Exercises:

1. Two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space V are **equivalent** if there exists positive constants c and C such that for all $v \in V \setminus \{0\}$
$$c \leq \frac{|v|_1}{|v|_2} \leq C.$$
   (a) Show that any two norms on a finite-dimensional vector space are equivalent.

   (b) Let $C([0,1])$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$. For $f \in C([0,1])$ consider the following two norms:
$$|f|_1 := \int_0^1 |f(t)|\ dt \qquad |f|_\infty := \max\{|f(t)| \colon 0 \leq t \leq 1\}.$$
   Show that these two norms are **not** equivalent.

   [**Note:** you do **not** need to verify that these are in fact norms.]

2. Let $C([0,1])$ be as in the previous exercise, equipped with the $|\cdot|_\infty$-norm. Let $C^1([0,1])$ be the (dense) subspace of $C([0,1])$ consisting of differentiable functions whose derivatives are continuous.
   (a) Define $T\colon C([0,1]) \to C^1([0,1])$ by
$$(Tf)(t) = \int_0^t f(x)\ dx.$$
   Show that T is continuous by determining its operator norm.

   (b) Define $S\colon C^1([0,1]) \to C([0,1])$ by $Sf = f'$. Show that S that is **not** continuous by showing its operator norm is infinite.

   [**Hint:** consider the monomials t^n, $n \in \mathbb{N}$.]

   (c) [**Not collected**] Convince yourself that
$$S \circ T\colon C([0,1]) \to C([0,1])$$
   is the identity map, and that
$$T \circ S\colon C^1([0,1]) \to C^1([0,1])$$
   is the identity map minus the linear transformation that evaluates a function at 0.

3. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ by
$$f(x_1, x_2, x_3) = (x_1x_2 + x_2x_3, x_3^3)$$
   For $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, use the definition of the derivative to show $(Df)_p = T_A$ where
$$A = \begin{bmatrix} p_2 & p_1 + p_3 & p_2 \\ 0 & 0 & 3p_3^2 \end{bmatrix}$$
   (i.e. show the Taylor remainder is sublinear).

4. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x_1, x_2) = \begin{cases} \dfrac{x_1x_2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq 0 \\ 0 & \text{if } (x_1, x_2) = 0 \end{cases}.$$
   Show that the partial derivatives $\frac{\partial f(0)}{\partial x_1}$ and $\frac{\partial f(0)}{\partial x_2}$ exist but $(Df)_0$ does not.

Solutions:

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1. (a) Write V_i for V equipped with $|\cdot|_i$, $i = 1,2$. Consider the linear transformation $T\colon V_1 \to V_2$ given by $Tv = v$. Clearly T is an isomorphism. Since V is finite dimensional, a theorem from lecture implies that T is a homeomorphism; that is, T and T^{-1} are continuous. By another theorem from class, this means $\|T\|$ and $\|T^{-1}\|$ are finite. For $v \in V \setminus \{0\}$, we then have

$$0 < \frac{|v|_2}{|v|_1} = \frac{|T(v)|_2}{|v|_1} \leq \|T\|,$$

and

$$\frac{|v|_1}{|v|_2} = \frac{|T^{-1}(v)|_1}{|v|_2} \leq \|T^{-1}\|.$$

Set $c = \frac{1}{\|T\|}$ and $C = \|T^{-1}\|$. □

(b) We consider the functions $f_n(x) := x^n$. Then $|f_n|_1 = \frac{1}{n+1}$ while $|f_n|_\infty = 1$. Thus the ratio

$$\frac{|f_n|_1}{|f_n|_\infty} = \frac{1}{n+1}$$

can be made arbitrarily small and is therefore not bounded below by some uniform positive constant c. □

2. (a) For $f \in C([0,1])$ we have for each $t \in [0,1]$

$$|(Tf)(t)| = \left| \int_0^t f(x)\ dx \right| \leq \int_0^t |f(x)|\ dx \leq \int_0^t |f|_\infty\ dx = t|f|_\infty.$$

