Exercises:

1. **Higher Order Chain Rule:** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, and let $f: U \to \mathbb{R}^m$ and $g: V \to \mathbb{R}^\ell$ be r-times differentiable functions with $f(U) \subset V$. Prove that $h := g \circ f: U \to \mathbb{R}^\ell$ is r-times differentiable and that for $p \in U$

$$(D^r h)_p = \sum_{k=1}^r \sum_\mu (D^k g)_{f(p)} \circ (D^\mu f)_p,$$

where the second sum is over partitions μ of $\{1, \ldots, r\}$ into k disjoint, non-empty subsets. If $\mu = \{\mu_1, \mu_2, \ldots, \mu_k\}$ then $(D^\mu f)_p$ is defined by

$$(D^\mu f)_p(v_1, \ldots, v_r) = \Big((D^{|\mu_1|} f)_p(v_{\mu_1}), \ldots, (D^{|\mu_k|} f)_p(v_{\mu_k})\Big), \qquad v_1, \ldots, v_r \in \mathbb{R}^n,$$

where if $\mu_j = \{i_1 < i_2 < \cdots < i_d\}$ then $|\mu_j| = d$ and $v_{\mu_j} = (v_{i_1}, \ldots, v_{i_d})$.

2. **Higher Order Product Rule:** Let $U \subset \mathbb{R}^n$ be an open set, and let $f, g: U \to \mathbb{R}^m$ be r-times differentiable functions. For $v, w \in \mathbb{R}^m$ let $\langle v, w\rangle$ denote their scalar product:

$$\langle v, w\rangle := v_1 w_1 + \cdots + v_m w_m.$$

Prove that $h := \langle f, g\rangle : U \to \mathbb{R}$ is r-times differentiable and that for $p \in U$ and $v_1, \ldots, v_r \in \mathbb{R}^n$

$$(D^r h)_p(v_1, \ldots, v_n) = \sum_{k=0}^r \sum_{|\mu|=k} \left\langle (D^k f)_p(v_\mu), (D^{r-k} g)_p(v_{\mu^c})\right\rangle,$$

where the second sum is over subsets $\mu \subset \{1, 2, \ldots, r\}$ of size k and v_μ, v_{μ^c} are as in the previous exercise.

3. **Continuous versus Smooth Paths:**
   (a) Construct a continuous map $f: [0,1] \to \mathbb{R}^2$ whose image is **not** a zero set using the following steps:
      i. Show that the subset $S \subset [0,1]$ consisting of all numbers having decimal expansions of the form
      $$0.a_1 b_1 0 a_2 b_2 0 a_3 b_3 0 \ldots, \qquad a_i, b_i \in \{0, 1, 2, \ldots, 9\}$$
      is closed.
      ii. Show that the functions $\alpha, \beta: S \to [0,1]$ defined by
      $$\begin{aligned} \alpha&: \ 0.a_1 b_1 0 a_2 b_2 0 a_3 b_3 0 \ldots \mapsto 0.a_1 a_2 a_3 \ldots \\ \beta&: \ 0.a_1 b_1 0 a_2 b_2 0 a_3 b_3 0 \ldots \mapsto 0.b_1 b_2 b_3 \ldots, \end{aligned}$$
      are continuous.
      iii. Show that there exist continuous extensions of α and β to $[0,1]$, denoted A and B respectively, which equal 0 at 1 and are linear on $[0,1] \setminus S$.
      [**Note:** you may use, without proof, that every open subsets of $\mathbb{R}$ is a disjoint union of open intervals.]
      iv. Show that $f: [0,1] \to [0,1]^2$ defined by $f(x) = (A(x), B(x))$ is continuous and surjective.

      [**Note:** this part of the exercise was adapted from Exercise IV.31 of Rosenlicht's *Introduction to Analysis*.]

   (b) Suppose $f: [0,1] \to \mathbb{R}^2$ is Lipschitz:
   $$\sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in [0,1],\ x \neq y\right\} < \infty.$$
   Show that $f([0,1])$ is a zero set.

(c) Show that for any smooth map $f\colon [0,1] \to \mathbb{R}^2$, $f([0,1])$ is a zero set.

