Exercises:

1. Let U and V be open subsets.

   (a) For $T\colon U \to V$ be a smooth map, show that the pullback
   $$T^*\colon \Omega^k(V) \to \Omega^k(U)$$
   satisfies $T^*(Z^k(V)) \subset Z^k(U)$ and $T^*(B^k(V)) \subset T^*(B^k(U))$.

   (b) Prove that if U and V are diffeomorphic, then $H^k(U) \cong H^k(V)$ as vector spaces.

2. Let $\delta = (\delta_1, \ldots, \delta_n) \in (0,\infty)^n$. For $A \subset \mathbb{R}^n$, define
   $$\delta A := \{(\delta_1 x_1, \ldots, \delta_n x_n)\colon (x_1, \ldots, x_n) \in A\}.$$
   Show that $m^*(\delta A) = \delta_1 \cdots \delta_n m^*(A)$.

3. For $A \subset \mathbb{R}^n$, the **Jordan content** of A is the quantity
   $$J^*(A) := \inf \left\{ \sum_{k=1}^N |B_k| \colon N < \infty,\ A \subset \bigcup_{k=1}^N B_k,\ B_k \text{ open boxes} \right\}.$$
   That is, in contrast with the outer measure, here the infimum is taken over **finite** coverings of A by open boxes.

   (a) Show that $m^*(A) \leq J^*(A)$ for all $A \subset \mathbb{R}^n$.

   (b) Show that for any subset $A \subset \mathbb{R}^n$, $J^*(A) = J^*(\overline{A})$.

   (c) Find a subset $A \subset \mathbb{R}$ such that $m^*(A) < J^*(A)$.

4. Let $S \subset \mathbb{R}^2$ be Riemann measurable. Show that $|S| = m^*(S) = J^*(S)$.
   [**Hint:** first show you can replace S by $\overline{S}$, then take advantage of compactness.]

5. [**Not Collected**] Consider the triangle
   $$T := \{(x,y) \in \mathbb{R}^2 \colon 0 \leq y \leq x \leq 1\}.$$
   Convince yourself that computing $m^*(T) = \frac{1}{2}$ directly from the definition of the outer measure is hard.
   [**Note:** the easy way to show this is by using Exercise 4 and Fubini's theorem.]

Solutions:

1. (a) Suppose $\omega \in Z^k(V)$, so that by definition $d\omega = 0$. Since pullbacks commute with the exterior derivative, we have
   $$d(T^*\omega) = T^*(d\omega) = 0.$$
   Thus $T^*\omega \in Z^k(U)$. If $\omega \in B^k(V)$, then there exists $\alpha \in \Omega^k(V)$ with $d\alpha = \omega$. We then have
   $$T^*\omega = T^*(d\alpha) = d(T^*\alpha).$$
   Hence $T^*\omega \in B^k(U)$. □

   (b) Let $T\colon U \to V$ be a diffeomorphism. By part (a),
   $$T^*(Z^k(V)) \subset Z^k(U).$$
   Moreover, for any $\beta \in Z^k(U)$ we have (again by part (a)), that $(T^{-1})^*(\beta) \in Z^k(V)$ and hence
   $$T^*((T^{-1})^*(\beta)) = (T \circ T^{-1})^*(\beta) = id^*\beta = \beta.$$

Thus $T^*(Z^k(V)) = Z^k(U)$. Similarly $T^*(B^k(V)) = B^k(U)$ and $(T^{-1})^*(B^k(U)) = B^k(V)$. Now, consider the (potentially ill-defined) map

$$\Psi : H^k(V) \ni \omega + B^k(V) \mapsto T^*(\omega) + B^k(U) \in H^k(U),$$

which we note is linear since T^* is linear. We will show that Ψ is a vector space isomorphism by checking it is (i) well-defined, (ii) injective, and (iii) surjective.

(i) Suppose $\alpha, \beta \in Z^k(V)$ satisfy $\alpha + B^k(V) = \beta + B^k(V)$. We need to show they have the same image under Ψ. The equality of these cosets implies $\alpha - \beta \in B^k(V)$, and so $T^*(\alpha) - T^*(\beta) = T^*(\alpha - \beta) \in B^k(U)$ by the first part of our proof. But then

$$\Psi(\alpha + B^k(V)) = T^*(\alpha) + B^k(U) = T^*(\beta) + B^k(U) = \Psi(\beta + B^k(V))$$

Thus Ψ is well-defined.

