Exercises:

1. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be of class C^1. Assume that $f(3,-1,2) = 0$ and

$$(Df)_{(3,-1,2)} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

   (a) Show that there is an open neighborhood $U \subset \mathbb{R}$ of 3 and a function $g\colon U \to \mathbb{R}^2$ of class C^1 such that $g(3) = (-1,2)$ and $f(x, g(x)) = 0$ for all $x \in U$.

   (b) Determine $(Dg)_3$.

2. Let $U \subset \mathbb{R}^m$ be open and let $f\colon U \to \mathbb{R}^m$ be of class C^1. Show that if $(Df)_x$ is invertible for all $x \in U$, then $f(U)$ is open.

3. Let $G \subset \mathcal{M}(n,n)$ denote the set of invertible $n \times n$ matrices, with metric given by the operator norm.

   (a) Show that G is open.
   [**Hint:** use the determinant but take care with the norms.]

   (b) Show that G contains 1, is closed under multiplication, and closed under the map $\mathrm{Inv}(A) := A^{-1}$; that is, show G is a group.
   [**Note:** this group is called the **general linear group** and is typically denoted $GL_n(\mathbb{R})$]

   (c) Show that $\mathrm{Inv}\colon G \to G$ is a homeomorphism.

   (d) Show that $\mathrm{Inv}\colon G \to G$ is a C^1 diffeomorphism with

$$(D\mathrm{Inv})_A(X) = -A^{-1}XA^{-1} \qquad A \in G,\ X \in \mathcal{M}(n,n).$$

   [**Note:** this should remind you of the formula $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$.]

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that its graph

$$G := \{(x, f(x))\colon x \in \mathbb{R}\} \subset \mathbb{R}^2$$

is a zero set.

Solutions:

1. (a) For $p \in \mathbb{R}^3$, write $p = (x,y)$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}^2$. Using this notation, the given value for $(Df)_{(3,-1,2)}$ implies

$$\left(\frac{\partial f_i(3,-1,2)}{\partial y_j}\right)_{1\leq i,j\leq 2} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix},$$

where we are invoking the theorem from class that tells us the entries of the total derivative are given by partial derivatives. The determinant of this matrix is $2\cdot 1 - 1\cdot(-1) = 3 \neq 0$, so this matrix is non-zero. Hence the Implicit Function Theorem implies the existance of this neighborhood B and C^1 function g.

   (b) Write

$$(Dg)_3 = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Using the chain rule, we have (since $f(x, g(x)) = 0$ is a constant function in x)

$$0 = (Df(x, g(x)))_3 = (Df)_{(3,-1,2)} \circ \begin{bmatrix} 1 \\ (Dg)_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \begin{bmatrix} 1 + 2a + b \\ 1 - a + b \end{bmatrix}.$$

This corresponds to the system of equations

$$\begin{aligned} 1 + 2a + b &= 0 \\ 1 - a + b &= 0 \end{aligned}$$

which has the solution $(a, b) = (0, -1)$. Thus $(Dg)_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. $\square$

2. Let $y \in f(U)$ so that $y = f(x)$ for some $x \in U$. Since $(Df)_x$ is invertible, there is an open neighborhood $U_x \subset U$ of x on which f is a C^1 diffeomorphism. In particular, on $f(U_x)$ the function f^{-1} is well defined and continuous. But then $f(U_x)$ is the preimage of the open set U_x under the continuous function f^{-1}; that is, $f(U_x)$ is open. Since $y = f(x) \in f(U_x)$, there exists a radius $r > 0$ such that the open ball $B(y, r) \subset f(U_x) \subset f(U)$. Since $y \in f(U)$ was arbitrary, we see that $f(U)$ is open. $\square$

3. (a) Since the determinant of a matrix is just a polynomial in its entries, it is continuous as a map from $\mathcal{M}(n, n)$ to $\mathbb{R}$ when the matrices are given the Euclidean norm via $\mathcal{M}(n, n) \cong \mathbb{R}^{n^2}$. Let's denote this norm by $|\cdot|$. However, $\mathcal{M}(n, n)$ is a finite-dimensional vector space, so all norms are equivalent. In particular, $|\cdot|$ is equivalent to the operator norm $\|\cdot\|$. Therefore there exists a constant $c > 0$ such that $|A| \leq c\|A\|$ for all $A \in \mathcal{M}(n, n)$. Now, since the determinant is continuous with respect to $|\cdot|$, given $\epsilon > 0$ we can find $\delta > 0$ such that if $|A - B| < \delta$ then

$$|\det(A) - \det(B)| < \epsilon.$$

But then if we require $\|A - B\| < \frac{\delta}{c}$ we have

$$|A - B| \leq c\|A - B\| < c\frac{\delta}{c} = \delta,$$

and so $|\det(A) - \det(B)| < \epsilon$ is this case as well. That is, the determinant is continuous on $\mathcal{M}(n, n)$ with the metric structure is determined by the operator norm. Then, since $\mathbb{R} \setminus \{0\}$ is an open set, we have $G = \det^{-1}(\mathbb{R} \setminus \{0\})$ is an open set. $\square$