4. **Unit Disc as a 2-Cell:**

(a) Show that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \exp(-1/t) & \text{otherwise} \end{cases}$$

is smooth.

(b) Let $h(t) := g(1-t)$. Show that

$$f(t) := \frac{g(t)}{g(t) + h(t)}$$

is a smooth function satisfying

$$\begin{cases} f(t) = 0 & \text{if } t \leq 0 \\ 0 < f(t) < 1 & \text{if } 0 < t < 1 \\ f(t) = 1 & \text{if } 1 \leq t \end{cases} .$$

(c) Let $\varphi\colon [0,1]^2 \to \mathbb{R}^2$ be defined by

$$\varphi(u) = \frac{f(|\psi(u)|)}{|\psi(u)|}\psi(u),$$

where $\psi(x,y) = (2x-1, 2y-1)$. Prove that φ is a 2-cell in $\mathbb{R}^2$ whose image is $\{v \in \mathbb{R}^2 \colon |v| \leq 1\}$.

(d) Compute $\partial\varphi$ and the image of this 1-chain.

Solutions:

1. We proceed by induction. The base case follows from the version of the chain rule that we proved in class. Assume the claimed formula holds for $r-1$. Then we compute for $v_r \in \mathbb{R}^n$:

$$\begin{aligned}
(D^{r-1}h)_{p+v_r} - (D^{r-1}h)_p &= \sum_{k=1}^{r-1}\sum_{\mu}(D^k g)_{f(p+v_r)} \circ (D^\mu f)_{p+v_r} - (D^k g)_{f(p)} \circ (D^\mu f)_p \\
&= \sum_{k=1}^{r-1}\sum_{\mu}(D^k g)_{f(p)} \circ [(D^\mu f)_{p+v_r} - (D^\mu f)_p] \\
&\quad + \sum_{k=1}^{r-1}\sum_{\mu}[(D^k g)_{f(p+v_r)} - (D^k g)_{f(p)}] \circ (D^\mu f)_p \\
&\quad + \sum_{k=1}^{r-1}\sum_{\mu}[(D^k g)_{f(p+v_r)} - (D^k g)_{f(p)}] \circ [(D^\mu f)_{p+v_r} - (D^\mu f)_p] .
\end{aligned}$$

Let us denote these three sums by S_1, S_2, and S_3, respectively.

We first analyze S_1. Let $\mu = \{\mu_1, \ldots, \mu_k\}$, then by telescoping and the differentiability of f we have

$$\begin{aligned}
&(D^\mu f)_{p+v_r} - (D^\mu f)_p \\
&= \sum_{j=1}^{k} \left((D^{|\mu_1|} f)_{p+v_r}, \ldots, (D^{|\mu_{j-1}|} f)_{p+v_r}, \left[(D^{|\mu_j|} f)_{p+v_r} - (D^{|\mu_j|} f)_p \right], (D^{|\mu_{j+1}|} f)_p, \ldots, (D^{|\mu_k|} f)_p \right) \\
&= \sum_{j=1}^{k} \left((D^{|\mu_1|} f)_{p+v_r}, \ldots, (D^{|\mu_{j-1}|} f)_{p+v_r}, R_j(v_r) + (D^{|\mu_j|+1} f)_p(v_r), (D^{|\mu_{j+1}|} f)_p, \ldots, (D^{|\mu_k|} f)_p \right) \\
&= \sum_{j=1}^{k} \left((D^{|\mu_1|} f)_{p+v_r}, \ldots, (D^{|\mu_{j-1}|} f)_{p+v_r}, (D^{|\mu_j|+1} f)_p(v_r), (D^{|\mu_{j+1}|} f)_p, \ldots, (D^{|\mu_k|} f)_p \right) \\
&\quad + \sum_{j=1}^{k} \left((D^{|\mu_1|} f)_{p+v_r}, \ldots, (D^{|\mu_{j-1}|} f)_{p+v_r}, R_j(v_r), (D^{|\mu_{j+1}|} f)_p, \ldots, (D^{|\mu_k|} f)_p \right)
\end{aligned}$$

where $R_1, \ldots, R_k$ are sublinear with respect to v_r. Let $\nu_j := \{\mu_1, \ldots, \mu_j \cup \{v_r\}, \ldots, \mu_k\}$, then the term in S_1 corresponding to k and μ equals

$$\sum_{j=1}^{k} (D^k g)_{f(p)} \circ (D^{\nu_j} f)_p(v_r) + \tilde{R}_1(v_r),$$

where $\tilde{R}_1$ is also sublinear with respect to v_r.

