Exercises:
1. For the following subsets, determine (without proof) their interiors, closures, and boundaries
   (a) $S = [0, 1]$ in $\mathbb{R}$ with the usual metric.
   (b) $S = (0, 1)$ in $\mathbb{R}$ with the usual metric.
   (c) $S = \mathbb{Z}$ in $\mathbb{R}$ with the usual metric.
   (d) $S = \mathbb{Q}$ in $\mathbb{R}$ with the usual metric.
   (e) $S = \mathbb{R}$ in $\mathbb{R}$ with the usual metric.
   (f) $S = [0, 1) \times [0, 1)$ in $\mathbb{R}^2$ with the 2-dimensional Euclidean metric.

2. Let $S \subset E$.
   (a) Show that $S^o$ is the union of all open subsets $U \subset S$.
   (b) Show that $S^o$ is open.
   (c) Show that $\partial S$ is closed.

3. Let $T \subset S \subset E$.
   (a) Show that $\overline{T} \subset \overline{S}$.
   (b) Show that $T^o \subset S^o$.

4. Let $S \subset E$.
   (a) Show that $S$ is open if and only if $S = S^o$.
   (b) Show that $S$ is closed if and only if $S = \overline{S}$.

5. Let $S \subset E$.
   (a) Show that $\overline{S} = ((S^c)^c)^c$.
   (b) Show that $S^o = \left(\overline{(S^c)}\right)^c$.
   (c) Show that $\partial S = \overline{S} \cap \overline{S^c}$.
   (d) Show $\partial S = \partial (S^c)$.

6. Let $S \subset E$.
   (a) Show $\overline{S} = \{x \in E: B(x, r) \cap S \neq \emptyset \ \forall r > 0\}$.
   (b) Show $\partial S = \{x \in E: B(x, r) \cap S \neq \emptyset \text{ and } B(x, r) \cap S^c \neq \emptyset \ \forall r > 0\}$.

7. For $S \subset E$ show that the following are equivalent:
   (i) $S$ is dense in $E$.
   (ii) $(S^c)^o = \emptyset$.
   (iii) $\overline{S} = E$.

8. For $S \subset E$, show that $E$ is the disjoint union of $S^o$, $\partial S$, and $(S^c)^o$.

9. Let $S \subset E$.
   (a) Show that $S$ is closed if and only if $\partial S \subset S$.
   (b) Show that $S$ is open if and only if $\partial S \cap S = \emptyset$.

10. Let $A, B \subset E$.
    (a) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
11. Let $S \subset E$ be a connected set. Suppose $T \subset E$ satisfies $S \subset T \subset \overline{S}$. Show that $T$ is also connected.

**Solutions:**

1. (a) $S^o = (0, 1)$, $\overline{S} = [0, 1]$, and $\partial S = \{0, 1\}$.

(b) $S^o = (0, 1)$, $\overline{S} = [0, 1]$, and $\partial S = \{0, 1\}$.

(c) $S^o = \emptyset$, $\overline{S} = \mathbb{Z}$, and $\partial S = \mathbb{Z}$.

(d) $S^o = \emptyset$, $\overline{S} = \mathbb{R}$, and $\partial S = \mathbb{R}$.

(e) $S^o = \mathbb{R}$, $\overline{S} = \mathbb{R}$, and $\partial S = \emptyset$.

(f) $S^o = (0, 1) \times (0, 1)$, $\overline{S} = [0, 1] \times [0, 1]$, and $\partial S = \{(x, y) \in \mathbb{R}^2 : \text{either } x \in \{0, 1\} \text{ or } y \in \{0, 1\} \}$. (or both)

2. (a) Denote by $T$ the union of all open subsets of $S$. Let $x \in S^o$. Since $x$ is an interior point, there exists $r > 0$ such that $B(x, r) \subset S$, but then $B(x, r)$ is an open subset of $S$. Hence $x \in T$, and since $x \in S^o$ was arbitrary we have $S^o \subset T$. Conversely, if $x \in T$ then $x \in U$ for some open subset $U \subset S$. Since $U$ is open, there exists $r > 0$ such that $B(x, r) \subset U \subset S$; that is, $x$ is an interior point of $S$ and therefore $x \in S^o$. Since $x \in T$ was arbitrary, we have $T \subset S^o$, which yields $T = S^o$.

