Exercises:

1. Let $A \subset \mathbb{R}$ satisfy $m^*(A) > 0$. Show that for every $\alpha \in (0,1)$ there exists an open interval I such that

$$m^*(A \cap I) \geq m^*(I)$$

[**Hint:** Use the definition of the outer measure to find an open set $U \supset A$ such that $m^*(A) \geq \alpha m^*(U)$. Then use the fact that every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.]

2. Let $E \in \mathcal{M}(\mathbb{R})$ with $m(E) > 0$. Consider the **difference set**

$$D(E) := \{x - y \colon x, y \in E\}.$$

Show that D contains an open interval centered at the origin.

[**Hint:** Invoke the previous exercise for $\alpha \in (\frac{1}{2}, 1)$. Also, for any set S, if $D(S)$ does not contain an open interval centered at the origin, then for all $\delta > 0$ the set $(-\delta, \delta) \setminus D(S)$ is non-empty. Think about the relation between S and its translation by an element of $(-\delta, \delta) \setminus D(S)$.]

3. Let $A \subset [0,1]$ be the set of numbers without the digit 4 appearing in their decimal expansion. Show that A is Lebesgue measurable and compute $m(A)$.

[**Hint:** consider for each $n \in \mathbb{N}$ the set of numbers without the digit 4 appearing in the first n digits of the decimal expansion.]

4. (a) Show that every closed set in $\mathbb{R}^d$ is both G_δ and F_σ.
   (b) Show that every open set in $\mathbb{R}^d$ is both G_δ and F_σ.
   (c) Show every Riemann measurable set in $\mathbb{R}^d$ is Lebesgue measurable.

5. [**The Borel–Cantelli Lemma**] Suppose $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d)$ satisfies

$$\sum_{n=1}^{\infty} m(E_n) < \infty.$$

Consider $E := \{x \in \mathbb{R}^d \colon x \in E_n \text{ for infinitely many } n \in \mathbb{N}\}$.

   (a) Show that

$$E = \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} E_n.$$

   (b) Show that $E \in \mathcal{M}(\mathbb{R}^d)$ with $m(E) = 0$.
   (c) Show that

$$\chi_E(x) = \limsup_{n\to\infty} \chi_{E_n}(x) \qquad \forall x \in \mathbb{R}^d$$

[**Note:** for this reason, E is typically denoted $\limsup\limits_{n\to\infty} E_n$.]

Solutions:

1. Fix $\alpha \in (0,1)$. First note that it suffices to assume $m^*(A) < \infty$. Indeed, by subaddivity we have

$$0 < m^*(A) \leq \sum_{n=1}^{\infty} m^*(A \cap [-n, n]),$$

so there exists $n \in \mathbb{N}$ with $m^*(A \cap [-n,n]) > 0$, and note that $m^*(A \cap [-n,n]) \leq m^*([-n,n]) = 2n$. Now, if we find an interval I such that $m^*(A \cap [-n,n] \cap I) \geq \alpha m^*(I)$, then by monotonicity we have $m^*(A \cap I) \geq \alpha m^*(I)$. Hence, by replacing A with $A \cap [-n,n]$ if necessary, we may assume $m^*(A) < \infty$.

[**Alternatively:** if $m^*(A) = \infty$ then we can simply take $I = \mathbb{R}$.]

Now, for $\epsilon = (\frac{1}{\alpha} - 1)m^*(A)$, let $\{B_k\}_{k\in\mathbb{N}}$ be a countable covering of A by open boxes (intervals in this case) satisfying

$$\sum_{k=1}^{\infty} |B_k| \leq m^*(A) + \epsilon.$$

Set $U = \bigcup_{k=1}^{\infty} B_k$, the U is open as the union of open sets. By countable subadditivity and our computation of the outer measure of boxes from class we have

$$m^*(U) \leq \sum_{k=1}^{\infty} m^*(B_k) = \sum_{k=1}^{\infty} |B_k| \leq m^*(A) + \epsilon = m^*(A) + (\frac{1}{\alpha} - 1)m^*(A) = \frac{1}{\alpha} m^*(A).$$

Hence $m^*(A) \geq \alpha m^*(U)$. Now, as an open set we can express U as the a countable union of disjoint open intervals: $U = \bigcup_{n=1}^{\infty} I_n$. Note that since open sets are measurable we have by countable additivity

$$\sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} m(I_n) = m(U) = m^*(U).$$