Next we analyze S_2. From the base case it follows

$$(D^k g)_{f(p+v_r)} - (D^k g)_{f(p)} = \tilde{R}_2(v_r) + D((D^k g) \circ f)_p(v_r) = \tilde{R}_2(v_r) + [(D^{k+1} g)_{f(p)} \circ (Df)](v_r)$$

where $\tilde{R}_2$ is sublinear with respect to v_r. Now, if $\mu = \{\mu_1, \ldots, \mu_k\}$ is a partition of $\{1, \ldots, r-1\}$, then $\nu := \{\mu_1, \ldots, \mu_k, \{r\}\}$ is a partition of $\{1, \ldots, r\}$ into $k+1$ subsets. Thus the terms in S_2 corresponding to k and μ equals

$$\tilde{R}_2(v_r, (D^\mu f)_p) + (D^{k+1} g)_{f(p)} \circ (D^\nu f)_p.$$

Every partition of $\{1, \ldots, r\}$ arises from a partition $\mu = \{\mu_1, \ldots, \mu_k\}$ of $\{1, \ldots, r-1\}$ by either adding $\{r\}$ to some μ_j or by letting $\{r\}$ be the $k+1$st subset. Thus $S_1 + S_2$ yields the claimed formula for $D^r h$ plus terms that are sublinear with respect to v_r. Consequently, it remains to show that S_3 is sublinear with respect to v_r. Indeed, by our previous analyses we have:

$$\begin{aligned}
&[(D^k g)_{f(p+v_r)} - (D^k g)_{f(p)}] \circ [(D^\mu f)_{p+v_r} - (D^\mu f)_p] \\
&= \tilde{R}_2(v_r, (D^\mu f)_{p+v_r} - (D^\mu f)_p) + (D^{k+1} g)_{f(p)}((Df)_p(v_r), (D^\mu f)_{p+v_r} - (D^\mu f)_p) \\
&= [\tilde{R}_2(v_r, (D^\mu f)_{p+v_r} - (D^\mu f)_p) \\
&\quad + \sum_{j=1}^{k} (D^{k+1} g)_{f(p)} \left((Df)_p(v_r), (D^{|\mu_1|} f)_{p+v_r}, \ldots, R_j(v_r), \ldots, (D^{|\mu_k|} f)_p \right) \\
&\quad + \sum_{j=1}^{k} (D^{k+1} g)_{f(p)} \left((Df)_p(v_r), (D^{|\mu_1|} f)_{p+v_r}, \ldots, (D^{|\mu_j|+1} f)_p(v_r), \ldots, (D^{|\mu_k|} f)_p \right)
\end{aligned}$$

The first term and the first sum are clearly sublinear. The second sum is sublinear with respect to v_r since two entries contain v_r. Thus S_3 is sublinear with respect to v_r and the claimed formula for $D^r h$ holds. □

2. We proceed by induction. The base case follows from the version of the product rule we proved in

class. Assume the claimed formula holds for $r-1$. Then we compute for $v_1, \ldots, v_r \in \mathbb{R}^n$

$$\begin{aligned}
&[(D^{r-1}h)_{p+v_r} - (D^{r-1}h)_p](v_1, \ldots, v_{r-1}) \\
&\quad = \sum_{k=0}^{r-1} \sum_{|\mu|=k} \left\langle (D^k f)_{p+v_r}(v_\mu), (D^{r-1-k}g)_{p+v_r}(v_{\mu^c}) \right\rangle - \left\langle (D^k f)_p(v_\mu), (D^{r-1-k}g)_p(v_{\mu^c}) \right\rangle \\
&\quad = \sum_{k=0}^{r-1} \sum_{|\mu|=k} \left\langle (D^k f)_{p+v_r}(v_\mu), [(D^{r-1-k}g)_{p+v_r} - (D^{r-1-k}g)_p](v_{\mu^c}) \right\rangle \\
&\qquad + \sum_{k=0}^{r-1} \sum_{|\mu|=k} \left\langle [(D^k f)_{p+v_r} - (D^k f)_p](v_\mu), (D^{r-1-k}g)_p(v_{\mu^c}) \right\rangle
\end{aligned}$$

