Exercises:

1. Find your book from Math 110.
2. Compute (without proof) the following matrices:
   (a) $\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 4 & -6 \end{bmatrix}$.
   (b) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -6 & 7 \end{bmatrix}$.
   (c) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 4 \end{bmatrix}^{\mathrm{T}}$ (here 'T' denotes the transpose of the matrix).
   (d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -2 & 2 & 1 \end{bmatrix}^{-1}$.
3. Let V be the vector space of degree three polynomials with real coefficients. Let W be the vector space of degree two polynomials with real coefficients. Consider the linear operator $T\colon V \to W$ which sends a polynomial to its derivative. Determine (without proof) the matrix representation of T with respect to the (ordered) bases $\{1, x, x^2, x^3\}$ for V and $\{1, x, x^2\}$ for W.
4. Denote $O = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.
   (a) Prove that $\langle Ox, Oy\rangle = \langle x, y\rangle$ for any vectors $x, y \in \mathbb{R}^2$. [Note: such a matrix is called an **orthogonal matrix**, and its column vectors yield an orthonormal basis for $\mathbb{R}^2$.]
   (b) Remind yourself what "orthonormal basis" means.
   (c) Prove that $\|O\| = 1$. [Hint: use part (a).]
5. Let $n \in \mathbb{N}$ and let $D \in M_{n\times n}(\mathbb{R}^n)$ be a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n \in \mathbb{R}$. Prove that $\|D\| = \max\limits_{1\leq i\leq n} |d_i|$.

Solutions:

1. It's at my parent's house. □
2. (a) $\begin{bmatrix} 7 & 1 \\ 1 & -6 \end{bmatrix}$.
   (b) $\begin{bmatrix} -4 & 8 \\ -28 & 35 \end{bmatrix}$.
   (c) $\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$.
   (d) $\begin{bmatrix} 1 & -2 & -1 \\ 3/2 & -7/2 & -1 \\ -1 & 3 & 1 \end{bmatrix}$.
3. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

4. (a) Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we compute:

$$\begin{aligned}\langle Ox, Oy\rangle &= \left\langle \left(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, -\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right), \left(\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}, -\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}\right)\right\rangle \\ &= \left(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right)\left(\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}\right) + \left(-\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right)\left(-\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}\right) \\ &= \frac{x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2}{2} + \frac{x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2}{2} \\ &= x_1y_1 + x_2y_2 = \langle x, y\rangle .\end{aligned}$$

Alternatively, we use $\langle Ox, Oy\rangle = \left\langle O^{\mathrm{T}}Ox, y\right\rangle$ and note that

$$O^{\mathrm{T}}O = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

(b) A *basis* is a set of vectors that is both linearly independent and spanning. A basis is *orthonormal* if each of the vectors has norm one and the vectors are pairwise orthogonal.

(c) Let $x \in \mathbb{R}^2$ and recall that $|x| = \langle x, x\rangle^{\frac{1}{2}}$. Thus by part (a), for all $x \neq 0$ we have

$$\frac{|Ox|}{|x|} = \frac{\langle Ox, Ox\rangle^{\frac{1}{2}}}{|x|} = \frac{\langle x, x\rangle^{\frac{1}{2}}}{|x|} = \frac{|x|}{|x|} = 1.$$

Thus

$$\|O\| = \sup\left\{\frac{|Ox|}{|x|} : x \neq 0\right\} = 1.$$

□

5. Fix $x \in \mathbb{R}^n$, $x \neq 0$, and write $x = (x_1, x_2, \ldots, x_n)$. Then $Dx = (d_1x_1, d_2x_2, \ldots, d_nx_n)$. Denote $d := \max_{1\leq i\leq n} |d_i|$. Observe that

$$|Dx|^2 = (d_1x_1)^2 + \cdots + (d_nx_n)^2 \leq d^2(x_1^2 + \cdots + x_n^2) = d^2|x|^2,$$

Thus $\|D\| \leq d$. On the other hand, if $i \in \{1, \ldots, n\}$ is such that $|d_i| = d$, then letting e_i be the vector whose entries are all zero except for a 1 in the *ith* component, then

$$\frac{|De_i|}{|e_i|} = \frac{\sqrt{0^2 + \cdots + (d_i)^2 + \cdots + 0^2}}{\sqrt{0^2 + \cdots + 1^2 + \cdots + 0^2}} = \frac{|d_i|}{1} = d.$$

Thus $\|D\| \geq d$, which yields $\|D\| = d$ when combined with the previous inequality. □