Exercises:

1. Consider $f\colon [0,1]^2 \to \mathbb{R}$ defined by
$$f(x,y) = \begin{cases} 1 - \frac{1}{q} & \text{if } x, y \in \mathbb{Q} \text{ with } y = \frac{p}{q} \text{ in lowest terms} \\ 1 & \text{otherwise} \end{cases}.$$
Prove that f is Riemann integrable with $\int_{[0,1]^2} f = 1$.

2. Let $0 \leq \epsilon \leq 1$ and suppose $a, b > 0$ satisfy
$$\frac{1}{(1+\epsilon)^n} \leq \frac{a}{b} \leq (1+\epsilon)^n.$$
Show that $|a - b| \leq 2^n b\epsilon$.

3. Let
$$S = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq a^2\},$$
and let $f\colon S \to \mathbb{R}$ be Riemann integrable. Prove that
$$\int_S f = \int_0^{2\pi} \int_0^a f(r\cos\theta, r\sin\theta) r \; dr d\theta,$$
despite the fact that the polar coordinates do not define a C^1-diffeomorphism on $R = [0,a] \times [0, 2\pi]$. [**Hint:** approximate S by keyholes.]

4. Let $\varphi_1, \varphi_2 \in C_1(\mathbb{R}^2)$ be C^1-diffeomorphisms such that $\varphi_1([0,1]) = \varphi_2([0,1])$.
   (a) Show that either
   $$(\varphi_1(0), \varphi_1(1)) = (\varphi_2(0), \varphi_2(1)) \qquad \text{or} \qquad (\varphi_1(0), \varphi_1(1)) = (\varphi_2(1), \varphi_2(0)).$$
   (b) Show that for any differential 1-form $\omega = f dx + g dy$ we have
   $$\omega(\varphi_1) = \pm \omega(\varphi_2),$$
   where we get a '+' if the first case in part (a) holds and we get '−' otherwise.

5. Find $\varphi \in C_3(\mathbb{R}^3)$, a 3-cell in $\mathbb{R}^3$, such that $\varphi(I^3)$ is the closed ball centered at $(0,0,0)$ with radius 1.

Solutions:

1. We claim that the discontinuity set D of f is $S = \{(x,y) \in \mathbb{R}^2 \colon 0 \leq x, y \leq 1 \text{ and } y \in \mathbb{Q}\}$. Indeed, for $x_0 \in [0,1]$ and $y_0 \in \mathbb{Q} \cap [0,1]$ with $y_0 = \frac{p}{q}$ in lowest terms we have (by the density of the rational and irrational numbers in $[0,1]$) that $\operatorname{osc}_{(x_0,y_0)}(f) \geq \frac{1}{q}$. Hence $(x_0, y_0) \in D$ and so $S \subset D$.

   On the other hand, let $(x_0, y_0) \in S$; that is, $x_0 \in [0,1]$ and $y_0 \in [0,1] \setminus \mathbb{Q}$. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$. Since y_0 is irrational, we can find $\delta > 0$ small enough so that $(y_0 - \delta, y_0 + \delta)$ does not contain any rationals of the form $\frac{p}{q}$ for $q \leq N$. Then for any $(x,y) \in \mathbb{R}^2$ with $|(x,y) - (x_0,y_0)| < \delta$, we have $|y - y_0| < \delta$. This means if y is rational, then it is of the form $\frac{p}{q}$ in lowest terms for $q > N$. In any case, we have $f(x,y) \geq 1 - \frac{1}{N}$ so that
   $$|f(x_0,y_0) - f(x,y)| = 1 - f(x,y) \leq 1 - (1 - \frac{1}{N}) = \frac{1}{N} < \epsilon.$$
   Thus f is continuous at (x_0, y_0) and $S^c \subset D^c$.

We have shown $D = S$ and we note that S is a zero set since it is the countable union of zero sets (line segments):

$$S = \bigcup_{y \in \mathbb{Q} \cap [0,1]} [0,1] \times \{y\}.$$

Thus f is Riemann integrable by the Riemann–Lebesgue theorem.

