Tensors are higher dimensional analogs of matrices. We will see that one way
to view a symmetric tensor is as a homogeneous polynomial. Basic attributes
of matrices, like eigenvectors, can be defined for tensors. This talk is split in
2 sections– symmetric and non symmetric tensors. For each, we review some
familiar aspects of matrices in preparation for the analogous concept for tensors.

Definition 1. A tensor is a \( d \)-dimensional array \( T = (t_{i_1, \ldots, i_d}) \). The entries are
elements of the ground field \( K \). The set of all tensors of format \( n_1 \times \cdots \times n_d \) form
a vector space of dimension \( n_1 \cdots n_d \) over \( K \).

1 Symmetric Tensors, Homogeneous Polynomials, Eigenvectors

1.1 Square Symmetric Matrices (d = 2)

Let \( K \) be a field. Recall that symmetric matrices correspond to quadratic forms.

Example 2. Let \( Q = 2x^2 + 7y^2 + 23z^2 + 6xy + 10xz + 22yz \). This is represented as
a symmetric \( 3 \times 3 \)-matrix as follows:

\[
Q = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

The gradient of a quadratic form is the vector of its partial derivatives. So, it
is a vector of linear forms, giving a map \( K^n \to K^n \).
Example 3. For the quadratic form we have from before, this is given by
\[ \nabla Q = \begin{pmatrix} \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial y} \\ \frac{\partial Q}{\partial z} \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \]

Then, a vector \( v \in \mathbb{K}^n \) is an \textbf{eigenvector} of \( Q \) if \( v \) is mapped to a scalar multiple of \( v \): \( \nabla Q v = \lambda v \), \( \lambda \in \mathbb{K} \). Replacing \( \mathbb{K}^n \) by projective space \( \mathbb{P}^{n-1} \), we obtain a rational self-map of projective space:
\[ \nabla Q : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}. \]

If \( Q \) is rank-deficient then the linear map has a kernel. These are places where the gradient \( \nabla Q \) vanishes. These are called \textbf{base points} of the map. If \( Q \) has full rank then \( \nabla Q \) is a regular map \( \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1} \) so it is defined on all of \( \mathbb{P}^{n-1} \).

Question 4. If you are interested in trying an example, do Question 1.

Remark 5. The eigenvectors of \( Q \) are the fixed points \((\lambda \neq 0)\) and base points \((\lambda = 0)\) of the gradient map \( \nabla Q \).

1.2 Symmetric Tensors

An \( n \times \cdots \times n \) tensor \( T = (t_{i_1, \ldots, i_d}) \) is called \textbf{symmetric} if it is unchanged after permuting the indices. Symmetric tensors correspond to homogeneous polynomials of degree \( d \) in \( n \) variables:
\[ T = \sum_{i_1, \ldots, i_d=1}^{n} t_{i_1, \ldots, i_d} \cdot x_{i_1} \cdots x_{i_d}. \]

Remark 6. For the rest of this section, it is more convenient to think of a tensor as a polynomial, NOT as an array.

As with matrices, the gradient of \( T \) defines a map \( \nabla T : \mathbb{K}^n \rightarrow \mathbb{K}^n \) (\( T \) is a homogeneous polynomial in \( n \) variables of degree \( d \)).

Definition 7. A vector \( v \in \mathbb{K}^n \) is an \textbf{eigenvector} of \( T \) if \((\nabla T)(v) = \lambda v \) for \( \lambda \in \mathbb{K} \).

Question 8. If you would like to compute an example, do Question 4.
If we again think of this map instead as a map on \( \mathbb{P}^{n-1} \), then the gradient map is a rational map from projective space to itself:

\[
\nabla T : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}.
\]

The eigenvectors of \( T \) are fixed points \((\lambda \neq 0)\) and base points \((\lambda = 0)\) of \( \nabla T \).

**Theorem 9** (Cartwright-Sturmfels). If \( K \) is algebraically closed, then the number of eigenvectors of a general \( d \)-dimensional \( n \times \cdots \times n \) symmetric tensor \( T \) is

\[
\frac{(d-1)^n - 1}{d-2} = \sum_{i=0}^{n-1} (d-1)^i.
\]

**Proof.** The proof is Question 5. \( \square \)

**Example 10.** \((n = d = 3)\) Consider the Fermat Cubic \( T = x^3 + y^3 + z^3 \). Its gradient map is the regular map that squares each coordinate:

\[
\nabla T : \mathbb{P}^2 \to \mathbb{P}^2, \ (x : y : z) \mapsto (x^2 : y^2 : z^2).
\]

This has \( 7 = 1 + 2 + 2^2 \) fixed points (all combinations of 1,0 minus all 0’s):

\[
(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 0), (1 : 0 : 1), (0 : 1 : 1), (1 : 1 : 1).
\]

Therefore, \( T \) has 7 eigenvectors, as the theorem predicts.

## 2 Rectangular Tensors, Multilinear Forms, Singular Vectors

### 2.1 Rectangular matrices \((d = 2)\)

For a rectangular matrix, one instead considers singular vectors. The number of singular vectors is equal to the smaller of the two matrix dimensions. Each rectangular matrix represents a bilinear form.

