Chromatic Number of the Kneser Graph

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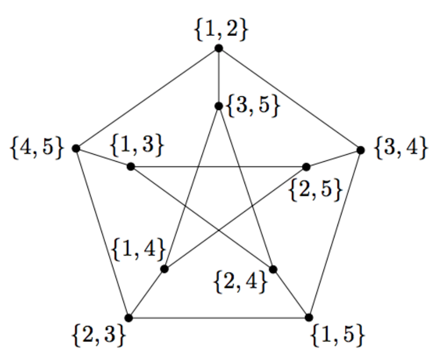
Introduction

Definition 1. A *proper coloring* of a graph G is a function $c : V(G) \to \{1, \ldots, t\}$ such that for any edge $e \in E(G)$, we have that $|c(e)| = 2$. In this case, we say that G is *t-colorable.* The *chromatic number* of a graph G is the smallest t such that G is t-colorable.

Definition 2. The *Kneser Graph* $K(n, k)$ is defined in the following way. The vertex set is the set of all n-subsets of the set $\{1, \ldots, 2n + k\}$, and two vertices v and v' are adjacent if and only if $v \cap v' = \emptyset$.

Example The *Petersen graph* is the Kneser graph $K(2, 1)$.



Conjecture 1 (Kneser, 1955)**.** *The chromatic number of the Kneser graph is* $k + 2$*.*

This was proved by Lovász in 1978 using the Borsuk-Ulam theorem [Lov78]. Since then, Bárány (1978) has shortened the proof in [Bár78] by using a result of Gale [Gal56]. In 2002, Greene gave a short proof in [Gre02] that relies only on the Borsuk-Ulam theorem, and in 2004 Matoušek gave a purely combinatorial proof in [Mat04].

Remark It is easy to show that $K(n, k)$ is $(k + 2)$-colorable. Let

$$C_i = \{v \in V \mid i \text{ is the smallest number in } v\},$$

and let the color classes be

$$\{C_1, \ldots, C_{k+1}, C_{k+2} \cup \cdots \cup C_{k+n+1}\}.$$

We claim that this is a proper $(k+2)$-coloring. There will only be a problem if there are two n-sets in the same color class which are disjoint. Clearly, this cannot happen for any of the $\{C_1, \ldots, C_{k+1}\}$. The final class, $C_{k+2} \cup \cdots \cup C_{k+n+1}$, is on $2n+k-(k+1) = 2n-1$ numbers, so by the pigeon hole principle, any two vertices in this class will intersect in at least one number.

So, what remains to be proved in the conjecture is that there is no proper $(k+1)$-coloring of $K(n, k)$.

Ingredients

We are going to present Greene's proof of Kneser's conjecture, but first we will discuss the ingredients: the Borsuk-Ulam theorem and some more definitions.

Theorem 1 (Borsuk-Ulam)**.** *If $S^k = F_1 \cup \cdots \cup F_{k+1}$, where $F_1, \ldots, F_{k+1}$ are each either open or closed subsets of S^k, then one of the F_i contains two antipodal points.*

We proved this for closed sets and $k = 2$ in class. We will assume it holds for all k, and show that we can assume the F_i are either open or closed. To do this, we will induct over the number t of closed sets in the cover of S^k.

For the base case, let $t = 0$. Then the sets $F_1, \ldots, F_{k+1}$ are open. Select a positive number λ such that for all $x \in S^k$, the closed ball $\bar{B}(x, \lambda)$ is in some F_j. By compactness, there exists a finite collection of points $\{x_1, \ldots, , x_s\}$ such that the open balls $B(x_i, \lambda)$ cover S^m. For each j, let H_j be the union of the balls $\bar{B}(x_i, \lambda)$ contained in F_j. Then H_j is closed, $H_j \subset F_j$, and the H_j cover S^k, so the regular Borsuk-Ulam theorem implies that one of the H_j, and hence one of the F_j, contains a pair of antipodes.

Now assume that $0 < t < k+1$ and that the theorem holds when we have fewer than t closed sets. Let $\{F_1, \ldots, F_{k+1}\}$ be a cover of S^k where exactly t sets are closed and the rest are open. Suppose F_1 is closed, and that F_1 does not contain a pair of antipodes. Then its diameter is $2-\epsilon$ for some ϵ (the diameter is defined as the maximal distance between two points in F_1). Let U be the open set containing all points in S^k whose distance from F_1 is less than $\epsilon/2$. Then $\{U, \ldots, F_{k+1}\}$ is a cover of S^k by $k+1$ sets with exactly $t-1$ closed sets. Then by the inductive hypothesis, one of them contains a pair of antipodes. By construction, U does not contain such a pair, and so some set in the original $\{F_1, \ldots, F_{k+1}\}$ must contain the antipodes, as desired.

Definition 3. Let $a \in S^k$. Then define

$$H(a) = \{x \in S^k \mid \langle x, a\rangle > 0\}.$$

This is the open hemisphere centered at a. Let

$$S(a) = \{x \in S^k \mid \langle x, a \rangle = 0\},$$

the boundary of $H(a)$. Call $S(a)$ a *great* $(k-1)$*-sphere* on S^k.

Definition 4. Let us say a placement of points on S^k is *in general position* if no $k+1$ points lie on a great $(k-1)$-sphere.

Proof of Kneser's Conjecture

We are now ready to discuss the following proof of Lovász' Conjecture, which was given in 2002 by Greene in [Gre02].

Let $W = \{1, \dots, 2n+k\}$, and distribute the points of W on S^{k+1} in general position. Think of elements of W as being in correspondence with the ground set of the Kneser graph $K(n,k)$. Suppose the $K(n,k)$ has a proper $(k+1)$-coloring, c. Let

$$C_i = \{x \in S^{k+1} \mid v \subset H(x),\ v \in V(K(n,k)),\ c(v) = i\}.$$

We claim that C_i is open for all $i \in \{1, \dots, k+1\}$. Let $x \in C_i$. There is some v such that $c(v) = i$, and $v = \{x_1, \dots, x_n\}$, where $\{x_1, \dots, x_n\} \subset H(x)$. Let l be the infimal distance from any x_i to any point on $S(a)$. Then $l > 0$, so if $y \in B(l, x)$, then $\{x_1, \dots, x_n\} \subset H(y)$, so that $y \in C_i$. Hence, C_i is open.

Now, let

$$F = S^{k+1} \backslash (C_1 \cup \cdots \cup C_{k+1}).$$

Then F is closed, and the collection $\{C_1, \dots, C_{k+1}, F\}$ covers S^{k+1}, so by the Borsuk-Ulam theorem, one of them contains antipodes $a, -a$. The antipodes cannot be contained in F, because if they were, then $H(a)$ and $H(-a)$ did not receive colors, so at most $n-1$ points of W in each, meaning that

$$2n + k - 2(n-1) = k + 2$$

elements of W are in the set $S^{k+1} \backslash (H(a) \cup H(-a)) = S(a)$, contradicting the general position. Therefore, for some $i \in \{1, \dots, k+1\}$, we have that $a, -a$ are in C_i. Then $H(a)$ and $H(-a)$ contain n-element subset of V which are of the same color class, but these are disjoint, contradicting the fact that c was a proper coloring.

References

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