Packing Polynomials on Sectors of $\mathbb{R}^2$

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Let $I \subset \mathbb{Z}^2$. A **packing polynomial** on $I$ is a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \mid_I$ is a bijection from $I$ to $\mathbb{N}$. 

**Introduction**
Let \( I \subset \mathbb{Z}^2 \). A **packing polynomial** on \( I \) is a polynomial \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( f \mid_I \) is a bijection from \( I \) to \( \mathbb{N} \).

The **Cantor Polynomials**:

\[
\begin{align*}
   f(x, y) &= \frac{(x+y)^2}{2} + \frac{x+3y}{2}, \\
   g(x, y) &= \frac{(x+y)^2}{2} + \frac{3x+y}{2}.
\end{align*}
\]

Fueter and Pólya proved that these are the only quadratic packing polynomials on \( \mathbb{N}^2 \).
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- $\alpha \in \mathbb{N}$: Solved by Stanton.
- $\alpha \not\in \mathbb{Q}$: Nathanson conjectured that there are no packing polynomials on $I(\alpha)$.
- $\alpha \in \mathbb{Q}$: we solved.
Example:
This is a packing polynomial on $I(8/5)$, and

$$p(x, y) = 4 \left( x - \frac{y}{2} \right)^2 - x + y.$$
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Definition: Let \( p \) be a quadratic packing polynomial on \( I\left(\frac{n}{m}\right) \). Then \( p \) is a \textit{k-stair} polynomial if for any two consecutive integral points \( r, s \) along a line with slope \( \frac{n}{m-1} \), we have \( p(r) - p(s) = \pm k \).
Theorem (Stanton)

Let $n/m \geq 1$, and $(n, m) = 1$. If $I(n/m)$ has a quadratic packing polynomial $p$, then $n | (m - 1)^2$ and

$$p(x, y) = \frac{n}{2} \left( x - \frac{m-1}{n} y \right)^2 + \text{linear terms.}$$
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This implies that all packing polynomials on sectors $I(n/m)$ are $k$-stair polynomials for some $k$. 

PREVIOUS RESULT
We will say that two packing polynomials \( p \) on \( I(\alpha) \) and \( q \) on \( I(\beta) \) are \textbf{equivalent} if there exists a linear bijection \( T \) from \( I(\alpha) \) to \( I(\beta) \) such that

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p = q \circ T.
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\( k \)-stair to \(-k\)-stair
**Equivalence**

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\( n/m < 1 \) to \( n/m \geq 1 \):
Properties of $k$-Stair Polynomials

A 3-stair packing polynomial on $I(12/7)$:
Main Result: Necessary Form

Let \( l = \gcd(n, m-1) \).

Theorem (Brandt)

Let \( p \) be a \( k \)-stair packing polynomial on \( I(n/m) \), where \( m \neq 1 \). Then (up to equivalence) \( k \equiv \frac{m-1}{l} \mod \frac{n}{l} \), and

\[
p(x, y) = \frac{n}{2} \left( x - \frac{m-1}{n} y \right)^2 + \left( 1 - \frac{kl}{2} \right) x + \frac{2(1-m)+kl(m+1)}{2n} y + c.
\]

The expression for \( p(x, y) \) only depends on \( n, m, \) and \( k \).
Theorem (Brandt)

The following results give the $k$-stair packing polynomials on sectors $I(\frac{n}{m})$ for all $k$ (up to equivalence).
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3. 3-stair polynomials: $m \equiv 10 \mod 27$ or $m \equiv 19 \mod 27$ and $n = \frac{1}{27} (m - 1)^2$. 
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3. 3-stair polynomials: $m \equiv 10 \mod 27$ or $m \equiv 19 \mod 27$ and $n = \frac{1}{27}(m - 1)^2$.
4. There are no $k$-stair packing polynomials for $k \geq 4$. 
1. Prove that there are no packing polynomials of degree greater than 2 on sectors of $\mathbb{R}^2$.

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2. Prove that there are no packing polynomials on irrational sectors. (Nathanson)
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