Now, since f and g are r-times differentiable, we can write

$$\begin{aligned}
[(D^{r-1-k}g)_{p+v_r} - (D^{r-1-k}g)_p](v_{\mu^c}) &= R_g(v_{\mu^c}, v_r) + (D^{r-k}g)_p(v_{\mu^c}, v_r) \\
[(D^k f)_{p+v_r} - (D^k f)_p](v_\mu) &= R_f(v_\mu, v_r) + (D^{k+1}f)_p(v_\mu, v_r),
\end{aligned}$$

where R_g and R_f are sublinear with respect to v_r. Continuing our previous computation we have

$$\begin{aligned}
&[(D^{r-1}h)_{p+v_r} - (D^{r-1}h)_p](v_1, \ldots, v_{r-1}) \\
&\quad = \sum_{k=0}^{r-1} \sum_{|\mu|=k} \left\langle (D^k f)_{p+v_r}(v_\mu), (D^{r-k}g)_p(v_{\mu^c}, v_r) \right\rangle \\
&\qquad + \sum_{k=0}^{r-1} \sum_{|\mu|=k} \left\langle (D^{k+1} f)_p(v_\mu, v_r), (D^{r-k-1}g)_p(v_{\mu^c}) \right\rangle + R(v_r),
\end{aligned}$$

where $R(v_r)$ is sublinear with respect to v_r. For $0 \leq k \leq r-1$, rewrite

$$\begin{aligned}
\left\langle (D^k f)_{p+v_r}(v_\mu), (D^{r-k}g)_p(v_{\mu^c}, v_r) \right\rangle = &\left\langle (D^k f)_p(v\mu), (D^{r-k}g)_p(v_{\mu^c}, v_r) \right\rangle \\
&+ \left\langle [(D^k f)_{p+v_r} - (D^k f)_p](v_\mu), (D^{r-k}g)_p(v_{\mu^c}, v_r) \right\rangle.
\end{aligned}$$

Since $k < r$, $D^k f$ is continuous since it is differentiable. Hence the second term above is sublinear with respect to v_r. So we may push our previous computation even further to write

$$\begin{aligned}
&[(D^{r-1}h)_{p+v_r} - (D^{r-1}h)_p](v_1, \ldots, v_{r-1}) \\
&\quad = \sum_{k=0}^{r-1} \sum_{|\mu|=k} \left\langle (D^k f)_p(v_\mu), (D^{r-k}g)_p(v_{\mu^c}, v_r) \right\rangle \\
&\qquad + \sum_{k=0}^{r-1} \sum_{|\mu|=k} \left\langle (D^{k+1} f)_p(v_\mu, v_r), (D^{r-k-1}g)_p(v_{\mu^c}) \right\rangle + \tilde{R}(v_r),
\end{aligned}$$

where $\tilde{R}(v_r)$ is some new quantity which is sublinear with respect to v_r. However, the rest of the last expression is precisely the formula for r:

$$\sum_{k=0}^{r} \sum_{|\nu|=k} \left\langle (D^k f)_p(v_\nu), (D^{r-k}g)_p(v_{\nu^c}) \right\rangle.$$

Indeed fix $\nu \subset \{1, \ldots, r\}$. If $r \notin \nu$, then for $\mu := \nu \subset \{1, \ldots, r-1\}$

$$\left\langle (D^k f)_p(v_\nu), (D^{r-k}g)_p(v_{\nu^c}) \right\rangle = \left\langle (D^k f)_p(v_\mu), (D^{r-k}g)_p(v_{\mu^c}, v_r) \right\rangle.$$

