## Exercises:

1. Fix $(x_0, y_0) \in \mathbb{R}^2$ and $a, b > 0$. Consider $\varphi \in C_1(\mathbb{R}^2)$ defined by
$$\varphi(t) = (a\cos(2\pi t) + x_0, b\sin(2\pi t) + y_0).$$
For $\omega = -y_2 dy_1 \in \Omega^1(\mathbb{R}^2)$, compute $\int_\varphi \omega$.

2. Let
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\},$$
so that $S^2 = \varphi(I^2)$ for $\varphi \in C_2(\mathbb{R}^3)$ defined by
$$\varphi(s, t) = (\cos(2\pi s)\sin(\pi t), \sin(2\pi s)\sin(\pi t), \cos(\pi t)).$$
   (a) Compute the $(2, 1)$-shadow area of $\varphi$: $\int_\varphi dy_{(2,1)}$.

   (b) For $\omega = y_2 dy_{(1,3)} \in \Omega^2(\mathbb{R}^3)$, compute $\int_\varphi \omega$.

3. Fix $y \in \mathbb{R}^n$, $r > 0$, and an ascending $A = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$. Define $\iota \in C_k(\mathbb{R}^n)$ by
$$\iota(u_1, \ldots, u_k) = y + r(u_1 e_{i_1} + \cdots + u_k e_{i_k}),$$
where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. Show that for $I \in \{1, \ldots, n\}^k$ and $u \in I^k$
$$\frac{\partial \iota_I(u)}{\partial u} = \begin{cases} \text{sgn}(\pi) r^k & \text{if } I = \pi A \text{ for some } \pi \in S_k \\ 0 & \text{otherwise} \end{cases}.$$

4. Let $\omega = \sum f_A dy_A \in \Omega^k(\mathbb{R}^n)$ be in ascending presentation. Define the symmetrization map symm: $\Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$ by
$$\text{symm}(\omega) = \frac{1}{k!} \sum_A \sum_{\pi \in S_k} f_A dy_{\pi A},$$
where if $A = (i_1, i_2, \ldots, i_k)$ then $\pi A = (i_{\pi(1)}, i_{\pi(2)}, \ldots, i_{\pi(k)})$. Show that if $k > 1$, then $\text{symm}(\omega) = 0$ for all $\omega \in \Omega^k(\mathbb{R}^n)$.

5. Let $\alpha, \beta \in \Omega^3(\mathbb{R}^6)$ be defined by
$$\alpha := a_1 dy_{(1,3,2)} + a_2 dy_{(4,2,3)} + a_3 dy_{(4,3,1)}$$
$$\beta := b_1 dy_{(3,5,6)} + b_2 dy_{(6,4,5)} + b_3 dy_{(2,5,3)},$$
where $a_i, b_j \colon \mathbb{R}^6 \to \mathbb{R}$ are smooth functions for $1 \le i, j \le 3$.
   (a) Determine the ascending presentations of $\alpha$ and $\beta$.

   (b) Compute $\alpha \wedge \beta$ using the ascending presentations.

   (c) Using the insensitivity to presentation, compute $\alpha \wedge \beta$ with their original presentations and verify that this agrees with the answer in part (b).

---

## Solutions:

1. We first note that
$$\frac{\partial \varphi_1(u)}{\partial u} = -2\pi a \sin(2\pi u).$$

Thus

$$\begin{aligned}\int_\varphi \omega &= \int_0^1 -\varphi_2(u)\frac{\partial \varphi_1(u)}{\partial u} du\\ &= 2\pi a \int_0^1 b\sin^2(2\pi u) + y_0 \sin(2\pi u)\ du\\ &= 2\pi a \int_0^1 b\frac{1-\cos(4\pi u)}{2} + y_0\sin(2\pi u)\ du\\ &= 2\pi a\left[\frac{b}{2}u - \frac{b}{8\pi}\sin(4\pi u) - \frac{y_0}{2\pi}\cos(2\pi u)\right]_0^1 = \pi ab.\end{aligned}$$

Note that this is precisely the area of the ellipse enclosed by the curve $\varphi([0,1])$. □

