Exercises:

1. Let $S \subset \mathbb{R}^2$ be a zero set. Show that its interior,
$$S^\circ := \{z \in S \colon \exists r > 0 \text{ such that } B(z,r) \subset S\},$$
is empty.

2. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be bounded. Recall that for $z \in \mathbb{R}^2$, the **oscillation** of f at z is the quantity
$$\operatorname{osc}_z(f) := \lim_{r \to 0} \left[\sup(f(B(z,r))) - \inf(f(B(z,r)))\right].$$
   (a) Show that f is continuous at $z \in \mathbb{R}^2$ if and only if $\operatorname{osc}_z(f) = 0$.
   (b) For $S \subset \mathbb{R}^2$, show that χ_S is discontinuous at z if and only if $z \in \partial S$.

3. Let $R \subset \mathbb{R}^2$ be a rectangle. Suppose $f, g \colon R \to \mathbb{R}$ are Riemann integrable over R and that
$$P := \{z \in R \colon |f(z) - g(z)| > 0\}$$
is a zero set. Show that $\int_R f = \int_R g$.
[**Hint:** first show that for $z \in P$, $\operatorname{osc}_z(|f-g|) \geq |f(z) - g(z)|$, then proceed as in the proof of the Riemann–Lebesgue Theorem.]

4. Let $S \subset \mathbb{R}^2$ be bounded.
   (a) Show that if S is Riemann measurable then so are S° (its interior) and $\overline{S}$ (its closure).
   (b) Show that if S° and $\overline{S}$ are Riemann measurable with $|S^\circ| = |\overline{S}|$, then S is Riemann integrable with the same area.
   (c) Show that the hypothesis $|S^\circ| = |\overline{S}|$ in the previous part is a necessary by considering $S = \mathbb{Q}^2 \cap [0,1]^2$.

5. Use the volume multiplier formula to prove that the area of a parallelogram is the length of its base times its height.

Solutions:

1. Suppose, towards a contradiction that there exists $z \in S^\circ$. Then by definition of the interior there exists $r > 0$ such that $B(z,r) \subset S$. In particular, this open ball contains the open square centered at z with side-length $\sqrt{2}r$. Hence any covering of S by open rectangles must necessarily cover this open square and therefore have total area at least $2r^2$, contradicting S being a zero set. □

2. (a) $(\Rightarrow)$: Suppose f is continuous at $z \in \mathbb{R}^2$. Let $\epsilon > 0$, then there exists $\delta > 0$ such that whenever $w \in B(z,\delta)$, $|f(w) - f(z)| < \frac{\epsilon}{2}$. Consequently, for any $w, w' \in B(z,\delta)$ we have
$$f(w) - f(w') \leq |f(w) - f(w')| \leq |f(w) - f(z)| + |f(z) - f(w')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
Taking the supremum over w and the infimum over w' yields
$$\sup(f(B(z,\delta))) - \inf(f(B(z,\delta))) \leq \epsilon.$$
This clearly holds for $r \leq \delta$ as well and hence $\operatorname{osc}_z(f) \leq \epsilon$. As $\epsilon > 0$ was arbitrary, we must have $\operatorname{osc}_z(f) = 0$.
$(\Leftarrow)$: Suppose $\operatorname{osc}_z(f) = 0$. Let $\epsilon > 0$. Take $r > 0$ small enough so that
$$\sup(f(B(z,r))) - \inf(f(B(z,r))) < \epsilon.$$
Then for $\delta = r$, if $w \in B(z,\delta)$ we have
$$|f(w) - f(z)| \leq \sup(f(B(z,r))) - \inf(f(B(z,r))) < \epsilon.$$
Thus f is continuous at z. □

(b) $(\Rightarrow)$: Suppose χ_S is discontinuous at z. If $z \notin \partial S$, then there exists $r > 0$ such that either $B(z,r) \cap S = \emptyset$ or $B(z,r) \cap S^c = \emptyset$. In the former case, $\chi_S(w) = 0$ for all $w \in B(z,r)$ which imply $\text{osc}_z(\chi_S) = 0$. In the latter case, $\chi_S(w) = 1$ for all $w \in B(z,r)$ which again implies $\text{osc}_z(\chi_S) = 0$. So in either case, by part (a), we contradict χ_S being discontinuous at z.

$(\Leftarrow)$: Suppose $z \in \partial S$. Then for every $r > 0$ we have $B(z,r) \cap S \neq \emptyset$ and $B(z,r) \cap S^c \neq \emptyset$. So leting $w \in B(z,r) \cap S$ and $w' \in B(z,r) \cap S^c$ we have

$$\sup(\chi_S(B(z,r))) - \inf(\chi_S(B(z,r))) \geq \chi_S(w) - \chi_S(w') = 1 - 0 = 1.$$

Since this holds for each $r > 0$, we have $\text{osc}_z(\chi_S) \geq 1$. (In fact, given the bounds on χ_S, it easily follows that $\text{osc}_z(\chi_S) = 1$.) $\square$

3. Denote $h = |f - g|$. Then h is Riemann integrable and

$$\left| \int_R f - \int_R g \right| \leq \int_R h.$$

So it suffices to show $\int_R h = 0$.

Observe that

$$P = \{z \in R \colon h(z) > 0\}.$$

For each $k \in \mathbb{N}$, define

$$P_k := \left\{ z \in R \colon h(z) \geq \frac{1}{k} \right\},$$

So that $P = \bigcup_{k \in \mathbb{N}} P_k$.

