The Tutte Polynomial

Madeline Brandt

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Introduction

The Tutte polynomial is a polynomial $T(x,y)$ in two variables which can be defined for graphs or matroids. Many problems about graphs can be reduced to problems of finding and evaluating the Tutte polynomial at certain values. In this talk, we will define the Tutte polynomial for graphs and discuss some results about evaluating the polynomial at certain points, and give two formulas for $T(1,y)$.

For this talk, let $G=(V,E)$ be a graph. If $A \subset E$, let the *rank* of A be $|V|-k(A)$, where $k(A)$ is the number of connected components of the graph (V,A). Note that for a graphic matroid, this is just the (matroid) rank of A.

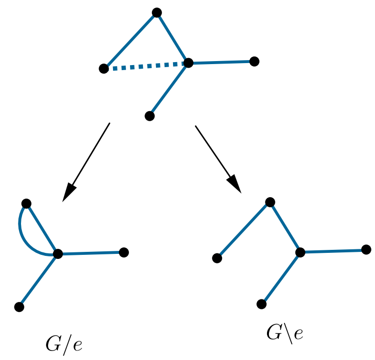
Definition 1. The *Tutte Polynomial* of a graph G is

$$T_G(x,y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Example If we let $G = K_3$, then

$$\begin{aligned} T(x,y) =& \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\ =& \underbrace{(x-1)^{2-2}(y-1)^{3-2}}_{\{e_1,e_2,e_3\}} + 3\underbrace{(x-1)^{2-2}(y-1)^{2-2}}_{\{e_i,e_j\}} \\ &+ 3\underbrace{(x-1)^{2-1}(y-1)^{1-1}}_{\{e_i\}} + \underbrace{(x-1)^{2}(y-1)^{0-0}}_{\emptyset} \\ =& x^2+x+y. \end{aligned}$$

Definition 2. We will define a *loop* in G to be an edge which starts and ends at the same vertex. A *bridge* in G will be an edge e such that if e is removed, then the resulting graph has more components than G. Let G/e be the graph in which the two vertices in e have been "glued" together. Let $G \backslash e$ be the graph in which the edge e has been deleted:



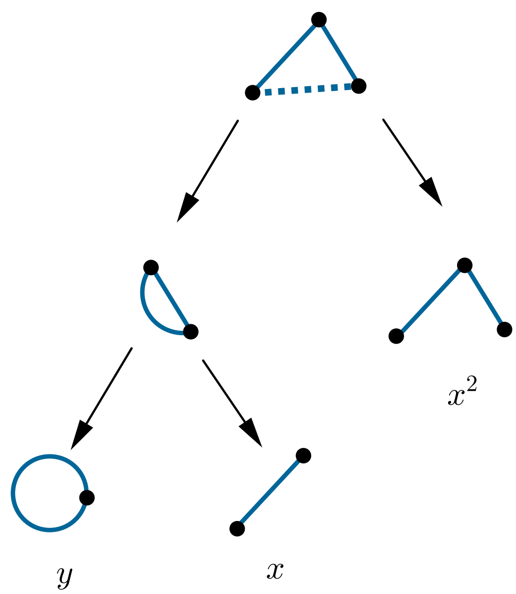
Theorem 1. *The Tutte polynomial can be defined recursively as follows:*

1. $T_{\emptyset}(x, y) = 1$
2. *If e is a loop, then* $T_G(x, y) = yT_{G\backslash e}(x, y)$
3. *If e is a bridge, then* $T_G(x, y) = xT_{G/e}(x, y)$
4. *If e is neither, then*

$$T_G(x, y) = T_{G\backslash e}(x, y) + T_{G/e}(x, y).$$

Proof. The proof is left as an exercise. □

Example We can use this to re-compute $T_{K_3}(x, y)$:



Observe that this is invariant under the order in which we delete and contract. There are several nice values of the Tutte polynomial.

Theorem 2. *We have the following values of T_G.*

1. $T_G(1,1)$ *is the number of maximal forests in* G.

2. $T_G(2,1)$ *is the number of forests in* G.

3. $T_G(1,2)$ *is the number of spanning subgraphs of* G.

4. $T_G(2,2)$ *is* $2^{|E|}$.