(b) By part (a), $S^o$ is a union of open sets and is therefore open.

(c) We have $\partial S = \overline{S} \setminus S^o = \overline{S} \cap (S^o)^c$. We know $\overline{S}$ is closed, and by part (b) $(S^o)^c$ is closed as the complement of an open set. Thus $\partial S$ is closed as an intersection of closed sets.

3. (a) Since $T \subset S \subset \overline{S}$, we have that $\overline{S}$ is a closed set containing $T$. Thus $\overline{T} \subset \overline{S}$.

(b) If $x \in T^o$, then there exists $r > 0$ such that $B(x, r) \subset T \subset S$. Hence $x$ is also an interior point of $S$ and so $x \in S^o$. Consequently $T^o \subset S^o$.

4. (a) $\Rightarrow$: If $S$ is open, then $U = S$ is an open subset of $S$. Hence $S \subset S^o$. The reverse inclusion always holds, so we have $S = S^o$.

$\Leftarrow$: By Exercise 3.(b), $S = S^o$ is open.

(b) $\Rightarrow$: If $S$ is closed, then $S$ is a closed set containing $S$. Hence $\overline{S} \subset S$. The reverse inclusion always holds, so we have $S = \overline{S}$.

$\Leftarrow$: $S = \overline{S}$ is closed as the intersection of closed sets.

5. (a) Denote $T := ((S^o)^c)^c$. We first note that $(S^c)^o$ is open and is contained in $S^c$. Consequently, its complement, $T$, is a closed set containing $S$. Hence $\overline{T} \subset T$. On the other hand, if $V$ is a closed set containing $S$, then we have that $V^c$ is an open subset of $S^c$. Hence $V^c \subset (S^c)^o$, which implies $V \supset T$. Since $V$ was an arbitrary closed set containing $S$, we have $T \subset \overline{S}$, and therefore $T = \overline{S}$.

(b) Denote $T = S^c$. Then using part (a) we have

$$\left(\overline{(S^o)}\right)^c = (T)^c = (((T^c)^c)^c)^c = (T^c)^o = S^o.$$ (or both)

(c) By definition we have

$$\partial S = \overline{S} \setminus S^o = \overline{S} \cap (S^o)^c.$$ (or both)
(d) From part (c), we have \( \partial S = \overline{S} \cap \overline{S^c} = \overline{S^c} \cap \overline{S} = \partial (S^c) \). \( \square \)

6. (a) Denote

\[
T = \{ x \in E : B(x, r) \cap S \neq \emptyset \forall r > 0 \}.
\]

Observe that \( x \in T^c \) iff \( \exists r > 0 \) such that \( B(x, r) \cap S = \emptyset \) iff \( x \in (S^c)^o \) iff (by Exercise 5.(a)) \( x \in \overline{S^c} \). Thus \( T^c = \overline{S^c} \) which implies \( T = \overline{S} \). \( \square \)

(b) By part (a),

\[
\{ x \in E : B(x, r) \cap S \neq \emptyset \text{ and } B(x, r) \cap S^c \neq \emptyset \forall r > 0 \} = \overline{S} \cap \overline{S^c},
\]

and this latter set equals \( \partial S \) by Exercise 5.(c). \( \square \)

7. [(i) \( \Rightarrow \) (ii)] Assume \( S \) is dense in \( E \). Then for every \( x \in E \) and every \( r > 0 \) we have \( B(x, r) \cap S \neq \emptyset \). In particular, this is true for \( x \in S^c \) and means that \( B(x, r) \not\subset S^c \) for any \( r > 0 \). Hence \( S^c \) has no interior points which means \( (S^c)^o = \emptyset \).

[(ii) \( \Rightarrow \) (iii)] Assume \( (S^c)^o = \emptyset \). Then by Exercise 5.(a) we have \( \overline{S} = (\emptyset)^c = E \).

