Exercises:

1. Let $f\colon \mathbb{R}^d \to \overline{\mathbb{R}}$. Show that f is measurable if and only if $f^{-1}(\{\infty\}) \in \mathcal{M}(\mathbb{R}^d)$, $f^{-1}(\{-\infty\}) \in \mathcal{M}(\mathbb{R}^d)$, and $f^{-1}(U) \in \mathcal{M}(\mathbb{R}^d)$ for every open $U \subset \mathbb{R}$.

2. Let $f\colon \mathbb{R}^d \to \overline{R}$ be measurable. Show that the collection

$$\mathcal{A} := \{S \subset \overline{\mathbb{R}} \colon f^{-1}(S) \in \mathcal{M}(\mathbb{R}^d)\},$$

is a σ-algebra:

   (i) $\emptyset \in \mathcal{A}$.

   (ii) If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

   (iii) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, then $\displaystyle\bigcup_{n=1}^\infty E_n \in \mathcal{A}$.

   Moreover, show that $\mathcal{A}$ contains every open subset, closed subset, G_δ subset, and F_σ subset of $\overline{\mathbb{R}}$.

3. Let $B \subset \mathbb{R}^d$ be a box, and suppose $f\colon \mathbb{R}^d \to [0,\infty)$ is Riemann integrable over B. Show that the Riemann integral of f over B is equal to $\int_B f\ dm$, the Lebesgue integral of f over B.

4. For $f \in L^1(m)$, show $m(\{x\colon |f(x)| = \infty\}) = 0$.

5. (a) Suppose $f \in L^1(\mathbb{R}, m)$ is uniformly continuous. Show that $\displaystyle\lim_{|x|\to\infty} f(x) = 0$.

   (b) Find a positive, continuous $f \in L^1(\mathbb{R}, m)$ such that $\displaystyle\limsup_{x\to\infty} f(x) = \infty$.

6. Let $\delta = (\delta_1, \ldots, \delta_n) \in (0,\infty)^n$. For $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, define

$$f^\delta(x_1, \ldots, x_n) = f(\delta_1 x_1, \ldots, \delta_n x_n).$$

   If $f \in L^1(\mathbb{R}^n, m)$, show that $f^\delta \in L^1(\mathbb{R}^n, m)$ with $\int f^\delta\ dm = \delta_1 \cdots \delta_n \int f\ dm$.

Solutions:

1. $(\Longrightarrow)$: If f is measurable, then for every open $U \subset \overline{\mathbb{R}}$, $f^{-1}(U) \in \mathcal{M}(\mathbb{R}^d)$. Since $\{\pm\infty\}$ are clopen sets, it follows that $f^{-1}(\{\pm\infty\}) \in \mathcal{M}(\mathbb{R}^d)$. Also, if $U \subset \mathbb{R}$ is open, then, thinking of $\mathbb{R}$ as a subset of $\overline{\mathbb{R}}$, U is open relative to $\mathbb{R}$. Hence there exists open $U' \subset \overline{\mathbb{R}}$ such that $U = U' \cap \mathbb{R}$. But since $\mathbb{R}$ is open (it is the complement of the closed set $\{-\infty, \infty\}$), this implies U is open in $\overline{\mathbb{R}}$. Hence $f^{-1}(U) \in \mathcal{M}(\mathbb{R}^d)$ by measurability of f.

   $(\Longleftarrow)$: Let $U \subset \overline{\mathbb{R}}$ be open. Define $U' := U \setminus \{-\infty, \infty\}$, which is open as the finite intersection of open sets U, $\{\infty\}^c$, and $\{-\infty\}^c$. Moreover, $U' \subset \mathbb{R}$ and so $f^{-1}(U') \in \mathcal{M}(\mathbb{R}^d)$ by assumption. Then $f^{-1}(U)$ is $f^{-1}(U')$ possibly unioned with $f^{-1}(\{-\infty\})$ and/or $f^{-1}(\{\infty\})$, depending on whether U contained $\pm\infty$. Since each of these sets is measurable, so is their finite union. Hence $f^{-1}(U) \in \mathcal{M}(\mathbb{R}^d)$, and so f is measurable. $\square$

2. By definition measurability, $\mathcal{A}$ contains all open subsets of $\overline{\mathbb{R}}$. In particular, $\emptyset \in \mathcal{A}$, and so (i) holds. Property (ii) follows from $f^{-1}(E^c) = f^{-1}(E)^c$, and property (iii) follows from

$$f^{-1}\left(\bigcup_{n=1}^\infty E_n\right) = \bigcup_{n=1}^\infty f^{-1}(E_n).$$

