \|a\|$, i.e, for $b \in n_w$, $a \in A$, $< L_a b, L_a b >_w \leq \|a\|^2 < b, b >_w$.*

Proof: $< L_a b, L_a b >_w = < ab, ab >_2 = w(b^*a^*ab) \leq w(b^* \|a^*a\| b) = \|a\|^2 < b, b >_w$. ■

Let m_w be the linear span of $\{a \in A, a \geq 0 and w(a) < \infty\}$ so m_w is a *-subspace of A. If $a \in A, a \geq 0$, $w(a) < \infty$, then $a^{1/2} \in n_w$ so $a \in n_w$. Thus $m_w \subseteq n_w$. If $a, b \in m_w$, $b^*a = 1/4 \sum i^k(a + i^k b)^*(a + i^k b)$.

Proposition 117 *m_w is a *-subalgebra of A. m_w is not an ideal.*