[(iii) \( \Rightarrow \) (i)] Assume \( \overline{S} = E \). Let \( x \in E = \overline{S} \). Then by Exercise 6.(a), \( B(x, r) \cap S \neq \emptyset \) for all \( r > 0 \). Since this holds for all \( x \in E \), we have that \( S \) is dense. \( \square \)

8. We know \( E \) is the disjoint union of \( \overline{S} \) and \( \overline{S^c} \). By definition of the boundary we see that \( \overline{S} \) is the disjoint union of \( S^o \) and \( \partial S \), and by Exercise 5.(a) we see that \( \overline{S^c} = (S^c)^o \). \( \square \)

9. (a) If \( S \) is closed then \( S = \overline{S} \) by Exercise 4.(b), but then \( \partial S \subset \overline{S} = S \). Conversely, if \( \partial S \subset S \) then \( \overline{S} = \partial S \cup S^o \subset S \subset \overline{S} \). Thus \( S = \overline{S} \), which implies \( S \) is closed. \( \square \)

(b) If \( S \) is open then \( S = S^o \) by Exercise 4.(a), and hence \( \partial S \cap S = \partial S \cap S^o = \emptyset \). Conversely, if \( \partial S \cap S = \emptyset \), then \( S \subset \partial S \cup (S^c)^o \) by Exercise 8. Since \( (S^c)^o \subset S^c \), we know that \( S \subset S^o \) and the reverse inclusion always holds. Thus \( S = S^o \) which implies \( S \) is open by Exercise 4.(a). \( \square \)

10. (a) Note that \( \overline{A} \cup \overline{B} \) is closed as the finite union of closed sets, and it contains \( A \cup B \). Hence \( \overline{A} \cup \overline{B} \subset \overline{A \cup B} \). On the other hand, if \( V \) is a closed set containing \( A \cup B \), then it is also a closed set containing \( A \). Hence \( \overline{A} \subset V \). Similarly, \( \overline{B} \subset V \), which means \( \overline{A} \cup \overline{B} \subset V \). Since \( V \) was an arbitrary closed set containing \( A \cup B \), we have \( \overline{A} \cup \overline{B} \subset \overline{A \cup B} \), which gives equality. \( \square \)

(b) We can prove this directly in a similar fashion to part (a), or we can appeal to part (a) along with Exercise 5.(b):

\[
(A \cap B)^o = (\overline{(A \cap B)})^c = (\overline{A^c \cup B^c})^c = (\overline{A^c} \cup \overline{B^c})^c = (A^c)^c \cap (B^c)^c = A^o \cap B^o.
\]

\( \square \)

(c) Consider \( A = [0, 1) \) and \( B = [1, 2] \). Then \( A \cap B = \emptyset \), which is closed so \( \overline{A \cap B} = \emptyset \). However, \( \overline{A} = [0, 1] \) and \( \overline{B} = [1, 2] \) so that \( \overline{A \cap B} = \{1\} \neq \emptyset \).

Also, \( A \cup B = [0, 2] \) which has \( (A \cup B)^o = (0, 2) \), while \( A^o = (0, 1) \) and \( B^o = (1, 2) \) so that \( A^o \cup B^o = (0, 1) \cup (1, 2) \neq (0, 2) \).

11. Suppose \( T = A \cup B \) for disjoint non-empty subsets \( A, B \subset T \) which are open relative to \( T \). Since \( S \subset T \), we have \( S = [A \cap S] \cup [B \cap S] \), and \( A \cap S \) and \( B \cap S \) are open relative to \( S \). As \( S \) is connected, we know either \( A \cap S \) or \( B \cap S \) is empty. Without loss of generality, assume \( B \cap S = \emptyset \) and hence \( A \cap S = S \). This means \( S \subset A \), and consequently \( A^c \subset S^c \). Thus

\[
B \subset A^c \cap T \subset S^c.
\]

On the other hand, \( B \subset T \subset \overline{S} \). Combining these two inclusions yields \( B \subset \overline{S} \subset S^c \). Since \( \overline{S} \subset S \subset \partial S \), we have \( B \subset \partial S \). Now, since \( B \) is open relative to \( T \), for each \( x \in B \) there exists \( r > 0 \) such that \( B(x, r) \cap T \subset B \). On the other hand, by Exercise 6.(b), \( x \in B \subset \partial S \) means this ball must intersect \( S \) and hence \( A \), a contradiction. Thus \( T \) is connected. \( \square \)