   Now, using (ii), we see that $\mathcal{A}$ also contains all closed sets. Since an F_σ set is a countable union of closed sets, (iii) then implies $\mathcal{A}$ contains all F_σ sets. Finally, using (ii), $\mathcal{A}$ contains all G_δ sets since these are just complements of F_σ sets. $\square$

3. Let G be a grid on B, and let $\{B_i : i \in I\}$ be the induced subboxes. Observe that

$$U(f,G) = \sum_{i\in I} (sup_{x\in B_i} f(x))\, |B_i| = \int \sum_{i\in I} \left(\sup_{x\in B_i} f(x) \right) \chi_{B_i}\ dm.$$

So for the simple function $\psi_U := \sum_{i\in I} \left(\sup_{x\in B_i} f(x)\right) \chi_{B_i}$, we have $\int \psi_U\ dm = U(f,G)$. Similarly, for the simple function $\psi_L := \sum_{i\in I} \left(\inf_{x\in B_i} f(x)\right) \chi_{B_i}$, we have $\int \psi_L\ dm = L(f,G)$. Moreover, $\psi_L \leq f \leq \psi_U$. Thus for any simple function $0 \leq \phi \leq f$, we have by monotonicity of the integral

$$\int \phi\ dm \leq \int \psi_U\ dm = U(f,G).$$

Consequently,

$$L(f,G) = \int \psi_L\ dm \leq \underbrace{\sup\left\{ \int \phi\ dm \colon 0 \leq \phi \leq f,\ \phi \text{ simple} \right\}}_{=\int f\ dm} \leq U(f,G).$$

By Riemann integrability of f, taking the letting mesh(G) tends to zero yields the desired equality. □

4. Suppose, towards a contradiction, that $m(\{x \colon |f(x)| = \infty\}) > 0$. Since $\{x \colon |f(x)| = \infty\}$ is the (disjoint) union of $\{x \colon f(x) = -\infty\}$ and $\{x \colon f(x) = \infty\}$, one of these subsets must (by subaddivity) have positive measure. By replacing f with $-f$, we may assume $m(\{x \colon f(x) = \infty\}) > 0$. Denote $E := \{x \colon f(x) = \infty\}$. Then for every $R > 0$, we have $R\chi_E \leq |f|$. So by monotonicity of the integral we have

$$\int |f|\ dm \geq \int R\chi_E\ dm = Rm(E).$$

Since $m(E)$, letting $R \to \infty$ forces $Rm(E) \to \infty$, contradicting $\int |f|\ dm < \infty$. □

5. (a) Recall that $\lim_{|x|\to\infty} f(x) = 0$ is equivalent to:

$$\forall \epsilon > 0,\ \exists R > 0,\ \text{such that } \forall x \in \mathbb{R} \text{ with } |x| \geq R,\ |f(x)| < \epsilon.$$

Assume, towards a contradiction, that this is not the case. Then there exists $\epsilon > 0$ such that for all $R > 0$ there exists $x \in \mathbb{R}$ with $|x| \geq R$ and $|f(x)| \geq \epsilon$. By uniform continuity of f, there exists $\delta > 0$ such that whenever $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{2}$. In particular, if $|x - y| < \delta$, then we have

$$|f(y)| = |f(x) + f(y) - f(x)| \geq |f(x)| - |f(y) - f(x)| > |f(x)| - \frac{\epsilon}{2}.$$

So if $|f(x)| \geq \epsilon$, then $|f(y)| > \frac{\epsilon}{2}$. For $R_1 = 1$, let $x_1 \in \mathbb{R}$ be such that $|x_1| \geq R_1$ and $|f(x_1)| \geq \epsilon$. For $R_2 = |x_1| + 2\delta$, let $x_2 \in \mathbb{R}$ be such that $|x_2| \geq R_2$ and $|f(x_2)| \geq \epsilon$. Observe that $(x_2 - \delta, x_2 + \delta)$ and $(x_1 - \delta, x_1 + \delta)$ are disjoint, and by the above argument we have $|f(y)| > \frac{\epsilon}{2}$ for any y in either interval. Iterating this argument, we produce a sequence $(x_n)_{n\in\mathbb{N}}$ such that the intervals $(x_n - \delta, x_n + \delta)$ are pairwise disjoint and $|f(y)| > \frac{\epsilon}{2}$ for each y in these intervals. Consequently, if we define

$$g_n := \sum_{i=1}^{n} \frac{\epsilon}{2} \chi_{(x_i - \delta, x_i + \delta)},$$

