Exercises:

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth function defined by
$$T(z_1, z_2) = (z_1^2 + z_2^2, z_1 z_2, z_1 - z_2^3).$$
For $\omega = y_2 dy_{(1,3)} \in \Omega^2(\mathbb{R}^3)$, compute the ascending presentation of $T^*\omega$.

2. Let $\varphi \in C_2(\mathbb{R}^3)$ be defined by
$$\varphi(u) = (\cos(2\pi u_1), \sin(2\pi u_1), u_2).$$
Compute the dipoles $\delta^1\varphi$ and $\delta^2\varphi$, and the boundary $\partial\varphi$. Describe these (using words or pictures), making sure to note the proper orientations.

3. Let $\omega \in \Omega^1(\mathbb{R}^2)$. Assume ω is **closed**: $d\omega = 0$.
   (a) For $p, q \in \mathbb{R}^2$, let $\varphi, \psi \in C_1(\mathbb{R}^2)$ be such that $\varphi(0) = \psi(0) = p$ and $\varphi(1) = \psi(1) = q$. Show that $\int_\varphi \omega = \int_\psi \omega$.
   [**Hint:** consider $\sigma \in C_2(\mathbb{R}^2)$ defined by $\sigma(s,t) = (1-s)\varphi(t) + s\psi(t)$.]
   (b) Fix $p \in \mathbb{R}^2$. Define $h\colon \mathbb{R}^2 \to \mathbb{R}$ by $h(q) := \int_\varphi \omega$ where φ is a 1-cell in $\mathbb{R}^2$ satisfying $\varphi(0) = p$ and $\varphi(1) = q$. Show that h is a smooth function with $dh = \omega$.

Solutions:

1. According to the formula we proved in class,
$$T^*\omega = y_2 \circ T dT_1 \wedge dT_3 = T_2 dT_1 \wedge dT_3 = z_1 z_2 dT_1 \wedge dT_3.$$
Now
$$dT_1 = \frac{\partial T_1}{\partial z_1} dz_1 + \frac{\partial T_1}{\partial z_2} dz_2 = 2z_1 dz_1 + 2z_2 dz_2$$
$$dT_3 = \frac{\partial T_3}{\partial z_1} dz_1 + \frac{\partial T_3}{\partial z_2} dz_2 = dz_1 - 3z_2^2 dz_2.$$
Thus using the distributive property and signed commutativity we have
$$\begin{aligned} T^*\omega &= z_1 z_2 \left(2z_1 dz_1 + 2z_2 dz_2\right) \wedge \left(dz_1 - 3z_2^2 dz_2\right) \\ &= z_1 z_2 \left(0 - 6z_1 z_2^2 dz_{(1,2)} + 2z_2 dz_{(2,1)} + 0\right) \\ &= z_1 z_2 (-6z_1 z_2^2 - 2z_2) dz_{(1,2)} \\ &= (-6z_1^2 z_2^3 - 2z_1 z_2^2) dz_{(1,2)} \end{aligned}$$
□

2. We first compute:
$$\begin{aligned} \delta^1\varphi(t) &= \varphi \circ \iota^{1,1}(t) - \varphi \circ \iota^{1,1}(t) \\ &= \varphi(1,t) - \varphi(0,t) \\ &= (\cos(2\pi), \sin(2\pi), t) - (\cos(0), \sin(0), t) \\ &= (1, 0, t) - (1, 0, t), \end{aligned}$$
which is equal to zero even as a formal difference. Let us visualize this formal difference: as t ranges over $[0,1]$, the first term yields the segment $[(1,0,0),(1,0,1)]$ (oriented upwards), while the second term yields the same segment but oriented downwards.

Next we have compute

$$\begin{aligned}\delta^2\varphi(t) &= \varphi\circ\iota^{2,1}(t) - \varphi\circ\iota^{2,0}(t)\\ &= \varphi(t,1)-\varphi(t,0)\\ &= (\cos(2\pi t),\sin(2\pi t),1) - (\cos(2\pi t),\sin(2\pi t),0).\end{aligned}$$

As t ranges over $[0,1]$, the first term traces out a circle with center $(0,0,1)$ and radius 1 in the plane $z=1$ in the counter-clockwise direction when viewed from above. The second term traces out a circle with center $(0,0,0)$ in the plane $z=0$ in the clockwise direction (because of the negative sign).