Proof. We will prove (1) as an example, and leave the rest as exercises. In order for the x term in the Tutte polynomial to be nonzero, we must have that the exponent be 0, or $R(E) = R(A)$. This implies that the number of connected components of A is the same as the number of connected components in E. This implies that A spans the vertices of G. In order for the y term to be nonzero, we must have $r(A) = |A|$, which implies that $|A| = |V| - k(A)$, which implies that A has no loops (since we already know that A spans the vertices). Hence, A is a maximal forest in G, and it gets weight 1 in the $T(1,1)$, while all other subsets will get weight 0. □

Merino's Theorem and Chip Firing

We will now discuss the chip firing game on graphs. Fix some $q \in V$ to be called the *sink vertex*, and let $\tilde{V} = V \backslash \{q\}$. Define a *configuration* on G to be an element of $\mathbb{Z}\tilde{V}$. We will refer to the coefficient of a vertex in a configuration as the number of chips at that vertex. The *degree* of the configuration will be the total number of chips on the graph. We will say that a vertex v *fires* if it loses $\deg(v)$ chips and all neighboring vertices gain one:

$$\sum_{w \in \tilde{V}} n_w w \to \sum_{w \in \tilde{V}} n_w w - \deg(v)v + \sum_{w \in N(v)} w.$$

We can also fire at q. In this case, all vertices neighboring q gain one chip and nothing happens at the vertex q, since we are not keeping track of the number of chips here.

We will say two configurations c and c' are *equivalent* (denoted $c \sim c'$) if we can get to one from the other via a sequence of chip firing moves.

Definition 3. A *superstable* configuration is one where every vertex is out of debt and there are no *legal set firings*, meaning that for all $S \subset \tilde{V}$ there is a $v \in S$ such that $c(v) < \mathrm{outdeg}_S(v)$.

Proposition 1. *Every configuration is linearly equivalent to exactly one superstable configuration.*

Theorem 3 (Merino)**.** *Let $I_G(x,y) = T_G(1,y)$. Let h_i be the number of superstables of G of degree i with respect to some sink vertex q. Then*

$$T_G(1,y) = \sum_{i=0}^{g} h_{g-i} y^i.$$

Proof. The proof is by induction on the number of edges, and uses the deletion/contraction definition of the Tutte polynomial. □

We note that this implies two important things about superstables:

1. All superstables have degree $\leq g$. This is because the x part of $T_G(x,y)$ must have a 0 exponent to contribute a nonzero term, requiring that A have the same number of connected components as G. Then $r(A) = |V| - k(G)$. Then the exponent on Y is $|A| - |V| + k(G)$, which is at most g.

2. The h_i do not depend on which vertex was chosen to be the sink.

3. The number of superstables is $T(1,1)$.

Example We know that the Tutte polynomial of C_3 is x^2+x+y, so $T_{C_3}(1,y) = 2+y$, which means that there are 2 superstables of degree 2 and 1 superstable of degree 0. Then we can easily find what these are.

Gessel's Formula

Theorem 4. *Let G be a connected graph with distinguished vertex v. If $V_i \subset V(G)\backslash\{v\}$, then let $G[V_i]$ be the induced subgraph of G at V_i. Let $\epsilon(V_i)$ be the number of edges from vertices in V_i to v. Then*

$$I_G(y) = \sum_{V_1,\ldots,V_k} \prod_{i=1}^{k} \left(1 + y + \cdots + y^{\epsilon(V_i)-1}\right) I_{G[V_i]}(y), \tag{1}$$

Where the sum is over all partitions $\{V_1, \ldots, V_k\}$ of $V(G)\backslash\{v\}$ such that $G[V_i]$ is connected for all i and $\epsilon(V_i) \geq 1$.

Example Compute $I_G(y)$ for the graph displayed below. We find:

Partition of $V\backslash\{v\}$	Contribution to $I_G(y)$
$\{\{1,2,3,4\}\}$	$(1+y)(y+2)$
$\{\{1,4\},\{2,3\}\}$	1
$\{\{1\},\{2,3,4\}\}$	1

We will demonstrate that we can compute Gessel's formula using the deletion-contraction method of computing the Tutte polynomial. We can compute $I_G(y)$ by performing the deletion-contraction process for computing the Tutte polynomial, and replacing all x's with 1's. Label the vertices of G with the numbers $1, \ldots, n$, and let n be the distinguished vertex. We will delete and contract edges of G in the following way:

1. Only delete or contract edges which do not contain the vertex n.

2. When an edge $\{i, j\}$ is contracted, label the resulting vertex $\{i, j\}$.

Once there are no more edges which do not contain the vertex n which can be deleted or contracted, stop.

