249 Replacement Week 14 Problems

May 2, 2016

1. (a) Prove that if $\varphi : G \to GL(V)$ is a complex linear representation of a finite group, and χ is the character, then $\chi(g)$ is the sum of $\dim(V)$ many $|g|^{\text{th}}$ roots of unity.

   (b) Prove that $\chi(g^{-1}) = \overline{\chi(g)}$. Conclude that $\chi(g)$ is real if g is conjugate to its inverse.

2. Let ρ be a one-dimensional representation of a finite group G and σ some other representation of G. Show that σ is irreducible if and only if $\rho \otimes \sigma$ is irreducible.

3. Compute the character tables of A_3, A_4, A_5 and S_5.

4. Show that G is abelian if and only if all the irreducible representations of G are one-dimensional.

5. Find the character tables of

   (a) the dihedral group D_8 of order 8;

   (b) the quaternion group $H_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = ijk = -1$, and signs multiply according to the usual rules.

   Show that H_8 and D_8 are not isomorphic, and conclude that the character table of a finite group need not determine the group.

6. Let G be a finite group and let $\rho : G \to GL_2(\mathbb{C})$ be a representation of G. Suppose that there are elements $g, h \in G$ such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.