Now, consider the constant function $g\colon [0,1]^2 \to \mathbb{R}$, $g(x,y) = 1$. Then

$$P = \{(x,y) \in [0,1]^2 \colon |f(x,y) - g(x,y)| > 0\} = \mathbb{Q}^2 \cap [0,1]^2,$$

which is a zero set (it is a subset of S above). Thus by Exercise 3 from Homework 6 we have

$$\int_{[0,1]^2} f = \int_{[0,1]^2} g = |[0,1]^2| = 1.$$

□

2. Since $b > 0$, we can multiply both inequalities by b to obtain:

$$\frac{b}{(1+\epsilon)^n} \leq a \leq b(1+\epsilon)^n.$$

Subtracting b from each inequality then yields

$$-b\left(1 - \frac{1}{(1+\epsilon)^n}\right) \leq a - b \leq b\left((1+\epsilon)^n - 1\right).$$

Note that

$$1 - \frac{1}{(1+\epsilon)^n} = \frac{(1+\epsilon)^n - 1}{(1+\epsilon)^n} \leq (1+\epsilon)^n - 1.$$

So we have

$$-b\left((1+\epsilon)^n - 1\right) \leq a - b \leq b\left((1+\epsilon)^n - 1\right).$$

Thus

$$\begin{aligned} |a-b| &\leq b\left((1+\epsilon)^n - 1\right) \\ &= b\epsilon\big((1+\epsilon)^{n-1} + (1+\epsilon)^{n-2} + \cdots + (1+\epsilon) + 1\big) \\ &\leq b\epsilon(2^{n-1} + 2^{n-2} + \cdots + 2 + 1) \\ &= b\epsilon\frac{2^n - 1}{2-1} \leq 2^n b\epsilon. \end{aligned}$$

□

3. Define $\varphi\colon R \to S$ by

$$\varphi(r,\theta) = (r\cos\theta, r\sin\theta).$$

While φ is not a C^1-diffeomorphism on R (it fails to be injective for example), it is C^1-diffeomorphism on $U = (0,\infty) \times (0, 2\pi)$. Indeed, it is easily seen to be injective on U and it is of class C^1 since its partial derivatives exist and are continuous. Moreover,

$$\mathrm{Jac}_{(r,\theta)}(\varphi) = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0.$$

The inverse function theorem implies φ^{-1} is locally of class C^1 at (r,θ), but since this holds for every point in U we obtain that $\varphi\colon U \to \varphi(U)$ is indeed a C^1-diffeomorphism.

Now, for $\epsilon > 0$ define $R_\epsilon = [\epsilon, a] \times [\epsilon, 2\pi - \epsilon] \subset U$. Then the change of variables formula along with Fubini's theorem implies

$$\int_{\varphi(R_\epsilon)} f = \int_{R_\epsilon} f \circ \varphi |\mathrm{Jac}(\varphi)| = \int_\epsilon^{2\pi-\epsilon} \int_\epsilon^a f(r\cos\theta, r\sin\theta) r \; dr d\theta.$$

We will show that letting ϵ tend to zero yields the desired formula. Let $M = \sup\{|f(z)| \colon z \in S\}$.

We first consider the left-hand side:

$$\left| \int_S f - \int_{\varphi(R_\epsilon)} f \right| \leq \int_{S \setminus \varphi(R_\epsilon)} |f| \leq M|S \setminus \varphi(R_\epsilon)|,$$

where $S \setminus \varphi(R_\epsilon)|$ is a keyhole. It is contained in the union of the closed ball centered centered at $(0,0)$ with radius ϵ and $\varphi(T_\epsilon)$ where

$$T_\epsilon = [\epsilon, a] \times [-\epsilon, \epsilon].$$

Since φ is (by the same argument as above) a C^1-diffeomorphism on $(0,\infty) \times (-\pi, \pi)$, the change of variables formula and Fubini's theorem implies

$$|\varphi(T_\epsilon)| = \int_{\varphi(T_\epsilon)} 1 = \int_{T_\epsilon} |\text{Jac}(\varphi)| = \int_{-\epsilon}^{\epsilon} \int_{\epsilon}^{a} r \; drd\theta = \epsilon[a^2 - \epsilon]$$

Thus

$$\left| \int_S f - \int_{\varphi(R_\epsilon)} f \right| \leq M \left(\pi\epsilon^2 + \epsilon[a^2 - \epsilon] \right),$$

which tends to zero as ϵ does.