Otherwise $r \in \nu$, so then for $\mu := \nu \setminus \{r\} \subset \{1, \ldots, r-1\}$

$$\left\langle (D^k f)_p(v_\nu), (D^{r-k}g)_p(v_{\nu^c}) \right\rangle = \left\langle (D^k f)_p(v_\mu, v_r), (D^{r-k}g)_p(v_{\mu^c}) \right\rangle.$$

Thus we have shown that $(D^{r-1}h)_{p+v_r} - (D^{r-1}h)_p$ minus the claimed the formula for $(D^r h)_p(v_r)$ is sublinear with respect to v_r. This precisely means the claimed formula gives $(D^r h)_p$. $\square$

3. (a) i. For each $n \in \mathbb{N}$, set

$$S_n := \bigcup_{a_1,b_1,\ldots,a_n,b_n=0}^{9} [0.a_1b_10a_2b_20\ldots0a_nb_n, 0.a_1b_10a_2b_20\ldots0a_nb_n1]$$

Then S_n is closed as the finite union of closed intervals. Then

$$S = \bigcap_{n=1}^{\infty} S_n,$$

so that S is closed as an intersection of closed sets. □

ii. Let $(x_n)_{n\in\mathbb{N}} \subset S$ be a sequence converging to x (which is necessarily in S since S is closed). To see that α and β are continuous, it suffices show show $\alpha(x_n) \to \alpha(x)$ and $\beta(x_n) \to \beta(x)$. Suppose

$$\begin{aligned} x &= 0.a_1, b_10a_2b_20\ldots \\ x_n &= 0.a_1^{(n)}b_1^{(n)}0a_2^{(n)}b_2^{(n)}0\ldots \qquad n \in \mathbb{N} \end{aligned}$$

We claim that for each $k \in \mathbb{N}$, the sequences $(a_k^{(n)})_{n\in\mathbb{N}}$ and $(b_k^{(n)})$ converge to a_k and b_k, respectively. Suppose, towards a contradiction, that this is not the case. Let $k \in \mathbb{N}$ be the smallest number for which either $(a_k^{(n)})_{n\in\mathbb{N}}$ or $(b_k^{(n)})_{n\in\mathbb{N}}$ does not converge to a_k or b_k, respectively. We will assume the former, the proof for the latter being similar. Then $(a_\ell^{(n)})_{n\in\mathbb{N}}$ and $(b_\ell^{(n)})_{n\in\mathbb{N}}$ converge to a_ℓ and b_ℓ, respectively, for each $\ell = 1, \ldots, k-1$. But since these are discrete sequences that means there exists $N_\ell \in \mathbb{N}$ such that for all $n \geq N_\ell$ we have $a_\ell^{(n)} = a_\ell$ and $b_\ell^{(n)} = b_\ell$. Set $N = \max\{N_1, \ldots, N_{k-1}\}$. Now, $(a_k^{(n)})_{n\in\mathbb{N}}$ not converging to a_k means there is some $a \in \{0, 1, 2, \ldots, 9\} \setminus a_k$ such that $a_k^{(n)} = a$ for infinitely many $n \in \mathbb{N}$. Thus we can find a subsequence $(a_k^{(n_j)})_{j\in\mathbb{N}}$ that is constantly equal to $a \neq a_k$. But then for $j \in \mathbb{N}$ large enough so that $n_j \geq N$, we have

$$|x_{n_j} - x| \geq \frac{1}{10^{3k-2}}|a - a_k| - 0.\underbrace{0\cdots0}_{3k-2 \text{ digits}}991 \geq 0.\underbrace{0\cdots0}_{3k-2 \text{ digits}}009.$$

Since this holds for all sufficiently large j, this contradicts x_n converging x. Thus we must have $a_k^{(n)} \to a_k$ and $b_k^{(n)} \to b_k$ for all $k \in \mathbb{N}$.
Now, let $\epsilon > 0$. Let $k \in \mathbb{N}$ be such that $\frac{1}{10^k} < \epsilon$. For each $\ell = 1, \ldots, k$, there exists N_ℓ such that for $n \geq N_\ell$ we have $a_\ell^{(n)} = a_\ell$. Let $N = \max\{N_1, \ldots, N_k\}$. Then for $n \geq N$ we have

$$|\alpha(x_n) - \alpha(x)| = \left| \sum_{\ell=k+1}^{\infty} \frac{a_\ell^{(n)} - a_\ell}{10^\ell} \right| \leq \sum_{\ell=k+1}^{\infty} \frac{9}{10^\ell} = \frac{1}{10^k} < \epsilon.$$

