Exercises:

1. Consider $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x_1, x_2, x_3) = (x_1x_2 + x_2x_3, x_3^3).$$

For $p \in \mathbb{R}^3$, determine the matrix representation for $(D^2f)_p$ with respect to the ordered basis

$$\{(e_1, e_1), (e_1, e_2), (e_1, e_3), (e_2, e_1), (e_2, e_2), (e_2, e_3), (e_3, e_1), (e_3, e_2), (e_3, e_3)\}$$

for $\mathbb{R}^{3^2}$. Then prove that the corresponding Taylor remainder for Df at p is sublinear.

[**Note:** you already computed $(Df)_p$ on Homework 2, so you do not need to rederive this.]

2. Let $\beta \in \mathcal{L}^k(\mathbb{R}^n, \mathbb{R}^m)$ be a k-linear map. The **symmetrization** of β is the k-linear map $\text{symm}(\beta) \in \mathcal{L}^k(\mathbb{R}^n, \mathbb{R}^m)$ defined by

$$\text{symm}(\beta)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \beta(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where S_k is the permutation group on k elements.

   (a) Show that $\text{symm}(\beta)$ is indeed symmetric.

   (b) Show that β is symmetric if and only if $\beta = \text{symm}(\beta)$.

3. Let $\beta \in \mathcal{L}^k(\mathbb{R}^n, \mathbb{R}^m)$ be a k-linear map. Define $f\colon \mathbb{R}^n \to \mathbb{R}^m$ by $f(x) = \beta(x, \ldots, x)$.

   (a) Show that for $p, v \in \mathbb{R}^n$

$$(Df)_p(v) = \beta(v, p, \ldots, p) + \beta(p, v, p, \ldots, p) + \cdots + \beta(p, \ldots, p, v).$$

   (b) Show that for $p, v_1, v_2 \in \mathbb{R}^n$

$$(D^2f)_p(v_1, w_2) = \sum_{1 \le i < j \le k} \sum_{\sigma \in S_2} \beta(\underbrace{p, \ldots, p}_{i-1}, v_{\sigma(1)}, \underbrace{p, \ldots, p}_{j-i-1}, v_{\sigma(2)}, p, \ldots, p).$$

   (c) Further assume that β is symmetric. For $r \ge 0$, show that for $p, v_1, \ldots, v_r \in \mathbb{R}^n$

$$(D^rf)_p(v_1, \ldots, v_r) = \begin{cases} \dfrac{k!}{(k-r)!}\beta(v_1, \ldots, v_r, p, \ldots, p) & \text{if } r \le k \\ 0 & \text{if } r > k \end{cases}.$$

   Note that this implies $(D^kf)_p = k!\text{symm}(\beta)$.

   [**Hint:** proceed by induction and note that the base case $r = 0$ is trivial.]

4. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \ne (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Show that the second partial derivatives exists everywhere, but that $\dfrac{\partial^2 f(0,0)}{\partial x \partial y} \ne \dfrac{\partial^2 f(0,0)}{\partial y \partial x}$.

5. Let $f\colon U \to \mathbb{R}^m$ be r-times differentiable at $p \in U$ with $r \ge 3$. Use induction to show that $(D^rf)_p$ is symmetric: first show for $v_1, \ldots, v_r \in \mathbb{R}^n$ that $(D^rf)_p$ is symmetric with respect to permutations of $v_2, \ldots, v_r$, and then use the fact that $r > 2$ to show that it is also invariant under permutations of v_1 and v_2.

Solutions:

1. By a result from class, we know the entries of $(D^2 f)_p$ are of the form

$$\frac{\partial^2 f_k(p)}{\partial x_i \partial x_j},$$

where k determines the row and (i,j) determines the column according to our ordered basis. For example, $\frac{\partial^2 f_1(p)}{\partial x_2 \partial x_3}$ will appear in the 1st row and, since (e_2, e_3) is the sixth basis vector, the $6th$ column. So we have

$$(D^2 f)_p = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6p_3 \end{bmatrix}$$