Next, for the right-hand side we have:

$$\begin{aligned} \left| \int_R f \circ \varphi |\text{Jac}(\varphi)| - \int_{R_\epsilon} f \circ \varphi |\text{Jac}(\varphi)| \right| &\leq \int_{R \setminus R_\epsilon} Ma \; drd\theta \\ &= Ma[2\pi a - (2\pi - 2\epsilon)(a - \epsilon)] \\ &= Ma[2\pi\epsilon + 2\epsilon a - 2\epsilon^2], \end{aligned}$$

which tends to zero as ϵ does. Thus

$$\int_R f \circ \varphi |\text{Jac}(\varphi)| = \lim_{\epsilon \to 0} \int_{R_\epsilon} f \circ \varphi |\text{Jac}(\varphi)| = \lim_{\epsilon \to 0} \int_{\varphi(R_\epsilon)} f = \int_S f.$$

Applying Fubini's theorem to the left-hand side above yields the claimed formula. □

4. (a) Let $t_0 = \varphi_1^{-1}(\varphi_2(0))$. Then

$$[0, t_0) \cup (t_0, 1] = \varphi_1^{-1}(\varphi_2((0,1])).$$

Since $(0,1]$ is connected, so is the above set. But this is only possible if $t_0 = 0$ or $t_0 = 1$. The same argument implies $t_1 := \varphi_1^{-1}(\varphi_2(1))$ is either 0 or 1. If $t_0 = 0$, then necessarily $t_1 = 1$ and hence $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(1) = \varphi_2(1)$. If $t_0 = 1$, we have $t_1 = 0$ and hence $\varphi_1(0) = \varphi_2(1)$ and $\varphi_1(1) = \varphi_2(0)$. □

(b) Define $\psi := \varphi_2^{-1} \circ \varphi_1$, which is a C^1-diffeomorphism from $[0,1]$ to itself. Write $\varphi_j(t) = (x_j(t), y_j(t))$, $j = 1, 2$. Then for a differential 1-form $fdx + gdy$ we have

$$\begin{aligned} (fdx + gdy)(\varphi_1) &= \int_0^1 f(\phi_1(t)) \frac{dx_1(t)}{dt} + g(\varphi_1(t)) \frac{dy_1(t)}{dt} dt \\ &= \int_0^1 f(\varphi_2(\psi(t))) \frac{dx_2(\psi(t))}{dt} + g(\varphi_2(\psi(t))) \frac{dy_2(\psi(t))}{dt} dt \\ &= \int_0^1 f(\varphi_2(\psi(t))) \frac{dx_2}{d\psi} \psi'(t) dt + g(\varphi_2(\psi(t))) \frac{dy_2}{d\psi} \psi'(t) dt \\ &= \pm \int_0^1 f(\varphi_2(s)) \frac{dx_2(s)}{ds} + g(\varphi_2(s)) \frac{dy_2(s)}{ds} ds \\ &= \pm (fdx + gdy)(\varphi_2), \end{aligned}$$

where the second to last equality follows by the one-dimensional change of variables formula. The sign is determined by whether $\psi(0) = 0$ and $\psi(1) = 1$ or $\psi(0) = 1$ and $\psi(1) = 0$ (which are the only two possibilities by part (b)). In the latter case, we pick up a minus sign by changing $\int_1^0 \cdot\, dt$ to $\int_0^1 \cdot\, dt$. □

5. We appeal to spherical coordinates. Define

$$\varphi(x_1, x_2, x_3) = (x_1 \cos(2\pi x_2)\sin(\pi x_3), x_1 \sin(2\pi x_2)\sin(\pi x_3), x_1 \cos(\pi x_3)).$$

This smooth since all of its mixed partials exist and are continuous. Thus $\varphi \in C_3(\mathbb{R}^3)$. Furthermore, it is clear that φ maps I^3 onto the desired set. □

[Note that φ is not injective, and hence not a diffeomorphism, but this does not preclude it from being a 3-cell.]