2. (a) We first compute

$$\begin{aligned}\frac{\partial \varphi_{(2,1)}(u)}{\partial u} &= \det\begin{bmatrix} 2\pi\cos(2\pi u_1)\sin(\pi u_2) & \pi\sin(2\pi u_1)\cos(\pi u_2)\\ -2\pi\sin(2\pi u_1)\sin(\pi u_2) & \pi\cos(2\pi u_1)\cos(\pi u_2)\end{bmatrix}\\ &= 2\pi^2\sin^2(2\pi u_1)\sin(\pi u_2)\cos(\pi u_2) + 2\pi^2\cos^2(2\pi u_1)\sin(\pi u_2)\cos(\pi u_1)\\ &= 2\pi^2\sin(\pi u_2)\cos(\pi u_2).\end{aligned}$$

Thus by Fubini's theorem

$$\begin{aligned}\int_\varphi dy_{(2,1)} &= \int_0^1\int_0^1 2\pi^2\sin(\pi u_2)\cos(\pi u_2)\ du_1 du_2\\ &= 2\pi^2\left[\frac{1}{2\pi}\sin^2(\pi u_2)\right]_0^1 = 0.\end{aligned}$$

Note that this is expected value since the top and bottom halves of $S^2$ are both projected to $(2,1)$-plane, with one half having negative area and so canceling the other half. □

(b) We first compute

$$\frac{\partial \varphi_{(1,3)}(u)}{\partial u} = \det\begin{bmatrix} -2\pi\sin(2\pi u_1)\sin(\pi u_2) & \pi\cos(2\pi u_1)\cos(\pi u_2)\\ 0 & -\pi\sin(\pi u_2)\end{bmatrix} = 2\pi^2\sin(2\pi u_1)\sin^2(\pi u_2).$$

Thus by Fubini's theorem we have

$$\begin{aligned}\int_\varphi \omega &= \int_0^1\int_0^1 (\sin(2\pi u_1)\sin(\pi u_2))(2\pi^2\sin(2\pi u_1)\sin^2(\pi u_2))\ du_1 du_2\\ &= 2\pi^2\int_0^1 \sin^2(2\pi u_1)\ du_1 \int_0^1 \sin^3(\pi u_2)\ du_2\\ &= 2\pi^2\left[\frac{u_1}{2} - \frac{1}{8\pi}\sin(4\pi u_1)\right]_0^1\left[-\frac{3}{4\pi}\cos(\pi u_2) + \frac{1}{12\pi}\cos(3\pi u_2)\right]_0^1\\ &= 2\pi^2\left[\frac{1}{2}\right]\left[\frac{3}{2\pi} - \frac{1}{6\pi}\right] = \frac{4\pi}{3}.\end{aligned}$$

Note that this is the volume of the sphere enclosed by $S^2$. □

3. For $I \in \{1,\ldots,k\}^k$, if $I$ has any repeated entries then we immediately have $\frac{\partial \iota_I}{\partial u} = 0$, which agrees with the claimed formula since it cannot be rearranged as an ascending $k$-tuple. Suppose $I$ has no repeated entries and let $\pi \in S_k$ be such that $B := \pi I$ is ascending. Then $\frac{\partial \iota_I}{\partial u} = \operatorname{sgn}(\pi)\frac{\partial \iota_B}{\partial u}$. Now, for $1 \le i \le n$ note that

$$\iota_i(u) = \begin{cases} y_i + ru_j & \text{if } i = i_j \in A \text{ for some } 1 \le j \le k\\ y_i & \text{otherwise}\end{cases},$$

 

so that

$$\frac{\partial \iota_i(u)}{\partial u_j} = \begin{cases} r & \text{if } i = i_j \\ 0 & \text{otherwise} \end{cases}.$$

Writing $B = (b_1, \ldots, b_k)$, the matrix

$$\begin{bmatrix} \frac{\partial \iota_{b_1}(u)}{\partial u_1} & \cdots & \frac{\partial \iota_{b_1}(u)}{\partial u_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \iota_{b_k}(u)}{\partial u_1} & \cdots & \frac{\partial \iota_{b_k}(u)}{\partial u_k} \end{bmatrix}$$

is $r$ times the identity matrix if $B = A$. Otherwise, if $b_j \in B \setminus A$ for some $1 \leq j \leq k$, the matrix has all zeros in its $j$th row. Hence

$$\frac{\partial \iota_B(u)}{\partial u} = \begin{cases} r^k & \text{if } B = A \\ 0 & \text{otherwise} \end{cases}.$$