Therefore, $|Tf|_\infty \leq 1 \cdot |f|_\infty = |f|_\infty$. Hence $\|T\| \leq 1$. On the other hand, for $f(x) = 1$ we have

$$\|T\| \geq \frac{|Tf|_\infty}{|f|_\infty} = \frac{|t|_\infty}{|1|_\infty} = \frac{1}{1} = 1.$$

Thus $\|T\| = 1$. □

(b) For $f_n(t) := t^n$ we have $(Sf_n)(t) = nt^{n-1}$. Thus

$$\frac{|Sf_n|_\infty}{|f_n|_\infty} = \frac{n}{1} = n.$$

Consequently $\|S\| \geq \sup(\mathbb{N}) = \infty$. □

(c) Indeed, this simply follows from the Fundamental Theorem(s) of Calculus. □

3. We begin by computing the Taylor remainder $R(v)$ for $v = (v_1, v_2, v_3) \in \mathbb{R}^3$:

$$\begin{aligned} R(v) &= f(p+v) - f(p) - Av \\ &= \begin{pmatrix} (p_1+v_1)(p_2+v_2) + (p_2+v_2)(p_3+v_3) \\ (p_3+v_3)^3 \end{pmatrix} - \begin{pmatrix} p_1p_2 + p_2p_3 \\ p_3^3 \end{pmatrix} - \begin{pmatrix} p_2v_1 + (p_1+p_3)v_2 + p_2v_3 \\ 3p_3^2v_3 \end{pmatrix} \\ &= \begin{pmatrix} v_1v_2 + v_2v_3 \\ 3p_3v_3^2 + v_3^3 \end{pmatrix}. \end{aligned}$$

Now, to estimate $\frac{|R(v)|}{|v|}$, recall that $|(x_1,x_2)| \leq |x_1| + |x_2|$ (to see this, square each side of the inequality). It then follows that

$$\frac{|R(v)|}{|v|} \leq \frac{|v_1v_2 + v_2v_3|}{|v|} + \frac{|3p_3v_3^2 + v_3^3|}{|v|}.$$

Next, recall that $|v| \geq |v_j|$ for each $j = 1,2,3$. We will apply this to the above estimate with $j = 2$ in the first term and $j = 3$ in the second term:

$$\frac{|R(v)|}{|v|} \leq \frac{v_1v_2 + v_2v_3|}{|v_2|} + \frac{|3p_3v_3^2 + v_3^3|}{|v_3|} = |v_1 + v_3| + |3p_3v_3 + v_3^2|.$$

This tends to zero as $v \to 0$ since each of the components of v tend to zero. Thus $R(v)$ is sublinear and $A = (Df)_p$ as claimed. □

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4. We compute

$$\frac{\partial f(0)}{\partial x_1} = \lim_{t\to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{0}{t^2+0} - 0}{t} = 0,$$

and similarly $\frac{\partial f(0)}{\partial x_2} = 0$.

Now, supposed towards a contradiction that $(Df)_0$ exists. Then by a theorem from class:

$$(Df)_0 = \begin{bmatrix} \frac{\partial f(0)}{\partial x_1} & \frac{\partial f(0)}{\partial x_2} \end{bmatrix} = [0 \;\; 0].$$

Consequently, the Taylor remainder for $v = (v_1, v_2) \in \mathbb{R}^2$ is

$$R(v) = f(0+v) - f(0) - [0 \;\; 0] \cdot v = f(v) = \frac{v_1 v_2}{v_1^2 + v_2^2},$$

and so

$$\frac{|R(v)|}{|v|} = \frac{|v_1 v_2|}{v_1^2 + v_2^2} \cdot \frac{1}{\sqrt{v_1^2 + v_2^2}} = \frac{|v_1 v_2|}{(v_1^2 + v_2^2)^{\frac{3}{2}}}.$$

This is not sublinear. Indeed, for $v = (t,t)$, which tends to the origin as $t \to 0$, we have

$$\frac{|R(v)|}{|v|} = \frac{t^2}{(2t^2)^{\frac{3}{2}}} = \frac{1}{(2)^{\frac{3}{2}}|t|},$$

which does not tend to zero as t does, a contradiction. □

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