Hence $\alpha(x_n) \to \alpha(x)$. Similarly $\beta(x_n) \to \beta(x)$. □

iii. Since S is closed, S^c is open and therefore a countable union of disjoint open intervals. Since $0 \in S$, we have, in particular, that

$$[0,1] \setminus S = (c_0, 1] \sqcup \bigsqcup_{n=1}^{\infty} (c_n, d_n),$$

where $c_0, c_1, d_1, c_2, d_2, \ldots \in S$. Thus we define A on $[0,1] \setminus S$ by

$$A(x) := \begin{cases} \alpha(c_n)\frac{d_n - x}{d_n - c_n} + \alpha(d_n)\frac{x - c_n}{d_n - c_n} & \text{if } x \in (c_n, d_n) \text{ for some } n \in \mathbb{N} \\ \alpha(c_0)\frac{1-x}{1-c_0} & \text{if } x \in (c_0, 1] \end{cases}.$$

Similarly for B. Then A and B are continuous on $[0,1]$ since they are continuous on S (by virtue of α and β being continuous), continuous on $S \setminus [0,1]$ (since they are linear), and agree on the common boundary points $c_0, c_1, d_1, \ldots$ by definition. □

iv. f is continuous since each of its coordinate functions are continuous. Let $(x, y) \in [0,1]^2$. Then there decimal expansions for x and y of the form $x = 0.a_1a_2\ldots$ and $y = 0.b_1b_2\ldots$ (we use $1 = 0.99\ldots$ if necessary). Thus

$$(x, y) = f(0.a_1b_10a_2b_20\ldots),$$

and so f is surjective. □

(b) Denote

$$L := \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in [0,1],\ x \neq y\right\}.$$

Observe that for any $x, y \in [0, 1]$, we have $|f(x) - f(y)| \leq L|x - y|$. In particular, for $\delta > 0$ if $B \subset \mathbb{R}^2$ is an open square with center x and sidelength $2L\delta$, then $f(y) \in B$ for all $y \in [0, 1]$ satisfying $|x - y| < \delta$.

Now, let $\epsilon > 0$. Let $0 < \delta < \min\{\frac{\epsilon}{12L^2}, 1\}$ and let $N \in \mathbb{N}$ be such that $1 \leq \delta N$ and $\delta(N + 1) \leq 3$. Set $x_0 = 0$ and $x_n = \max\{x_0 + \delta n, 1\}$ for $n = 1, \ldots, N$. Then

$$[0, 1] \subset \bigcup_{n=0}^{N}(x_n - \delta, x_n + \delta).$$

Consequently, if we let B_n be the open square with center $f(x_n)$ and sidelength $2L\delta$, then

$$f([0, 1]) \subset \bigcup_{n=0}^{N} B_n.$$

We have

$$\sum_{n=0}^{N}|B_n| = \sum_{n=0}^{N} 4L^2\delta^2 = 4(N + 1)L^2\delta^2 \leq 12L^2\delta < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have that $f([0, 1])$ is a zero set. □

(c) By part (c), it suffices to show that f is Lipschitz. Since f is smooth, Df is continuous. Then Df is bounded on $[0, 1]$ since it is a compact set. Thus by the Mean Value Theorem we see that f is Lipschitz. □

4. (a) g is clearly smooth on $\mathbb{R} \setminus \{0\}$, so it suffices to check smoothness at 0. Note that by the chain rule and product rule, for any $n \in \mathbb{N}$ and $t > 0$ $g^{(n)}(t) = p(\frac{1}{t})e^{-1/t}$ where p is some polynomial. Since $e^{-1/t}$ tends to zero as $t \to 0$ faster than any polynomial, we have that

$$\lim_{x\to 0^+} \frac{g^{(n)}(t) - g^{(n)}(0)}{t - 0} = \lim_{x\to 0^+} \frac{p(\frac{1}{t})e^{-1/t}}{t} = \lim_{x\to 0^+} \frac{1}{t}p(\frac{1}{t})e^{-1/t} = 0,$$

which clearly agrees with the left-hand limit. Thus for each $n \in \mathbb{N}$, $g^{(n)}(0) = 0$. In particular, g is smooth. □