(ii) If $\Psi(\alpha + B^k(V)) = \Psi(\beta + B^k(V))$ for $\alpha, \beta \in Z^k(V)$, then $T^*(\alpha) + B^k(U) = T^*(\beta) + B^k(U)$ so that $T^*(\alpha - \beta) = T^*(\alpha) - T^*(\beta) \in B^k(U)$. But then the first part of our proof implies $\alpha - \beta \in B^k(V)$. Thus $\alpha + B^k(V) = \beta + B^k(V)$, and so Ψ is injective.

(iii) This follows immediately from $T^*(Z^k(V)) = Z^k(U)$.

Thus $H^k(V) \cong H^k(U)$ via the vector space isomorphism Ψ. □

2. First observe that for any open box $B = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$, we have

$$\delta B = (\delta_1 a_1, \delta_1 b_1) \times \cdots \times (\delta_n a_n, \delta_n b_n).$$

Thus $|\delta B| = \delta_1 \cdots \delta_n |B|$. Now, if $\{B_k\}_{k\in\mathbb{N}}$ is a countable collection of open boxes that covers A, then $\{\delta B_k\}_{k\in\mathbb{N}}$ is a countable collection of open boxes. Furthermore, it covers δA. Indeed, every element of δA is of the form $(\delta_1 x_1, \ldots, \delta_n x_n)$ for some $(x_1, \ldots, x_n) \in A$. There exists some k such that $(x_1, \ldots, x_n) \in B_k$ and hence $(\delta_1 x_1, \ldots, \delta_n x_n) \in \delta B_k$. This implies

$$m^*(\delta A) \leq \sum_{k=1}^{\infty} |\delta B_k| = \delta_1 \cdots \delta_n \sum_{k=1}^{\infty} |B_k|.$$

Since this holds for all countable coverings of A by open boxes, we obtain $m^*(\delta A) \leq m^*(A)$. The reverse inequality holds by considering $A' := \delta A$, $\delta' = (\delta_1^{-1}, \ldots, \delta_n^{-1})$, and $\delta' A'$. □

3. (a) Since every finite covering of A by open boxes is in particular a countable covering, we immediately have $m^*(A) \leq J^*(A)$. □

(b) Since $A \subset \overline{A}$, any covering of $\overline{A}$ by finitely many open boxes is a covering for A. Hence $J^*(A) \leq J^*(\overline{A})$. To see the reverse inequality let $\epsilon > 0$. Let $\{B_1, \ldots, B_N\}$ be a finite collection of open boxes covering A. Observe that

$$\bigcup_{k=1}^{N} \overline{B_k}$$

is a closed set (as a finite union of closed sets) containing A. Hence $\overline{A}$ is contained in the above union. For each $k = 1, \ldots, N$, if

$$B_k = (a_1, b_1) \times \cdots \otimes (a_n, b_n),$$

define B_k' to be the dilation of B_k about its center point $\left(\frac{a_1+b_1}{2}, \ldots, \frac{a_n+b_n}{2}\right)$ by a factor of $(1+\epsilon)$. That is,

$$B_k' := \left(\frac{a_1+b_1}{2} - (1+\epsilon)\frac{b_1-a_1}{2}, \frac{a_1+b_1}{2} + (1+\epsilon)\frac{b_1-a_1}{2}\right) \times \cdots$$
$$\cdots \times \left(\frac{a_n+b_n}{2} - (1+\epsilon)\frac{b_n-a_n}{2}, \frac{a_n+b_n}{2} + (1+\epsilon)\frac{b_n-a_n}{2}\right).$$

Then $|B_k'| = (1+\epsilon)^n|B_k|$. Also, $\overline{B_k} \subset B_k'$ so that

$$\bigcup_{k=1}^N B_k' \supset \bigcup_{k=1}^N \overline{B_k} \supset \overline{A}.$$