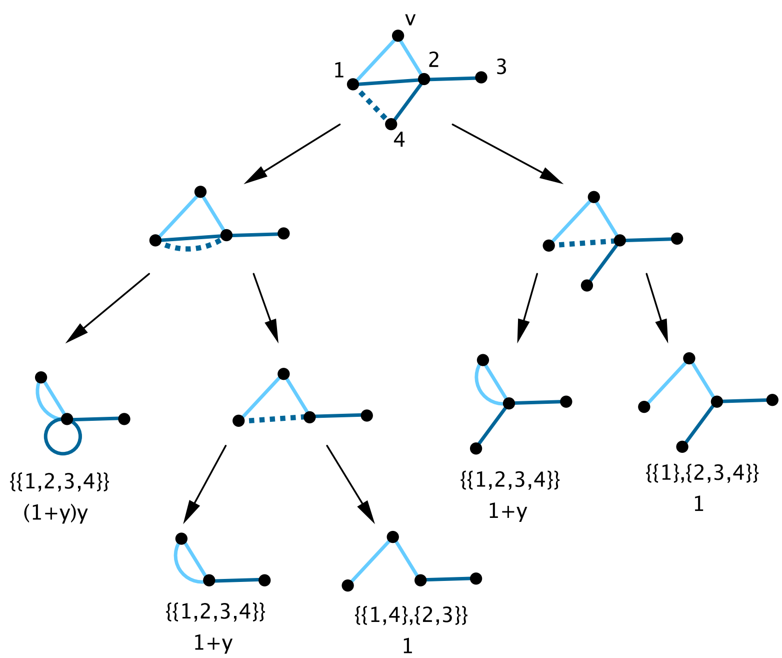


Figure 1: Computation of the Tutte polynomial using no edges connected to v. In the end, each graph is labeled with the partition $\{V_1, \ldots, V_2\}$ that it corresponds to, as well as its contribution to $I_G(y)$.

For each graph H at the end of a path in the computation of the Tutte polynomial, associate to it a partition $\{V_1, \ldots, V_k\}$ of $V \backslash \{v\}$ in the following way. Remove the vertex n from the graph. Then take the union of all vertices in any connected component of the resulting graph, and let this be one part in the partition.

If we add together all $I(y)$ which correspond to the same partition, we get the corresponding component in the sum of Gessel's formula. So, for the example, we have:

Partition of $V\backslash\{v\}$	Contribution to $I_G(y)$
$\{\{1,2,3,4\}\}$	$(1+y)(y+2)$
$\{\{1,4\},\{2,3\}\}$	1
$\{\{1\},\{2,3,4\}\}$	1

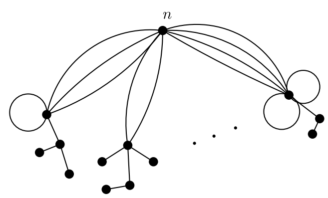
Which is the same as what we had before!

Why did this happen?

Let B_i be the graph with two vertices and i edges between them. Then we can show by induction that

$$T_{B_i} = x + y + \cdots y^{i-1}.$$

Graphs at the end of each path in the computation of the Tutte polynomial will be of the following form:



It will be a sequence of graphs $B_{i_1}, \ldots, B_{i_s}$ joined together at the vertex n with trees and loops at the other vertex. (This is because we usually get trees with loops, but since we are ignoring edges to the special vertex, we may also have multi edges from the usual terminal graphs to n). If n_j is the number of loops attached to one end of B_{i_j}, then the $I(y)$ polynomial of this graph is

$$\prod_{j=1}^{s} I_{B_{i_j}}(y) y^{n_j} = \prod_{j=1}^{s} (1 + y + \cdots + y^{i_j - 1}) y^{n_j}.$$

Then, $I_G(y)$ is the sum of these. Why is it that summing over terminal graphs corresponding to the same partition gives the competent of $I_G(y)$ corresponding to that partition?

We must consider what happens in the branches of the Tutte computation that give the partition $\{V_1, \ldots, V_k\}$. To obtain $\{V_1, \ldots, V_k\}$, all edges from a vertex of V_i to a vertex of V_j must be deleted. From that point, the possible deletions and contractions which preserve the partition are exactly those which are either also legal in $G[V_i]$ or do not alter the polynomial $I(y)$.

So, we have shown that the paths in the computation of $I_G(y)$ which preserve the partition $\{V_1, \ldots, V_k\}$ are in direct correspondence with the paths of the computation of $I_{G[V_1]\cup\cdots\cup G[V_k]}(y)$. Furthermore, contributions from the graphs in each path in both computations will be the same.

References

[Ges95] Ira M. Gessel. Enumerative applications of a decomposition for graphs and digraphs. *Discrete Mathematics*, 139(1–3):257 – 271, 1995.

[PC15] David Perkinson and Scott Cory. *Divisors and Sandpiles*. In progress, 2015.