(b) Since $g(t) = 0$ for $t \leq 0$, we have that $f(t) = 0$. For $0 < t < 1$ we have $g(t), h(t) > 0$ and so

$$0 < \frac{g(t)}{g(t) + h(t)} < \frac{g(t)}{g(t)} = 1.$$

For $t \geq 1$ we have $h(t) = g(1 - t) = 0$ so that

$$f(t) = \frac{g(t)}{g(t)} = 1.$$

Finally, f is smooth since g and $g + h$ are smooth and $g + h > 0$. □

(c) We first note that

$$\mathbb{R} \ni t \mapsto \frac{f(\sqrt{|t|})}{\sqrt{|t|}}$$

is smooth. Indeed, for $t \neq 0$ this is clear, and for $t = 0$ it follows from the fact that g and hence f decays exponentially. By the product rule, $|\psi(u)|^2 = \langle \psi(u), \psi(u) \rangle$ is smooth and thus

$$\frac{f(|\psi(u)|}{|\psi(u)|} = \frac{\sqrt{|\psi(u)|^2}}{\sqrt{|\psi(u)|^2}},$$

is smooth by the chain rule. Another application of the product rule yields that φ is smooth and hence a 2-cell in $\mathbb{R}^2$.

To see that φ has the claimed image, first note that $f\,|_{[0,1]}$ is onto $[0,1]$ by the intermediate value theorem. Let $v \in \mathbb{R}^2$ satisfy $|v| \leq 1$, and let $s \in [0,1]$ be such that $f(s) = |v|$. Then for $w = \frac{sv}{|v|}$, $|w| = s$ and so

$$\frac{f(|w|)}{|w|} w = \frac{f(s)}{s} \frac{sv}{|v|} = |v| \frac{v}{|v|} = v.$$

Since ψ is onto $[-1,1]^2$, there exists $u \in [0,1]^2$ such that $\psi(u) = w$ and so φ has the claimed image. □

(d) We first compute the dipoles

$$\begin{aligned}
\delta^1\varphi(t) = \varphi(1,t) - \varphi(0,t) &= \frac{f(|(1,2t-1)|)}{|(1,2t-1)|}(1,2t-1) - \frac{f(|(-1,2t-1)|)}{|-1,2t-1|}(-1,2t-1) \\
&= \frac{f(\sqrt{1+(2t-1)^2})}{|(1,2t-1)|}(1,2t-1) - \frac{f(\sqrt{1+(2t-1)^2})}{|(-1,2t-1)|}(-1,2t-1) \\
&= \frac{(1,2t-1)}{|(1,2t-1)|} - \frac{(-1,2t-1)}{|(-1,2t-1)|}
\end{aligned}$$

Similarly

$$\delta^2\varphi(t) = \frac{(2t-1,1)}{|(2t-1,1)|} - \frac{(2t-1,-1)}{|(2t-1,-1)|}$$

Thus

$$\partial\varphi(t) = \frac{(1,2t-1)}{|(1,2t-1)|} - \frac{(-1,2t-1)}{|(-1,2t-1)|} - \frac{(2t-1,1)}{|(2t-1,1)|} + \frac{(2t-1,-1)}{|(2t-1,-1)|}$$

Observe that each 1-cell in this 1-chain consists of units vectors. Hence the image $\partial\varphi$ is the set $\{v \in \mathbb{R}^2 \colon |v| = 1\}$. To be more precise, the first term travels from $(1,-1)$ to $(1,1)$ counter-clockwise around the unit circle, the second term travels from $(-1,1)$ to $(-1,-1)$, the third term from $(1,1)$ to $(-1,1)$, and the fourth term from $(-1,-1)$ to $(1,-1)$. □