Using $\frac{\partial \iota_I}{\partial u} = \text{sgn}(\pi)\frac{\partial \iota_B}{\partial u}$ completes the proof. □

4. By signed commutativity we have

$$\begin{aligned} \text{symm}(\omega) &= \frac{1}{k!}\sum_A \sum_{\pi \in S_k} \text{sgn}(\pi) f_A dy_A \\ &= \frac{1}{k!}\sum_A f_A \left(\sum_{\pi \in S_k} \text{sgn}(\pi)\right) dy_A. \end{aligned}$$

We claim for $k > 1$ that $\sum_{\pi \in S_k} \text{sgn}(\pi) = 0$, which will complete the proof. Indeed, define two sets $S_k^{\pm} := \{\pi \in S_k : \text{sgn}(\pi)\}$. Then

$$\sum_{\pi \in S_k} \text{sgn}(\pi) = \sum_{\pi S_k^+} 1 + \sum_{\pi \in S_k^-} (-1) = |S_k^+| - |S_k^-|,$$

and so we must show $S_k^+$ and $S_k^-$ have the same cardinality. Since $k > 1$, the transposition $(1, 2) \in S_k$. The map $S_k \ni \pi \mapsto (1, 2)\pi$ is a bijection (it is its own inverse even), and since

$$\text{sgn}((1, 2)\pi) = \text{sgn}((1, 2))\text{sgn}(\pi) = -\text{sgn}(\pi)$$

the map sends $S_k^+$ to $S_k^-$ and vice versa. Thus, $|S_k^-| = |(1, 2)S_k^+| = |S_k^+|$. □

5. (a) Using associativity of the wedge product followed by signed commutativity we have

$$\begin{aligned} \alpha &= a_1 dy_1 \wedge dy_3 \wedge dy_2 + a_2 dy_4 \wedge dy_2 \wedge dy_3 + a_3 dy_4 \wedge dy_3 \wedge dy_1 \\ &= (-1)^1 a_1 dy_1 \wedge dy_2 \wedge dy_3 + (-1)^2 a_2 dy_2 \wedge dy_3 \wedge dy_4 + (-1)^3 dy_1 \wedge dy_3 \wedge dy_4 \\ &= -a_1 dy_{(1,2,3)} + a_2 dy_{(2,3,4)} - a_3 dy_{(1,3,4)}. \end{aligned}$$

Similarly, we obtain

$$\beta = b_1 dy_{(3,5,6)} + b_2 dy_{(4,5,6)} - b_3 dy_{(2,3,5)}.$$

□

(b) In $\alpha$, note that $dy_{(2,3,4)}$ shares indices with all the simple 3-forms appearing in $\beta$, and similarly for $dy_{(1,3,4)}$. Consequently their wedge products with $\beta$ will be zero. Using distributivity of $\wedge$ we have

$$\alpha \wedge \beta = (-a_1 dy_{(1,2,3)}) \wedge \beta.$$

Now, of the simple 3-forms appearing in $\beta$, only $b_2 dy_{(4,5,6)}$ shares no indices with $dy_{(1,2,3)}$. Thus

$$\alpha \wedge \beta = 0 + (-a_1 dy_{(1,2,3)}) \wedge (b_2 dy_{(4,5,6)}) + 0 = -a_1 b_2 dy_{(1,2,3,4,5,6)}.$$

□

(c) Computing $\alpha \wedge \beta$ directly, the same argument as in part (b) shows that after distributing, the only term that survives is

$$\alpha \wedge \beta = (a_1 dy_{(1,3,2)}) \wedge (b_2 dy_{(6,4,5)}) = a_1 b_2 dy_{(1,3,2,6,4,5)}.$$

Note that permuting $(1,3,2,6,4,5)$ to $(1,2,3,4,5,6)$ involves 3 inversions $(3,2) \mapsto (2,3)$, $(6,4) \mapsto (4,6)$ followed by $(6,5) \mapsto (5,6)$. Hence

$$a_1 b_2 dy_{(1,3,2,6,4,5)} = (-1)^3 a_1 b_2 dy_{(1,2,3,4,5,6)} = -a_1 b_2 dy_{(1,2,3,4,5,6)},$$

which agrees with our answer from part (b). □