Exercises:
1. Prove that if \( S \subseteq \mathbb{R} \) is non-empty and bounded below, then it has an infimum.

2. For \( S \subseteq \mathbb{R} \) a non-empty subset that is bounded above and \( x \in \mathbb{R} \), let \( xS \) be the set \( \{xs: s \in S\} \).
   (a) Show that if \( x > 0 \), then \( \sup(xS) = x \sup(S) \).
   (b) Show that if \( x < 0 \), then \( \inf(xS) = x \inf(S) \).

3. Let \( S, T \subseteq \mathbb{R} \) be non-empty subsets that are bounded from above, and define \( S + T = \{s + t: s \in S, t \in T\} \). Show
   \[ \sup(S + T) = \sup(S) + \sup(T) \]
   Then, use this to prove that if \( x \in \mathbb{R} \) and \( S + x \) is the set \( \{s + x: s \in S\} \), then \( \sup(S + x) = \sup(S) + x \).

4. Recall that we say \( S \subseteq \mathbb{R} \) is dense if for any \( x \in \mathbb{R} \) and every \( \epsilon > 0 \) there exists \( s \in S \) such that \( |s - x| < \epsilon \).
   (a) Show that a set \( S \subseteq \mathbb{R} \) is dense if and only if for any \( a, b \in \mathbb{R} \) there exists \( s \in S \) with \( a < s < b \).
   (b) Show that the set of irrational numbers \( \mathbb{R} \setminus \mathbb{Q} \) is dense.

Solutions:
1. Let \( S \subseteq \mathbb{R} \) be a non-empty subset that is bounded from below. Then there exists \( a \in \mathbb{R} \) such that \( a \leq s \) for all \( s \in S \). This implies that \( -a \geq -s \) for all \( s \in S \). Hence \( -a \) is an upper bound for the set \( -S := \{-s: s \in S\} \). By the Least Upper Bound Property, \( -S \) has a supremum which we will denote by \( x \). We claim that \( -x \) is the infimum of \( S \). First note that since \( x \) is an upper bound for \( -S \) we have \( x \geq -s \) for all \( s \in S \), and hence \( -x \leq s \) for all \( s \in S \). That is, \( -x \) is a lower bound for \( S \). Now let \( b \) be any other lower bound for \( S \). Just as we showed for \( a \) above, \( -b \) is an upper bound for \( -S \). Since \( x \) is the least upper bound for \( -S \), we must have \( x \leq -b \). But then \( -x \geq b \). Hence \( -x \) is the greatest lower bound, meaning \( -x = \inf(S) \).

2. (a) Suppose \( x > 0 \). Then since \( s \leq \sup(S) \) for all \( s \in S \), multiplying by \( x \) yields \( xs \leq x \sup(S) \).
   Hence \( x \sup(S) \) is an upper bound for \( xS \) and therefore \( \sup(xS) \leq x \sup(S) \).
   On the other hand, for any \( s \in S \), we have
   \[ s = \frac{1}{x}xs \leq \frac{1}{x} \sup(xS) \]
   Thus \( \frac{1}{x} \sup(xS) \) is an upper bound for \( S \), so that \( \sup(S) \leq \frac{1}{x} \sup(xS) \) or \( x \sup(S) \leq \sup(xS) \).
   This combined with the previous inequality gives \( \sup(xS) = x \sup(S) \). \( \square \)
   (b) Suppose \( x < 0 \). Then since \( s \leq \sup(S) \) for all \( s \in S \), multiplying by \( x \) yields \( xs \geq x \sup(S) \).
   Hence \( x \sup(S) \) is a lower bound for \( xS \) and therefore \( \inf(xS) \geq x \sup(S) \).
   On the other hand, for any \( s \in S \), \( \inf(xS) \leq xs \) implies
   \[ \frac{1}{x} \inf(xS) \geq \frac{1}{x}xs = s \]
   since \( \frac{1}{x} < 0 \). Thus \( \frac{1}{x} \inf(xS) \) is an upper bound for \( S \), so that \( \sup(S) \leq \frac{1}{x} \inf(xS) \) or \( x \sup(S) \geq \inf(xS) \).
   This combined with the previous inequality gives \( \inf(xS) = x \sup(S) \). \( \square \)