Now, the Taylor remainder at p is given by

$$R(v) = (Df)_{p+v} - (Df)_p - (D^2 f)_p(v, \cdot),$$

where we are thinking of $(D^2 f)_p \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{R}^2)$ so that $(D^2 f)_p(v, \cdot) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$. Also, observe that $R(v) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$. Thus, to check sublinearity, we must show:

$$\lim_{v \to 0} \frac{\|R(v)\|}{|v|} = 0.$$

By definition of the operator norm, it suffices to show for any $w \in \mathbb{R}^3 \setminus \{0\}$ that

$$\lim_{v \to 0} \frac{|(R(v))(w)|}{|v||w|} = 0.$$

Using Exercise 3 from Homework 2, we first compute

$$\begin{aligned} (Df)_{p+v} - (Df)_p &= \begin{bmatrix} p_2 + v_2 & (p_1 + v_1) + (p_3 + v_3) & p_2 + v_2 \\ 0 & 0 & 3(p_3 + v_3)^2 \end{bmatrix} - \begin{bmatrix} p_2 & p_1 + p_3 & p_2 \\ 0 & 0 & 3p_3^2 \end{bmatrix} \\ &= \begin{bmatrix} v_2 & v_1 + v_3 & v_2 \\ 0 & 0 & 6p_3 v_3 + 3v_3^2 \end{bmatrix}. \end{aligned}$$

Using this, our formula for $(D2^f)_p$, and writing $(v, w) = \sum_{i,j=1}^{3} v_i w_j (e_i, e_j)$ we can compute

$$\begin{aligned} (R(v))(w) &= \begin{bmatrix} v_2 & v_1 + v_3 & v_2 \\ 0 & 0 & 6p_3 v_3 + 3v_3^2 \end{bmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - (D^2 f)_p(v, w) \\ &= \begin{pmatrix} v_2 w_1 + (v_1 + v_3) w_2 + v_2 w_3 \\ 6p_3 v_3 w_3 + 3v_3^2 w_3 \end{pmatrix} - \begin{pmatrix} v_1 w_2 + v_2 w_1 + v_2 w_3 + v_3 w_2 \\ 6p_3 v_3 w_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 3v_3^2 w_3 \end{pmatrix}. \end{aligned}$$

Thus

$$\frac{|(R(v))(w)|}{|v||w|} = \frac{|3v_3^2 w_3|}{|v||w|} \leq \frac{3|v|^2 |w|}{|v||w|} = 3|v|,$$

which clearly tends to zero as v does. Thus $R(v)$ is sublinear. □

2. (a) Fix $\pi \in S_k$ and $v_1, \ldots, v_k \in \mathbb{R}^n$, and write $w_j := v_{\pi(j)}$ for $j = 1, \ldots, k$. Observe that

$$S_k \ni \sigma \mapsto \sigma \circ \pi \in S_k$$

is a bijection since π is invertible (S_k is a group after all). Hence by the change of index given by the above map, we have

$$\begin{aligned}\text{symm}(\beta)(v_{\pi(1)},\ldots,v_{\pi(k)}) &= \text{symm}(\beta)(w_1,\ldots,w_k)\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\beta(w_{\sigma(1)},\ldots,w_{\sigma(k)})\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\beta(v_{\pi(\sigma(1))},\ldots,v_{\pi(\sigma(k))})\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\beta(v_{\sigma(1)},\ldots,v_{\sigma(k)}) = \text{symm}(\beta)(v_1,\ldots,v_k).\end{aligned}$$

That is, symm(β) is symmetric. □

(b) If $\beta = \text{symm}(\beta)$, then part (a) shows that it is symmetric. On the other hand, if β is already symmetric, then we have for $v_1,\ldots,v_k\in\mathbb{R}^n$

$$\text{symm}(\beta)(v_1,\ldots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\beta(v_{\sigma(1)},\ldots,v_{\sigma(k)}) = \frac{1}{k!}\sum_{\sigma\in S_k}\beta(v_1,\ldots,v_k) = \beta(v_1,\ldots,v_k),$$

where we have used $|S_k| = k!$. □

3. (a) We will show the Taylor remainder is sublinear. Indeed, expanding $f(p+v)$ in each entry of β, $f(p+v)-f(p)$ is the sum of β evaluated in v's and p's where each term has v in at least one entry. Then $R(v)$ is the sum of β evaluated in v's and p's where each term has v in **at least two entries**. Consequently,

$$|\mathbb{R}(v)| \leq \|\beta\|\sum_{i=2}^{k}\binom{k}{i}|v|^i|p|^{k-i},$$

where $\|\beta\|$ is the norm of β as a k-linear map. Since each term in the above estimate as at least two factors of $|v|$, we see that R is indeed sublinear. □