Fix $k \in \mathbb{N}$. We claim $P_k \subset D_k$ where

$$D_k := \left\{ z \in R \colon \text{osc}_z(h) \geq \frac{1}{k} \right\}.$$

Indeed, by Exercise 1 we have $P^\circ = \emptyset$. Therefore, for every every $z \in P_k \subset P$, $B(z,r) \cap P^c \neq \emptyset$ for every $r > 0$. In particular, for every $r > 0$ there exists $w \in B(z,r) \cap P^c$ and so

$$\sup(h(B(z,r))) - \inf(h(B(z,r))) \geq h(z) - h(w) \geq \frac{1}{k} - 0 = \frac{1}{k}.$$

Thus $\text{osc}_z(h) \geq \frac{1}{k}$ and the claim follows. Observe that the claim implies $D_k^c \subset P_k^c$ and so for every $z \in D_k^c$ we have $h(z) < \frac{1}{k}$.

Let $\epsilon > 0$. Since h is Riemann integrable, D_k is necessarily a zero set which we can therefore cover with open rectangles $\{S_\ell\}_\ell$ satisfying $\sum |S_\ell| < \epsilon$. For every $z \in D_k^c$ there exists an open neighborhood W_z such that

$$|h(w) - h(w')| < \frac{1}{k}$$

for all $w, w' \in W_z$. Note that for any $w \in W_z$ we have

$$h(w) = h(w) - h(z) + h(z) \leq |h(w) - h(z)| + h(z) < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}.$$

The open rectangles S_ℓ along with the open neighborhoods W_z form an open cover for the compact set R. Hence there is a positive Lebesgue number $\lambda > 0$ associated with this open cover.

Let G be a grid on R with $\text{mesh}(G) < \lambda$. Then

$$\int_R h \leq U(h,G) = \sum M_{ij} |R_{ij}|.$$

Now, since the diameter of R_{ij} is less than $\mathrm{mesh}(G) < \lambda$, each R_{ij} is contained in either some S_ℓ or some W_z. Splitting the above sum into two sums according to this have

$$\int_R h \leq M \sum |S_\ell| + \sum_{R_{ij}\subset W_z} \frac{2}{k}|R_{ij}| \leq M\epsilon + \frac{2}{k}|R|,$$

where $M = \sup(h(R))$ (which is finite because h is Riemann integrable and therefore bounded). Since $\epsilon > 0$ was arbitrary, we have $\int_R h \leq \frac{2}{k}|R|$ and letting k tend to infinity we have $\int_R h \leq 0$. Since $h \geq 0$, we immediately have the other inequality and so $\int_R h = 0$. □

4. (a) We claim $\partial(S^\circ), \partial(\overline{S}) \subset \partial S$. Indeed, the inclusions $S^\circ \subset S \subset \overline{S}$ imply $\overline{S^\circ} \subset \overline{S}$ and $S^\circ \subset (\overline{S})^\circ$. Hence

$$\partial(S^\circ) = \overline{S^\circ} \setminus (S^\circ)^\circ = \overline{S^\circ} \setminus S^\circ \subset \overline{S} \setminus S^\circ = \partial S,$$

and

$$\partial(\overline{S}) = \overline{\overline{S}} \setminus (\overline{S})^\circ = \overline{S} \setminus (\overline{S})^\circ \subset \overline{S} \setminus S^\circ = \partial S.$$

Thus, since S is Riemann measurable we must have ∂S is a zero set and therefore the boundaries of S° and $\overline{S}$ are zero sets and hence Riemann measurable. □

(b) These sets being Riemann measurable with the same area is equivalent to $f_1 := \chi_{S^\circ}$ and $f_2 := \chi_{\overline{S}}$ being Riemann integrable with $\int f_1 = |S^\circ| = |\overline{S}| = \int f_2$. Denote this common value by A. Consequently, for $\epsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that whenever G is a grid on R with $\mathrm{mesh}(G) < \delta_1$ we have

$$L(f_1, G) \geq A - \epsilon,$$

and whenever $\mathrm{mesh}(G) < \delta_2$ we have

$$U(f_2, G) \leq A + \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$ and let G be a grid on R with $\mathrm{mesh}(G) < \delta$. Then using $f_1 \leq \chi_S \leq f_2$ we have

$$U(\chi_S, G) \leq U(f_1, G) \leq A + \epsilon \leq L(f_1, G) + 2\epsilon \leq L(\chi_S, G) + 2\epsilon \leq U(\chi_S, G) + 2\epsilon.$$

Letting ϵ tend to zero yields equalities in the above and hence χ_S is Riemann integrable with $|S| = \int \chi_S = A = |S^\circ| = |\overline{S}|$. □

(c) Note that $S = \mathbb{Q}^2 \cap [0,1]^2$ is countable and dense in $[0,1]^2$. The former implies it is a zero set and so by Exercise 1, $S^\circ = \emptyset$. The latter implies $\overline{S} = [0,1]^2$. Hence $|S^\circ| = 0 \neq 1 = |\overline{S}|$. Moreover, we have $\partial S = \overline{S} \setminus S^\circ = [0,1]^2$ which is **not** a zero set. Hence S is not Riemann measurable. □

5. Consider an arbitrary parallelogram P with base length b and height h. We can place such a parallelogram in $\mathbb{R}^2$ with its base on the x-axis and bottom left vertex at the origin. Its vertices are then on the coordinates $(0,0)$, $(b,0)$, (x,h), and $(x+b,h)$ for some $x \in \mathbb{R}$. Observe that P is the image of the rectangle $[0,b] \times [0,h]$ under the linear transformation given by

$$T = \begin{bmatrix} 1 & \frac{x}{h} \\ 0 & 1 \end{bmatrix},$$

since the T sends the vectors $(b,0)$ and $(0,h)$ to $(b,0)$ and (x,h). Since $\det(T) = 1$, it follows that $|P| = |[0,b] \times [0,h]| = bh$. □