(b) The identity matrix 1 is its own inverse, so $1 \in G$. Given $A, B \in G$ we have $(AB)^{-1} = B^{-1}A^{-1}$, so $AB \in G$ and G is closed under multiplication. Finally, if $A \in G$, then $(A^{-1})^{-1} = A$ so $A^{-1} \in G$ and G is closed under taking inverses. $\square$

(c) By Cramer's rule, the entries of $\mathrm{Inv}(A)$ are given in terms of polynomials divided by $\det(A)$. Since these functions are continuous (and $\det(A) \neq 0$ on G), it follows that Inv is continuous when $\mathcal{M}(n, n)$ is given the Euclidean norm as in part (a). However, as we saw in part (a), the Euclidean norm is equivalent to the operator norm and hence Inv is continuous with respect to the operator norm. Indeed, as before let $|\cdot|$ denote the Euclidean norm. Then there exists constants $c_1, c_2 > 0$ such that $c_1\|A\| \leq |A| \leq c_2\|A\|$. Given $\epsilon > 0$, we can find $\delta > 0$ so that if $|A - B| < \delta$ then

$$|\mathrm{Inv}(A) - \mathrm{Inv}(B)| < c_1\epsilon.$$

But then if we require $\|A - B\| < \frac{\delta}{c_2}$ we have

$$|A - B| < c_2\|A - B\| < \delta,$$

so that

$$\|\mathrm{Inv}(A) - \mathrm{Inv}(B)\| \leq \frac{1}{c_1}|\mathrm{Inv}(A) - \mathrm{Inv}(B)| < \frac{1}{c_1}c_1\epsilon = \epsilon.$$

Thus Inv is continuous with respect to the operator norm. Since it is its own inverse, we have in fact shown that it is a homeomorphism. $\square$

Alternate Proof: We can actually avoid using Cramer's rule and the finite dimensionality using the following proof. Fixing $A \in G$, we will show Inv is continuous at A. Given $B \in G$ we observe that

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.$$

Since A is fixed and $B-A$ can be made small in operator norm, it suffices to show that $\|B^{-1}\|$ can be bounded in terms of A. For this we will use a trick involving the geometric series. For now, let us assume $\|A-B\| < \frac{1}{\|A^{-1}\|}$. Then we observe that

$$B = (B-A) + A = A(A^{-1}(B-A)+1) = A(1 - A^{-1}(A-B)).$$

We claim that $1-A^{-1}(A-B)$ is invertible. Indeed, we have $\|A^{-1}(A-B)\| \leq \|A^{-1}\|\|A-B\| < 1$ and hence the following series converges to an operator:

$$\sum_{k=0}^{\infty}[A^{-1}(A-B)]^k.$$

We compute

$$(1-A^{-1}(A-B))\sum_{k=0}^{\infty}[A^{-1}(A-B)]^k = \sum_{k=0}^{\infty}[A^{-1}(A-B)]^k - \sum_{k=0}^{\infty}[A^{-1}(A-B)]^{k+1} = 1.$$

Thus $(1-A^{-1}(A-B))$ is invertible with inverse $\sum[A^{-1}(A-B)]^k$. Consequently, we have

$$B^{-1} = (1-A^{-1}(A-B))^{-1}A^{-1} = \sum_{k=0}^{\infty}[A^{-1}(A-B)]^k A^{-1}.$$

Now, using this we can estimate

$$\|B^{-1}\| \leq \|A^{-1}\| \sum_{k=0}^{\infty}[\|A^{-1}\|\|A-B\|]^k.$$

If we require $\|A-B\| < \frac{1}{2\|A^{-1}\|}$, then the sum in the above expression will be bounded by 2 (using the geometric series for $\frac{1}{2}$), so that $\|B^{-1}\| \leq 2\|A^{-1}\|$.
Now, given $\epsilon > 0$ let

$$\delta = \min\left\{\frac{\epsilon}{2\|A^{-1}\|^2}, \frac{1}{2\|A^{-1}\|}\right\}.$$