(b) Again, we will show the Taylor remainder is sublinear. Using the formula from part (a), we see that $(Df)_{p+v_1}(v_2) - (Df)_p(v_2)$ (after expanding $p+v_1$ in the etries they appear in in β) is the sum of β evaluated in v_1's, v_2's, and p's, where v_2 appears exactly once, and v_1 appears at least once. Then $R(v_1,v_2)$ is the sum of β evaluated in v_1's, v_2's, and p's, where v_2 appears exactly once, and v_1 appears at least twice. Consequently,

$$|R(v_1,v_2)| \leq \|\beta\|\sum_{i=2}^{k-1}k|v_2|\binom{k-1}{i}|v_1|^i|p|^{k-1-i},$$

since each term has at least two factors of $|v_1|$, we have

$$\lim_{v_1\to 0}\frac{|R(v_1,v_2)|}{|v_1||v_2|} = 0,$$

which implies

$$\lim_{v_1\to 0}\frac{\|R(v_1,\cdot)\|}{|v_1|},$$

where we think of $R(v_1,\cdot)\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$. Hence R is sublinear. □

(c) Note that the case $r=0$ is simply saying $f(p)=\beta(p,\ldots,p)$, which holds. Now suppose the formula has been established for $r<k$ (note the strict inequality). Then for $v_{r+1}\in\mathbb{R}^n$ we have

$$\begin{aligned}&[(D^rf)_{p+v_1} - (D^rf)_p](v_2,\ldots,v_{r+1})\\ &=\frac{k!}{(k-r)!}\left[\beta(v_2,\ldots,v_{r+1},p+v_1,\ldots,p+v_1) - \beta(v_2,\ldots,v_{r+1},p,\ldots,p)\right]\\ &=\frac{k!}{(k-r)!}\sum_{i=1}^{k-r}\binom{k-r}{i}\beta(\underbrace{v_1,\ldots,v_1}_{i\text{ times}},v_2,\ldots,v_{r+1},p,\ldots,p),\end{aligned}$$

where we have used the symmetry of β. Note that the term corresponding to $i = 1$ in the above sum is exactly the claimed formula for $r + 1$ since $\binom{k-r}{1} = \frac{(k-r)!}{(k-r-1)!}$. Hence the Taylor remainder is given by

$$R(v_1, \dots, v_{r+1}) = \frac{k!}{(k-r)!} \sum_{i=2}^{k-r} \binom{k-r}{i} \beta(\underbrace{v_1, \dots, v_1}_{i \text{ times}}, v_2, \dots, v_{r+1}, p, \dots, p)$$

(i.e. the sum starts at $i = 2$). Since each term has v_1 appearing at least twice, we have

$$\lim_{v_1 \to 0} \frac{|R(v_1, \dots, v_{r+1}|}{|v_1| \cdots |v_{r+1}|} = 0$$

which implies

$$\lim_{v_1 \to 0} \frac{\|R(v_1, \cdot, \dots, \cdot)\|}{|v_1|} = 0,$$

when we think of $R(v_1, \cdot, \dots, \cdot) \in \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m)$. Thus the claimed formula holds for $r + 1$, and so by induction we have proven the formula for $0 \leq r \leq k$.

