249 Replacement Week 12 Problems

April 18, 2016

1. Recall that given a poset P, we let $\hat{P}$ denote P with a $\hat{1}$ and $\hat{0}$ adjoined, and we let $E(P)$ denote the set of edges of the Hasse diagram of P. Let P be a finite poset such that $\lambda : E(\hat{P}) \to \Lambda$ is an EL-labelling for $\hat{P}$ (Λ any poset).

   (a) Let $\mathbf{m} : \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ be a maximal chain. Denote $\lambda(\mathbf{m}) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{n-1}, x_n))$. Show that $\lambda(\mathbf{m})$ has a descent at $e \in [n-1]$ (i.e. $\lambda(x_{e-1}, x_e) > \lambda(x_e, x_{e+1})$) iff there is some maximal chain $\mathbf{h}$ such that $\mathbf{h} \cap \mathbf{m} = \mathbf{m} - \{x_e\}$ and $\lambda(\mathbf{h}) < \lambda(\mathbf{m})$ in the lexicographic order.

   (b) Prove that $\Delta(\hat{P})$ is shellable with the lexicographic ordering of the maximal chains. More precisely, prove that if $\mathbf{k}, \mathbf{m}$ are maximal chains with $\mathbf{k} < \mathbf{m}$, then show that there is a maximal chain $\mathbf{h} < \mathbf{m}$ such that $\mathbf{k} \cap \mathbf{m} \subseteq \mathbf{h} \cap \mathbf{m}$ and $|\mathbf{h} \cap \mathbf{m}| = |\mathbf{m}| - 1$.

2. Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$ and of rank n with rank function $\rho : P \to [0, n]$. Given $S \subseteq [0, n]$, define the poset

$$P_S = \{t \in P \mid \rho(t) \in S\}$$

   For example, $P_\varnothing = \varnothing$ and $P_{[0,n]} = P$.

   (a) Let $\alpha_P(S)$ be the number of maximal chains of P_S. Define

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T) \quad \xleftrightarrow{\text{equiv. by } \mu\text{-inv.}} \quad \alpha_P(S) = \sum_{T \subseteq S} \beta_P(T)$$

   Show that $\beta_P(S)$ is equal to the number of maximal chains $\mathbf{m}$ of $\hat{P}$ for which the sequence $\lambda(\mathbf{m})$ has descent set S. (Hint: consider a function $M_S \to M$ where M_S, M denote the set of maximal chains of P_S, P, respectively. What is the image?).

   (b) Recall (or prove) Philip Hall's theorem:

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots$$

   where c_i is the number of chains $\hat{0} = t_0 < t_1 < \cdots < t_i = \hat{1}$ of length i between $\hat{0}$ and $\hat{1}$. Using this, show that $\beta_P(S) = (-1)^{|S|-1} \mu_S(\hat{0}, \hat{1})$, where μ_S denotes the Möbius function of $\hat{P}_S = P_S \cup \{\hat{0}, \hat{1}\}$. Conclude that $(-1)^n \mu_P(\hat{0}, \hat{1})$ is the number of maximal chains $\mathbf{m}$ for which $\lambda(\mathbf{m})$ is non-increasing.

3. Recall that for any positive integer n, the partition lattice $\prod_n$ is the poset of all partitions of $[n]$ (into blocks), where we define $\pi \leq \sigma$ in $\prod_n$ if and only if each block of π is contained in a block of σ. (In other words, π is a refinement of σ.)

(a) Find an EL-labeling of $\prod_n$ (and prove that it is one). Then identify the homotopy-type of the order complex $\Delta(\prod_n - \hat{0} - \hat{1})$.

(b) Use the previous problem to calculate $\mu(\prod_n)$.

4. Consider S_n with the Bruhat ordering. Let $\Lambda = \{(i,j) \in [n] \times [n] \mid i < j\}$, totally ordered by $(i,j) < (r,s)$ if $i < r$ or if $i = r$ and $j < s$. Let $\lambda : E(S_n) \to \Lambda$ be the labelling $\lambda(\sigma,\tau) = (i,j)$ if i,j are interchanged in σ to obtain τ and $i < j$. For example in S_3 we have $\lambda(123, 213) = (1,2)$ and $\lambda(213, 312) = (2,3)$. Show that λ is an EL-labelling.