Bounding Projective Dimension

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1 Introduction

For the entire talk we let K be an algebraically closed field, and R be a polynomial ring in N variables over K. Let the degree of each variable be 1. We call a homogeneous polynomial a *form*.

During this talk, we will discuss the *complexity* of an ideal, and we will approach this problem through the perspective of computing a minimal free resolution for the ideal. The projective dimension of the ideal is one invariant of such a resolution, and counts the number of steps needed to find a minimal resolution. The following question due to Stillman asks about how to bound this dimension.

Conjecture 1 (Stillman's Question)**.** *Given an ideal I generated by n forms of degree at most d in R, is there a bound of the projective dimension $pd_R(R/I)$ which depends on n, d, but not N?*

We follow slides from a talk given at Berkeley on July 17, as well as [MS13] for an overview of known results. The content of this talk and some subsequent talks is based on the work done in the paper [AH16]. A good reference for any background material is [Eis95].

Definition 1. Given a finitely generated R-moudule M, a *free resolution* $F_\bullet$ of M is an exact sequence of the form

$$\cdots \xrightarrow{d_{s+1}} F_s \xrightarrow{d_s} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

Where all F_i are free and $M = F_0/\mathrm{im}(d_1)$. The *length* of a resolution if the greatest integer n such that $F_n \neq 0$.

Why would one study free resolutions? For one, you can read off the Hilbert polynomial, degree of the variety, Betti numbers, and many other useful things.

Example. Let $R = k[x, y, z]$ and take $I = (x, y, z)$. Then a free resolution for I is given by

$$0 \to R \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} R.$$

This is an example of a *Koszul complex*.

Remark. One can easily compute free resolutions symbolically using Gröbner bases, using your favorite computer algebra system, like `Macaulay2`. The algorithm works in a clever way due to Schreyer [Sch91]. Essentially, in computing a Gröbner bases for I one computes something called *S-pairs*, and these pairs provide you with a generating set for the relations on the generators of I. Iterating this process gives the full resolution. For more details, see [Bra15] or [Eis95].

Definition 2. The *projective dimension* of a module M, denoted $pd(M)$ is the minumum length of all free resolutions of M.

Remark. Over R every finitely generated projective module is free, which is why we are calling the length of a free module the projective dimension. Usually, one would make this definition using resolutions by projective modules.

Audience Participation. (continued) What is the projective dimension of the ideal I from the previous example? The projective dimension of I from the previous example is 3.

Audience Participation. What does it mean if a finitely generated module has projective dimension 0? It means that the module is free.

Theorem 1 (The Hilbert Syzygy Theorem,1890)**.** *Every finitely generated graded module M over R has a graded free resolution of length $\leq N$, the number of variables of R. Hence, $pd(M) \leq N$.*

Contrast this to the statement of Stillman's conjecture. The bound given here is tight; for example, the graded maximal ideal $m = (x_1, \ldots, x_N)$ has projective dimension N and minimal free resolution given by the Koszul complex.

Clarification. We will be studying homogeneous ideals I in R. We will always speak of $pd(R/I)$ and note that $pd(R/I) = pd(I) + 1$ for all ideals I.

Remark. The *Castelnuovo-Mumford regularity* is another way to measure the complexity of M, and is the largest degree of a generator of any of the F_i. There is a question which is equivalent (due to Caviglia) to Stillman's conjecture, which asks if there is a bound on the regularity of R/I depending only on the degrees of the generators of I. There is a doubly exponential bound in terms of N.

History. Pretend for a moment that the current date is October 23, 2016. We now discuss some cases in which Stillman's Question is answered in the affirmative (none of the below involve the number of variables N)....

1. When I is principal, what is the $pd(R/I)$? It is 1.

2. When there are 2 generators, one can show that $pd(R/I) = 2$ (What is the free resolution?).

3. When I is a monomial ideal, the projective dimension is $\leq n$, where n is the number of monomials.

4. When I is generated by 3 quadratic forms the projective dimension is at most 4 (Eisenbud-Huneke, unpublished). The bound is also tight.