We then have for the boundary:

$$\partial\varphi = \delta^1\varphi - \delta^2\varphi = 0 - \delta^2\varphi = (\cos(2\pi t),\sin(2\pi t),0) - (\cos(2\pi t),\sin(2\pi t),1),$$

which can visualized exactly as $\delta^2\varphi$ but with the orientations reversed. □

3. (a) Letting σ be as in the hint, we have

$$\begin{aligned}\delta^1\sigma(t) &= \sigma\circ\iota^{1,1}(t) - \sigma\circ\iota^{1,0}(t)\\ &= \sigma(1,t)-\sigma(0,t)\\ &= \psi(t)-\varphi(t),\end{aligned}$$

while

$$\begin{aligned}\delta^2\sigma(s) &= \sigma\circ\iota^{2,1}(s) - \sigma\circ\iota^{2,0}(s)\\ &= \sigma(s,1)-\sigma(s,0)\\ &= [(1-s)\varphi(1)+s\psi(1)] - [(1-s)\varphi(0)+s\psi(0)]\\ &= [(1-s)q+sq]-[(1-s)p+sp]\\ &= \{q\}-\{p\}.\end{aligned}$$

Now, since $d\omega=0$, we have by the general Stokes' formula:

$$0=\int_\sigma d\omega = \int_{\partial\sigma}\omega = \int_{\delta^1\varphi}\omega - \int_{\delta^2\varphi}\omega = \int_\psi\omega - \int_\varphi\omega - \int_{\delta^2\varphi}\omega.$$

Since $\delta^2\varphi$ is a difference of two constant 1-cells, the last term in the last expression above vanishes. Hence $\int_\psi\omega = \int_\varphi\omega$ as claimed. □

(b) For $q\in\mathbb{R}^2$, part (a) implies $h(q)$ does not actually depend on φ and hence we may replace it with any 1-cell in $\mathbb{R}^2$. So for $q\in\mathbb{R}^2$ consider $\psi_q\in C_1(\mathbb{R}^2)$ defined by

$$\psi_q(t) = (1-t)p+tq.$$

Note that $\frac{\partial(\psi_q)_i(t)}{\partial t} = q_i - p_i$, $i=1,2$, where $p=(p_1,p_2)$ and $q=(q_1,q_2)$.
Now, since $\omega\in\Omega^1(\mathbb{R}^2)$, we can write $\omega = f dx + g dy$ for some smooth functions $f,g\colon\mathbb{R}^2\to\mathbb{R}$. We then compute

$$\begin{aligned}h(q) &= \int_{\psi_q}\omega = \int_{\psi_q} f dx + g dy\\ &= \int_0^1 f\circ\psi_q(t)(q_1-p_1) + g\circ\psi_q(t)(q_2-p_2)\ dt.\end{aligned}$$

By a theorem from class, we can take partial derivatives under the integral. Hence, f and g being smooth functions implies all orders of partial derivatives of h exist. In particular, for each $r\in\mathbb{N}$ all the order r partial derivatives of h exist and are continuous (since the order $r+1$ partials exist), which implies h is of class C^r. Since this holds for all $r\in\mathbb{N}$, h is smooth.

Finally, we compute dh.

$$\begin{aligned}\frac{\partial h}{\partial q_1} &= \left(\int_0^1 \frac{\partial}{\partial q_1}\left[f\circ\psi_q(t)(q_1-p_1)\right] + \frac{\partial}{\partial q_2}\left[g\circ\psi_q(t)(q_2-p_2)\right]\ dt\right) dq_2\\ &= \int_0^1 \frac{\partial f(\psi_q(t))}{\partial x} t(q_1-p_1) + f\circ\psi_q(t) + \frac{\partial g(\psi_q(t))}{\partial x} t(q_2-p_2)\ dt\end{aligned}$$

Note that $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$ by virtue of $d\omega = 0$. Thus we can continue the above computation with

$$\begin{aligned}\frac{\partial h}{\partial q_1} &= \int_0^1 (Df)_{\psi_q(t)}(q-p)t + f\circ\psi_q(t)\ dt\\ &= \int_0^1 \frac{d}{dt}\left[f\circ\psi_q(t)t\right]\ dt = f\circ\psi_q(1) - 0 = f(q),\end{aligned}$$

where we have invoked the fundamental theorem of calculus in the second-to-last equality. A similar computation yields $\frac{\partial h}{\partial q_2} = g(q)$. Thus $dh = \omega$. □