We claim that there exists $n \in \mathbb{N}$ such that $m^*(A \cap I_n) \geq \alpha m^*(I_n)$ (in which case the proof is complete). Indeed, if any $m^*(A \cap I_n)$ is infinite, then we are done. Otherwise, assume $m^*(A \cap I_n)$ is finite and that $m^*(A \cap I_n) < \alpha m^*(I_n)$ for all $n \in \mathbb{N}$. Consequently

$$m^*(A) = m^*(A \cap U) = m^*\left(\bigcup_{n=1}^{\infty} A \cap I_n\right) \leq \sum_{n=1}^{\infty} m^*(A \cap I_n) < \sum_{n=1}^{\infty} \alpha m^*(I_n) = \alpha m^*(U),$$

contradicting $m^*(A) \geq \alpha m^*(U)$. □

2. By the same reduction at the beginning of the previous solution, we may assume $m(E) < \infty$. Fix any $\alpha \in (\frac{1}{2}, 1)$, and note that $0 < 2\alpha - 1 < 1$. By the previous exercise, there exists an open interval I such that $m(E \cap I) \geq \alpha m(I)$ (in particular, $m(I) < \infty$). Set $E_0 := E \cap I$, and note that $D(E_0) \subset D(E)$ so that it suffices to show $D(E_0)$ contains an open interval centered at the origin. If this is not the case, then for every $\delta > 0$ there exists $a \in (-\delta, \delta) \setminus D(E_0)$. We claim that for such a, E_0 and $E_0 + a$ are disjoint. Indeed, if not then there exists $x \in E_0$ such that $x = y + a$ for some $y \in E_0$. But then $a = x - y \in D(E_0)$, a contradiction. Now, fix such an a for $\delta = (2\alpha - 1)m(I)$, and note that $0 < \delta < m(I)$ by our choice of α. Also note that $E_0 \cup (E_0 + a) \subset I \cup (I + a)$. Since $|a|$ is smaller than $m(I)$ (the length of I), $I \cup (I + a)$ is an open interval having length $m(I) + |a|$. Thus, by our choice of I, translation invariance, countable additivity, monotonicity, and our choice of δ (in that order) we have

$$\begin{aligned} 2\alpha m(I) &\leq 2m(E_0) = m(E_0) + m(E_0 + a) = m(E_0 \cup (E_0 + a)) \\ &\leq m(I \cup (I + a)) = m(I) + |a| < m(I) + (2\alpha - 1)m(I) = 2\alpha m(I), \end{aligned}$$

(note that we are using $m(I) < \infty$ here) a contradiction. Thus $D(E_0)$ must contain an open interval centered at the origin, and consequently so must $D(E)$. □

3. For $n \in \mathbb{N}$ and digits $a_1, \ldots, a_n \in \{0, 1, 2, \ldots, 9\}$, observe that the half-open interval $I(a_1, \ldots, a_n) := [0.a_1a_2 \cdots a_n, 0.a_1a_2 \cdots (a_n + 1))$ is the set of numbers in $[0, 1]$ whose decimal expansion begins with $a_1a_2 \cdots a_n$. Furthermore, as a half-open interval it is measurable with

$$m(I(a_1, \ldots, a_n)) = \frac{1}{10^n}.$$

Also note that for any distinct choice of digits $b_1, \ldots, b_n$ we have $I(a_1, \ldots, a_n) \cap I(b_1, \ldots, b_n) = \emptyset$. Now, define $A_n \subset [0, 1]$ to be the set numbers without the digit 4 appearing in the first n-digits of its decimal expansion. Then

$$A_n = \bigcup_{a_1,\ldots,a_n \neq 4} I(a_1, \ldots, a_n)$$

and so A_n is measurable as a finite union of measurable sets. Moreover, there are 9^n choices for $a_1, \ldots, a_n$ which avoid the digit 4. Therefore

$$m(A_n) = \sum_{a_1,\ldots,a_n \neq 4} m(I(a_1, \ldots, a_n)) = \sum_{a_1,\ldots,a_n \neq 4} \frac{1}{10^n} = \frac{9^n}{10^n}.$$

Finally, we note that $A_1 \supset A_2 \supset \cdots$ and the intersection of these sets is precisely A. Hence A is measurable as the countable intersection of measurable sets, and by continuity from above (note that $m(A_1) < \infty$) we have

$$m(A) = m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} \frac{9^n}{10^n} = 0.$$