Finally, to see the formula for $r > k$, note that $(D^k)_p = k!\text{symm}(\beta)$ for all $p \in \mathbb{R}^n$. Hence

$$R(v) = (D^k f)_{p+v} - (D^k f)_p - 0 = 0,$$

which is clearly sublinear. Therefore $(D^{k+1} f)_p = (D(D^k f))_p = 0$. Since all derivatives of the zero map are the zero map, we obtain $(D^r f)_p = 0$ for all $r \geq k + 1$. □

4. For $(x, y) \neq (0, 0)$, the following formulas follow from one-dimensional differentiation rules:

$$\frac{\partial^2 f(x,y)}{\partial x^2} = -\frac{4xy^3(x^2 - 3y^2)}{(x^2 + y^2)^3} \qquad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3} \qquad \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{4x^3y(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

We take a little bit more care with computing the second partials at $(0, 0)$. First note that the partial derivatives are:

$$\frac{\partial f(x,y)}{\partial x} = \begin{cases} \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

$$\frac{\partial f(x,y)}{\partial y} = \begin{cases} \dfrac{x(x^4 + 4x2y^2 - y^4)}{(x^2 + y^2)^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then we have

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \lim_{h \to 0} \frac{\frac{\partial f(h,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1,$$

but

$$\frac{\partial^2 f(0,0)}{\partial y \partial x} = \lim_{h \to 0} \frac{\frac{\partial f(0,h)}{\partial x} - \frac{\partial f(0,0)}{\partial x}}{h} = \lim_{h \to 0} \frac{-h - 0}{h} = -1.$$

Thus the mixed second partials of f at $(0, 0)$ do not agree.

We remark that this implies $(D^2 f)_{(0,0)}$ cannot exist, since it would necessarily imply (as seen in class) that the mixed second partials are equal. □

5. The base case of $r = 2$ was done in class. Suppose we have shown $(D^{r-1}f)_p$ is symmetric. Then, if $\sigma \in S_r$ is such that $\sigma(1) = 1$, we have

$$\begin{aligned}
(D^r f)_p(v_{\sigma(1)}, \dots, v_{\sigma(r)}) &= (D^r f)_p(v_1, v_{\sigma(2)}, \dots v_{\sigma(r)}) \\
&= ((D(D^{r-1}f))_p(v_1))(v_{\sigma(2)}, \dots, v_{\sigma(r)}) \\
&= \left(\lim_{t\to 0} \frac{(D^{r-1}f)_{p+tv_1} - (D^{r-1}f)_p}{t}\right)(v_{\sigma(2)}, \dots, v_{\sigma(r)}) \\
&= \lim_{t\to 0} \frac{(D^{r-1}f)_{p+tv_1}(v_{\sigma(2)}, \dots, v_{\sigma(r)}) - (D^{r-1}f)_p(v_{\sigma(2)}, \dots, v_{\sigma(r)})}{t} \\
&= \lim_{t\to 0} \frac{(D^{r-1}f)_{p+tv_1}(v_2, \dots, v_r) - (D^{r-1}f)_p(v_2, \dots, v_r)}{t} = (D^r f)_p(v_1, \dots, v_r),
\end{aligned}$$

where in the second to last equality we have used the induction hypothesis.

Next, consider $g\colon U \to \mathcal{L}^{r-2}(\mathbb{R}^n, \mathbb{R}^m)$ defined by $g(p) = (D^{r-2}f)_p$. (Note that this is well-defined since $r \geq 2$). Then

$$(D^2 g)_p(v_1, v_2) = (D^r f)_p(v_1, v_2, \cdot, \dots, \cdot) \in \mathcal{L}^{r-2}(\mathbb{R}^n, \mathbb{R}^m).$$

By our base case, we have

$$(D^r f)_p(v_2, v_1, \cdot, \dots, \cdot) = (D^2 g)_p(v_2, v_1) = (D^2 g)_p(v_1, v_2) = (D^r f)_p(v_1, v_2, \cdot, \dots, \cdot),$$

thus $(D^r f)_p$ is symmetric with respect to permutations of v_1, v_2.

Combining the two above arguments, to complete the induction argument it suffices to show that any $\sigma \in S_k$ can be written as a product of permutations that fix 1 and permutations that only act on 1 and 2. But for $\sigma \in S_r$, if $\sigma(1) = k$ then it it is easy to check that

$$\sigma = (1, 2)(2, k)(1, 2)(1, k)\sigma$$

(where (i, j) represents that the transposition switching i and j), and observe that $(1, k)\sigma$ and $(2, k)$ fix 1. Thus, by induction if follows that $(D^r f)_p$ is symmetric. □