3. For any \( s \in S \) and \( t \in T \), we have \( s + t \leq \sup(S) + \sup(T) \). Hence \( \sup(S) + \sup(T) \) is an upper bound for the set \( S + T \) and therefore \( \sup(S + T) \leq \sup(S) + \sup(T) \). To prove the reverse inequality, we shall give ourselves an \( \epsilon \) of room. Let \( \epsilon > 0 \), and recall from class that we can find \( t_0 \in T \) such that \( \sup(T) - \epsilon < t_0 \leq \sup(T) \). Then for any \( s \in S \) we have
   \[ s = s + t_0 - t_0 \leq \sup(S + T) - t_0 \].
Thus $\sup(S + T) - t_0$ is an upper bound for $S$, which means $\sup(S) \leq \sup(S + T) - t_0$ or $\sup(S) + t_0 \leq \sup(S + T)$. Since $t_0 > \sup(T) - \epsilon$, we then have

$$\sup(S + T) \geq \sup(S) + t_0 > \sup(S) + \sup(T) - \epsilon,$$

or $\sup(S) + \sup(T) < \sup(S + T) + \epsilon$. Since $\epsilon > 0$ was arbitrary, the “$\epsilon$ of room” theorem from class implies $\sup(S) + \sup(T) \leq \sup(S + T)$. Combined with the previous inequality, we have $\sup(S + T) = \sup(S) + \sup(T)$.

Now, let $x \in \mathbb{R}$. Define $T = \{x\}$, so that as sets we have $S + T = S + x$. Note that $x$ is an upper bound for $T$, and if $a \in \mathbb{R}$ is any other upper bound, we have (by definition) $x \leq a$. Hence $x = \sup(T)$. So by the first part of the exercise we have

$$\sup(S + x) = \sup(S + T) = \sup(S) + \sup(T) = \sup(S) + x.$$

□

4. (a) ($\Rightarrow$) : Suppose $S$ is dense and let $a, b \in \mathbb{R}$ be such that $a < b$. Let $x = \frac{a + b}{2}$ and let $\epsilon = \frac{b - a}{2}$. Then by the density of $S$ we can find $s \in S$ such that $|s - x| < \epsilon$. We claim that in fact, $a < s < b$. Indeed, $|s - x| < \epsilon$ is equivalent to $x - \epsilon < s < x + \epsilon$. But

$$x + \epsilon = \frac{a + b}{2} + \frac{b - a}{2} = b$$

and

$$x - \epsilon = \frac{a + b}{2} - \frac{b - a}{2} = a.$$

Hence $a = x - \epsilon < s < x + \epsilon = b$.

($\Leftarrow$) : Now assume that for any $a, b \in \mathbb{R}$ with $a < b$, we can find $s \in S$ such that $a < s < b$. We must show $S$ is dense. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Then letting $a = x - \epsilon$ and $b = x + \epsilon$, we can use the assumed property of $S$ to find $s \in S$ such that $x - \epsilon < s < x + \epsilon$. But this is equivalent to $|s - x| < \epsilon$. □

(b) Recall from class that $\sqrt{2}$ is irrational. Also note that for any $x \in \mathbb{Q}$, $x + \sqrt{2}$ must also be irrational. If not, then $x + \sqrt{2} = y$ for some $y \in \mathbb{Q}$. But then $\sqrt{2} = y - x$, and since $x$ and $y$ are both rational, so is their difference $y - x$. This contradicts $\sqrt{2}$ being irrational. Hence $x + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Now, let $a, b \in \mathbb{R}$ be such that $a < b$. Then $a - \sqrt{2} < b - \sqrt{2}$. Using the density of $\mathbb{Q}$ and part (a), we can find $x \in \mathbb{Q}$ such that $a - \sqrt{2} < x < b - \sqrt{2}$. Adding $\sqrt{2}$ to these inequalities gives $a < x + \sqrt{2} < b$. As noted above, $x + \sqrt{2}$ is irrational and thus by part (a) again we have shown $\mathbb{R} \setminus \mathbb{Q}$ is dense.