Hence

$$J^*(\overline{A}) \leq \sum_{k=1}^N |B_k'| = (1+\epsilon)\sum_{k=1}^N |B_k|.$$

Since $\{B_1, \ldots, B_N\}$ was an arbitrary finite covering of A by open boxes, we obtain $J^*(\overline{A}) \leq (1+\epsilon)J^*(A)$. Since $\epsilon > 0$ was arbitrary, we have $J^*(\overline{A}) \leq J^*(A)$, which establishes the claimed equality. □

(c) Let $A = \mathbb{Q} \cap [0,1]$. Then A is a countable set, hence a zero set: $m^*(A) = 0$. On the other hand, by part (b) we have

$$J^*(A) = J^*(\overline{A}) = J^*([0,1]).$$

One can argue directly from the definition of the Jordan content that $J^*([0,1]) = 1$, but here simply appeal to Exercise 4 and note $J^*([0,1]) = |[0,1]| = 1$. Thus $J^*(A) = 1 > 0 = m^*(A)$. □

4. We first note that we may assume S is closed by replacing S by $\overline{S}$ is necessary. Indeed, by part (b) of the previous exercise we have $J^*(S) = J^*(\overline{S})$. Also, since S is Riemann measurable, ∂S is a zero set. Hence by subadditivity and monotonicity of the outer measure we have

$$m^*(\overline{S}) = m^*(S \cup \partial S) \leq m^*(S) + m^*(\partial S) = m^*(S) \leq m^*(\overline{S}),$$

so that $m^*(S) = m^*(\overline{S})$. Finally, we have

$$\partial\overline{S} = \overline{S} \setminus (\overline{S})^\circ \subset \overline{S} \setminus S^\circ = \partial S,$$

so that $m^*(\partial\overline{S}) = 0$, which means $\overline{S}$ is Riemann measurable. Since $\chi_{\overline{S}}$ and χ_S agree everywhere except possibly on ∂S, a zero set, we have

$$|\overline{S}| = \int \chi_{\overline{S}} = \int \chi_S = |S|,$$

by Exercise 3 on Homework 6. Hence all concerned quantities are unchanged when we replace S by $\overline{S}$ and so we may assume S is closed.

We will prove the following series of inequalities:

$$|S| \leq m^*(S) \leq J^*(S) \leq |S|.$$

Let $\{B_k\}_{k\in\mathbb{N}}$ be an countable covering of S by open boxes. Since S is bounded (it is Riemann measurable) and closed, it is compact by the Heine–Borel theorem. Hence we can reduce this open covering to a finite one: $\{B_{k_1}, \ldots, B_{k_N}\}$. We then have by monotonicity of the Riemann integral that

$$|S| = \int \chi_S \leq \int \sum_{i=1}^N \chi_{B_{k_i}} = \sum_{i=1}^N |B_{k_i}| \leq \sum_{k=1}^\infty |B_k|.$$

Since this holds for any countable covering $\{B_k\}_{k\in\mathbb{N}}$, we have $|S| \leq m^*(S)$. Next, the inequality $m^*(S) \leq J * (S)$ follows from part (a) of the previous exercise. Finally, $\epsilon > 0$, let $R \subset \mathbb{R}^2$ be a rectangle containing S, and let G be a grid with mesh(G) small enough so that

$$U(\chi_S, G) \leq \int \chi_S + \epsilon = |S| + \epsilon.$$

Let $R_1, \ldots, R_N$ be the subrectangles of R from the grid G which intersect S. Then $S \subset \bigcup_{k=1}^N R_k$ and

$$U(\chi_S, G) = \sum_{k=1}^N 1 \cdot |R_k|.$$

To relate this to $J^*(S)$, we must replace these closed subrectangles by open rectangles. For each $k = 1, \ldots, N$, define B_k to be the interior of the dilation of R_k about its center point by a factor of $(1+\epsilon)$. Then each B_k is an open rectangle containing R_k with $|B_k| = (1+\epsilon)|R_k|$. Hence $\{B_1, \ldots, B_k\}$ is a finite covering of S by open rectangles and so

$$J^*(S) \leq \sum_{k=1}^N |B_k| = (1+\epsilon) \sum_{k=1}^N |R_k| \leq (1+\epsilon)(|S| + \epsilon).$$

Letting $\epsilon \to 0$ yields the final inequality and hence the desired equalities. □