**Example 11.** Consider the following bilinear form.

\[
B = 2ux + 3uy + 5uz + 3vx + 7vy + 11vz = (u \ v) \left( \begin{array}{ccc}
2 & 3 & 5 \\
3 & 7 & 11 \\
\end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]
Its gradient defines an endomorphism on the direct sum of the row space and the column space. We get a map $\nabla B : \mathbb{K}^2 \oplus \mathbb{K}^3 \to \mathbb{K}^2 \oplus \mathbb{K}^3$ sending the pair

$$((u,v),(x,y,z)) \mapsto \left(\left(\frac{\partial B}{\partial u}, \frac{\partial B}{\partial v}\right), \left(\frac{\partial B}{\partial x}, \frac{\partial B}{\partial y}, \frac{\partial B}{\partial z}\right)\right) = \left(\left(2x + 3y + 5z, 3x + 7y + 11z\right), \left(2u + 3v, 3u + 7v, 5u + 11v\right)\right)$$

Let $B$ be an $m \times n$ matrix over $\mathbb{K}$. Consider the equations $Bx = \lambda y$, $B^t y = \lambda x$ for $\lambda \in \mathbb{K}$, $x \in \mathbb{K}^n$, $y \in \mathbb{K}^m$. Given a solution to these equations, we see that $x$ is an eigenvector of $B^t B$, $y$ is an eigenvector of $BB^t$, and $\lambda^2$ is a common eigenvalue. We call $x, y$ the right and left singular vector.

**Remark 12.** The singular pairs $(x, y)$ of a rectangular matrix $B$ are fixed points of the gradient map $\nabla B$ of the associated bilinear form. This is now a self-map on the product of projective spaces:

$$\nabla B : \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$$

**Question 13.** For those interested in computing an example, do Question 2.

### 2.2 Rectangular Tensors

Consider now a $d$-dimensional tensor $T$ in $\mathbb{K}^{n_1 \times \cdots \times n_d}$. It corresponds to a multi-linear form.

**Definition 14.** The singular vector tuples of $T$ are the fixed points of the gradient map

$$\nabla T : \mathbb{P}^{n_1-2} \times \cdots \times \mathbb{P}^{n_d-2} \to \mathbb{P}^{n_1-2} \times \cdots \times \mathbb{P}^{n_d-2}.$$ 

**Example 15.** The trilinear form $T = x_1y_1z_1 + x_2y_2z_2$ is interpreted as a $2 \times 2 \times 2$ tensor. The gradient $\nabla T$ of this trilinear form is the rational map

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

$$((x_1 : x_2), (y_1 : y_2), (z_1 : z_2)) \mapsto ((y_1z_1 : y_2z_2), (x_1z_1 : x_2z_2), (x_1y_1 : x_2y_2)).$$

This map has six fixed points, for example $((1 : 0), (1 : 0), (1 : 0))$, and others. These are the singular vector triples of the tensor $T$. 

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The expected number of singular vector triples is predicted by the following theorem.

**Theorem 16** (Friedland and Ottaviani). For a general $n_1 \times \cdots \times n_d$-tensor $T$ over an algebraically closed field $K$, the number of singular vector tuples is the coefficient of the monomial $z_1^{n_1-1} \cdots z_d^{n_d-1}$ in the polynomial

$$
\prod_{i=1}^d \frac{(\hat{z}_i)^{n_i} - z_i^{n_i}}{\hat{z}_i - z_i},
$$

where $\hat{z}_i = z_1 + \cdots + z_{i-1} + z_{i+1} + \cdots + z_d$.

**Example 17.** (Question 3) Consider the $3 \times 3 \times 2 \times 2$ tensor defined by the multilinear form $T = x_1 y_1 z_1 w_1 + x_2 y_2 z_2 w_2$.

Computing the polynomial in the above theorem and examining the coefficient of the monomial $x_1^2 y_1^2 z_1 w_1$, we expect that there are 98 singular vector tuples for $T$.

We will now determine all singular vectors of $T$. The gradient map sends

$$
((x_1 : x_2 : x_3), (y_1 : y_2 : y_3), (z_1 : z_2), (w_1 : w_2)) \mapsto
((y_1 z_1 w_1 : y_2 z_2 w_2 : 0), (x_1 z_1 w_1 : x_2 z_2 w_2 : 0), (x_1 y_1 w_1 : x_2 y_2 w_2), (x_1 y_1 z_1 : x_2 y_2 z_2)).
$$

What are the fixed points of this map? First, we observe that $x_3, y_3 = 0$.

If $x_1 = 0$: Then $y_1 = z_1 = w_1 = 0$, so the only solution is $((0, 1, 0), (0, 1, 0), (0, 1), (0, 1))$.

If $x_1 \neq 0$: Then $y_1 z_1 w_1 \neq 0$. So, we may set $x_1 = y_1 = z_1 = w_1 = 1$. Then we obtain:

$$
((1 : x_2 : 0), (1 : y_2 : 0), (1 : z_2), (1 : w_2))
= ((1 : y_2 z_2 w_2 : 0), (1 : x_2 z_2 w_2 : 0), (1 : x_2 y_2 w_2), (1 : x_2 y_2 z_2)).
$$

Macaulay2 (degree + primary decomposition) reveals that there are 17 solutions, and 9 of them are real. So in total, we have 18 singular vector tuples.