5. When I is generated by 3 cubic forms, the projective dimension is at most 36 (Engheta, in 3 papers). This bound is described as "likely not tight". The largest known projective dimension for an ideal of this form is 5.

6. When I is an ideal generated by m linear and n quadratic polynomials, the projective dimension is bounded with asymptotic order of magnitude $2(m+n)^{2(m+n)}$ (Ananyan & Hochster). These bounds are not tight.

2 AH Theorem

Definition 3. A *regular sequence* in R is an ordered collection of elements $r_1, \ldots, r_B$ in R such that r_i is a nonzero divisor on $R/(r_1, \ldots, r_{i+1})$.

Theorem 2 (AH)**.** *There is a bound $B_d(n)$ such that any n given forms of degree at most d are contained in a polynomial K-subalgebra A of R generated by $B_d(n)$ forms of degree at most d that form a regular sequence.*

$AH \Rightarrow SC$. Let I be a homogeneous ideal in R generated by n forms of degree at most d. Let $r_1, \ldots r_B$ be the $B_d(n)$ forms of degree at most d forming a regular sequence in R, with the property that $I \subset (r_1, \ldots r_B) := A$. Since $I \subset A$, and A has $B_d(n)$ variables, the length of the minimal free resolution $F.$ of A/I over A is at most $B_d(n)$ (by Hilbert Syzygy Theorem). If we take $F. \otimes R$, this will be exact since R is free over A, and this gives a resolution of length at most $B_d(n)$ for $R \otimes A/I = R/I$. So summarize,

$$pd_R(R/I) \leq pd_A(A/I) \leq B_d(n).$$

□

Example. Why is the degree needed when giving bounds? There is a theorem by Burch and Kohn which says that for $2N$ variables, there exists an ideal $I = (f, g, h)$ with projective dimension $N + 2$. Here is one such choice.

Let $R = K[x_1, \dots, x_N, y_1, \dots, y_N]$ and let

$$f = \prod_{i=1}^{N} x_i, \quad g = \prod_{i=1}^{N} y_i, \quad h = \sum_{i=1}^{N} \prod_{j=1, j\neq i}^{N} x_j y_j.$$

For example, if $N = 3$ then we have

$$I = (x_1x_2x_3, y_1y_2y_3, x_2y_2x_3y_3 + x_1y_1x_3y_3 + x_1y_1x_2y_2).$$

The projective dimension of R/I is 5, and it has resolution

$$0 \to R \to R^6 \to R^{12} \to R^9 \to R^3 \to R.$$

There are even examples of families of ideals with exponentially growing projective dimension relative to the number of generators.

3 K-strong

One of the main tools used was the notion of *collapse* and *strength*.

We say that a non-linear form F has *k-collapse* if it is contained in an ideal generated by k or fewer forms of strictly smaller positive degree. Then, we can write

$$F = G_1H_1 + \cdots + G_kH_k.$$

If F does not exhibit k-collapse, then we say that F is *k-strong*. Then, F has *strength* at least k. The *strength* of F is the largest integer k such that F is k-strong. We define the strength of a linear form to be ∞.

Audience Participation. When does a form have a 1-collapse?

Answer: if and only if it is reducible. So, x_1x_2 is not 1-strong. We do have $x_1x_2+x_3^2$ is 1-strong.

In this way, we can think of strength as measuring how irreducible a form is.

Example. The form $x_1x_2+x_3x_4+x_5x_6$ is 2-strong because it is irreducible. However, it is not 3-strong since it has a 3-collapse.

We say that a K-vector space in R consisting of forms of the same degree is *k-strong* if all of its nonzero elements are k-strong. We impose this condition element-wise.

A finite dimensional graded vector space is k strong if every graded component is k-strong. So, every nonzero form in it is k strong.

Example. Take n linear combinations of $x_1, \dots, x_k + 1$ with new indeterminates as coefficients. All elements of the vector space they span are k-strong, but not $k + 1$ strong.

References

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[Sch91] Frank-Olaf Schreyer, *A standard basis approach to syzygies of canonical curves*, J. Reine Angew. Math. **421** (1991), 83–123. MR 1129577 (92j:14040)