then $|f| \geq g_n$ for every $n \in \mathbb{N}$. But then

$$\int |f|\ dm \geq \int g_n\ dm = \sum_{i=1}^{n} \frac{\epsilon}{2} 2\delta = n\epsilon\delta,$$

and letting $n \to \infty$ contradicts $\int |f|\ dm < \infty$. □

(b) Define a continuous function by

$$f(x) := \begin{cases} 2^{3n+1}(x-n) & \text{if } n \leq x \leq n + 2^{-2n-1} \text{ for some } n \in \mathbb{N} \\ -2^{3n+1}(x - n - 2^{-2n}) & \text{if } n + 2^{-2n-1} < x \leq n + 2^{-2n} \text{ for some } n \in \mathbb{N}. \\ 0 & \text{otherwise} \end{cases}.$$

Then the graph of is x-axis except for a series of disjoint triangles, one contained in each interval $[n, n+1]$ for $n \in \mathbb{N}$ with base length 2^{-2n} and height 2^n. Hence

$$\int f \ dm = \sum_{n=1}^{\infty} \frac{1}{2} 2^{-2n} 2^n = \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2},$$

so that $f \in L^1(\mathbb{R}, m)$. However,

$$\limsup_{x \to \infty} f(x) \geq \limsup_{n \to \infty} f(n + 2^{-2n-1}) = \limsup_{n \to \infty} 2^n = \infty.$$

$\square$

6. By noting that $(f^\delta)_\pm = (f_\pm)^\delta$, it suffices to assume f is valued in $[0, \infty]$.

We first verify that f^δ is measurable. Let $U \subset \mathbb{R}$ be open. Denoting $\frac{1}{\delta} := (\frac{1}{\delta_1}, \ldots, \frac{1}{\delta_n})$, it is easy to see that

$$(f^\delta)^{-1}(U) = \frac{1}{\delta} f^{-1}(U),$$

where we are using the notation from Exercise 2 of Homework 2. Since f is a measurable function, $E := f^{-1}(U) \in \mathcal{M}(\mathbb{R}^n)$. In particular, E satisfies the Carathéodory condition: for all $X \subset \mathbb{R}^n$

$$m^*(X) = m^*(X \cap E) + m^*(X \cap E^c).$$

By Exercise 2 from Homework 2 we then have for all $X \subset \mathbb{R}^n$

$$\begin{aligned} m^*(\frac{1}{\delta} X) = \delta_1 \cdots \delta_n m^*(X) &= \delta_1 \cdots \delta_n m^*(X \cap E) + \delta_1 \cdots \delta_n m^*(X \cap E^c) \\ &= m^*(\frac{1}{\delta}(X \cap E)) + m^*(\frac{1}{\delta}(X \cap E^c)) \\ &= m^*((\frac{1}{\delta} X) \cap (\frac{1}{\delta} E)) + m^*((\frac{1}{\delta}) \cap (\frac{1}{\delta} E)^c). \end{aligned}$$

As X ranges over all subsets $\mathbb{R}^n$, so does $\frac{1}{\delta} X$, and so the above shows that $\frac{1}{\delta} E = (f^\delta)^{-1}(U)$ is measurable. Hence f^δ is a measurable function.

Now, observe that for any $E \subset \mathcal{M}(\mathbb{R}^n)$, $\chi_E^\delta = \chi_{\frac{1}{\delta} E}$. Consider a simple function $\phi = \sum_j a_j \chi_{E_j}$ satisfying $0 \leq \phi \leq f$. Then clearly $\phi^\delta \leq f^\delta$ and so by Exercise 2 on Homework 10

$$\begin{aligned} \int f^\delta \ dm \geq \int \phi^\delta \ dm = \int \sum_j a_j \chi_{E_j}^\delta \ dm &= \int \sum_j a_j \chi_{\frac{1}{\delta} E_j} \ dm \\ = \sum_j a_j m(\frac{1}{\delta} E_j) = \sum_j a_j \delta_1 \cdots \delta_n m(E_j) &= \delta_1 \cdots \delta_n \int \phi \ dm. \end{aligned}$$

Taking the supremum over all simple functions $0 \leq \phi \leq f$ yields $\delta_1 \cdots \delta_n \int f \ dm \leq \int f^\delta dm$. But then $\int f \ dm = \int (f^\delta)^{\frac{1}{\delta}} \ dm \geq \frac{1}{\delta_1} \cdots \frac{1}{\delta_n} \int f^\delta \ dm$, yielding the desired equality. $\square$