Then if $\|A-B\| < \delta$ we have

$$\|A^{-1} - B^{-1}\| = \|A^{-1}(B-A)B^{-1}\| \leq \|A^{-1}\|\|B-A\|2\|A^{-1}\| < \epsilon.$$

Thus Inv is continuous at A. Since $A \in G$ was arbitrary, we have that the map is continuous on all of G. Finally, since Inv is its own inverse we see that it is a homemorphism. □

(d) That Inv is a bijection is clear. Let us verify the formula for the derivative by computing the Taylor remainder:

$$\begin{aligned}
R(X) &= \mathrm{Inv}(A+X) - \mathrm{Inv}(A) - (-A^{-1}XA^{-1}) \\
&= (A+X)^{-1} - A^{-1} + A^{-1}XA^{-1} \\
&= (A+X)^{-1}(A-(A+X))A^{-1} + A^{-1}XA^{-1} \\
&= -(A+X)^{-1}XA^{-1} + A^{-1}XA^{-1} \\
&= [A^{-1} - (A+X)^{-1}]XA^{-1}.
\end{aligned}$$

Thus

$$\frac{\|R(X)\|}{\|X\|} \leq \|A^{-1} - (A+X)^{-1}\|\|A^{-1}\|,$$

which tends to zero as X does by part (c). Thus Inv is differentiable on G with the claimed derivative.

Let us now check that it is C^1. We have for $A, B \in G$ and $X \in \mathcal{M}(n,n)$

$$\begin{aligned}\frac{\|(D\mathrm{Inv})_A(X) - (D\mathrm{Inv})_B(X)\|}{\|X\|} &= \frac{\|A^{-1}XA^{-1} - B^{-1}XA^{-1} + B^{-1}XA^{-1} - B^{-1}XB^{-1}\|}{\|X\|} \\ &\leq \|A^{-1} - B^{-1}\|\|A^{-1}\| + \|B^{-1}\|\|A^{-1} - B^{-1}\|.\end{aligned}$$

Using part (c), this quantity tends to zero as $B \to A$, independent of X. Hence $\|(D\mathrm{Inv})_A - (D\mathrm{Inv})_B\|$ tends to zero as $B \to A$. Thus Inv is C^1.

Finally, since Inv is its own inverse, the above shows that it is C^1 diffeomorphism.

[Note that we could also apply the Inverse Function Theorem to finish this proof. At each $A \in G$, $(D\mathrm{Inv})_A$ is invertible with inverse $X \mapsto AXA$, and so locally Inv is a C^1 diffeomorphism. But since this holds for all $A \in G$, we have that Inv is globally (i.e. on all of G) a C^1 diffeomorphism.]

4. For $n \in \mathbb{Z}$, let
$$G_n := \{(x, f(x)) \colon n \leq x \leq n+1\}.$$
Since G is the (countable) union of G_n, it suffices to show that each G_n is a zero set. Fix $n \in \mathbb{Z}$ and let $\epsilon > 0$. Since $[n, n+1]$ is a compact set, f is uniformly continuous on this interval. So there exists $\delta > 0$ such that if $x, y \in [n, n+1]$ satisfy $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Replacing δ with 1 if necessary, we assume $\delta \leq 1$. $N \in \mathbb{N}$ be such that $N\delta \leq 1 < (N+1)\delta$, and note that $(N+1)\delta = N\delta + \delta \leq 2$ by our assumption on δ. Now, define open rectangles
$$S_i = (n + (i-1)\delta, n + (i+1)\delta) \times (f(n + i\delta) - \epsilon, f(n + i\delta) + \epsilon) \qquad i = 0, 1, 2, \ldots, N.$$
We claim that
$$G_n \subset \bigcup_{i=0}^{N} S_i.$$
Indeed, by the choice of N we have
$$[n, n+1] \subset \bigcup_{i=0}^{N} (n + (i-1)\delta, n + (i+1)\delta).$$
Then if $x \in (n + (i-1)\delta, n + (i+1)\delta)$ we have $|x - (n + i\delta)| < \delta$ so that $|f(x) - f(n + i\delta)| < \epsilon$, which implies $f(x) \in (f(n + i\delta) - \epsilon, f(n + i\delta) + \epsilon)$. That is, $(x, f(x)) \in S_i$ and hence the claim follows.

Now, observe that
$$\sum_{i=0}^{N} |S_i| = \sum_{i=0}^{N} 2\delta 2\epsilon = 4(N+1)\delta\epsilon \leq 8\epsilon$$
Since $\epsilon > 0$ is arbitrary, we have that G_n is a zero set. $\square$