□

4. (a) Let $V \subset \mathbb{R}^d$ be closed. Then V is the countable union of closed sets (namely the union over the collection containing just V), and so V is an F_σ set. To see that V is a G_δ set, defined for each $n \in \mathbb{N}$,

$$U_n := \bigcup_{x\in V} B(x, \frac{1}{n}).$$

Then U_n is an open set containing V. Suppose $y \in U_n$ for all $n \in \mathbb{N}$. Fix $r > 0$ and let $n \in \mathbb{N}$ be such that $\frac{1}{n} < r$. Since $y \in U_n$, there exists $x \in V$ with $y \in B(x, \frac{1}{n})$, but this also implies $x \in B(y, \frac{1}{n}) \subset B(y, r)$. Thus $B(y, r) \cap V \neq \emptyset$. Since $r > 0$ was arbitrary, this shows $y \in \overline{V} = V$. Hence we have shown

$$\bigcap_{n=1}^{\infty} U_n \subset V,$$

while the other inclusion is immediate. Thus V is the countable intersection of open sets and therefore is a G_δ set. □

(b) Let $U \subset \mathbb{R}^d$ be open. Then U^c is closed and so by part (a) is both an G_δ set and an F_σ set. Thus its complement, U is an F_σ set and an G_δ set. □

(c) Let $S \subset \mathbb{R}^d$ be Riemann measurable. Then its interior $F := S^\circ$ is an F_σ set by part (b), and its closure $G := \overline{S}$ is a G_δ set by part (a). Since S is Riemann measurable, its boundary $\partial S = \overline{S} \setminus S^\circ$ is a zero set. Thus we have found an F_σ set F and an G_δ set G satisfying $F \subset S \subset G$ and $m(G \setminus F) = m(\partial S) = 0$. By a result from lecture this implies S is Lebesgue measurable. □

5. (a) Let $x \in E$. Since $x \in E_n$ for infinitely many $n \in \mathbb{N}$, for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $x \in E_n$. This implies that for all $N \in \mathbb{N}$, $x \in \bigcup_{n\geq N} E_n$. Hence $x \in \displaystyle\bigcap_{N=1}^{\infty} \bigcup_{n\geq N} E_n$. Conversely, suppose $x \in \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} E_n$. Consequently, for all $N \geq N$ we have $x \in \bigcup_{n\geq N} E_n$. For $N = 1$, let $n_1 \geq 1$ be such that $x \in E_{n_1}$. For $N = n_1 + 1$, let $n_2 \geq n_1 + 1$ be such that $x \in E_{n_2}$. Continuiing in this way, we can demonstrate that $x \in E_n$ for infinitely many $n \in \mathbb{N}$. Thus the claimed equality holds. □

(b) For each $N \in \mathbb{N}$, $\bigcup_{n\geq N} E_n \in \mathcal{M}$ as the countable union of measurable sets. Then $E \in \mathcal{M}$ as the countable intersection of measurable sets. Now, for each $N \in \mathbb{N}$ define $F_n := \bigcup_{n\geq N} E_n$. Then $F_1 \supset F_2 \supset \cdots$ and by countable subadditivity we have

$$m(F_1) \leq \sum_{n=1}^{\infty} m(E_n),$$

which is finite by assumption. Thus, by continuity from above we have

$$m(E) = m\left(\bigcap_{N=1}^{\infty} F_N\right) = \lim_{N\to\infty} m(F_N) \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} m(E_n).$$

This last limit is zero since the tail of a convergent series tends to zero. Thus $m(E) = 0$. □

(c) Let $x \in \mathbb{R}^d$. For $N \in \mathbb{N}$, we note that $\sup\{\chi_{E_n}(x) \colon n \geq N\}$ is 1 if $x \in \bigcup_{n\geq N} E_n$ and is zero otherwise. Thus if $x \in E$, then by part (b) we have

$$\lim_{N\to\infty} \sup\{\chi_{E_n}(x) \colon n \geq N\} = 1 = \chi_E(x).$$

If $x \notin E$, then there exists $N \in \mathbb{N}$ such that $x \notin \bigcup_{n\geq N} E_n$. Note that this implies $x \notin \bigcup_{n\geq M} E_n$ for any $M \geq N$. Hence

$$\lim_{N\to\infty} \sup\{\chi_{E_n}(x) \colon n \geq N\} = 0 = \chi_E(x).$$

□