Symmetric Functions (Problem Set)

Problem 1. *(Stanley, Problem 7.3) Expand the power series* $\prod_{i\geq 1}(1+x_i+x_i^2)$ *in terms of the elementary symmetric functions.*

Problem 2. *(Stanley, Problem 7.4) Show that*

$$h_r(x_1,\dots,x_n) = \sum_{k=1}^{n} x_k^{n-1-r} \prod_{i\neq k}(x_k - x_i)^{-1}.$$

Problem 3. *(Stanley, Problem 7.8) Let* $f \in \Lambda^n$*, and for any* $g \in \Lambda^n$ *define* $g_k \in \Lambda^{nk}$ *by*

$$g_k(x_1, x_2, \dots) = g(x_1^k, x_2^k, \dots).$$

Show that

$$\omega f_k = (-1)^{n(k-1)}(\omega f)_k.$$

Problem 4. *(Stanley, Problem 7.9) Let* λ *be a partition of* n *of length* ℓ*. Define the* forgotten symmetric function f_λ *by*

$$f_\lambda = \epsilon_\lambda \omega(m_\lambda),$$

where $\epsilon_\lambda = (-1)^{n-\ell}$ *as usual. Let* $f_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$*. Show that* $a_{\lambda\mu}$ *equals the number of distinct permutations* $(\alpha_1,\dots,\alpha_\ell)$ *of the sequence* $(\lambda_1,\dots,\lambda_\ell)$ *such that*

$$\{\alpha_1+\cdots+\alpha_i : 1\leq i\leq \ell\} \supseteq \{\mu_1+\cdots+\mu_j : 1\leq j\leq \ell(\mu)\}.$$

Problem 5. *For* $\lambda \vdash n$*, let* $h_\lambda = \sum_{\mu\vdash n} N_{\lambda\mu} m_\mu$*. Show that*

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda\in \mathrm{Par}} m_\lambda(x) m_\lambda(y).$$