

249 Replacement Week 7 Problems

March 7, 2016

Problems from Stanley, EC Volume I.

3.33, 3.66abc, 3.72a, 3.74

Other problems (taken from previous courses by L. Williams and M. Haiman).

1. Let P be a finite poset and $m \in \mathbb{N}$. Let \underline{m} be the chain poset $1 < 2 < \dots < m$. Show that the following numbers are equal:

- (a) The number of surjective order-preserving maps $\sigma : P \rightarrow \underline{m}$.
- (b) The number of chains $\hat{0} = I_0 < I_1 < \dots < I_m = \hat{1}$ of length $m \in J(P)$.

2. (a) Given a finite lattice L , define $I(L)$ to be the poset of all intervals $[x, y] \subseteq L$, plus the empty set, ordered by containment. Show that $I(L)$ is a lattice and describe its meet and join.

- (b) Show that the Möbius function in $I(L)$ is given by

$$\begin{aligned}\mu([x, y], [w, z]) &= \mu_L(w, x)\mu_L(y, z) \quad \text{for nonempty } [x, y], \\ \mu(\emptyset, [w, z]) &= -\mu_L(w, z).\end{aligned}$$

- (c) Prove *Crapo's lemma*: let X be a subset of L , and let n_k be the number of k -element subsets of X with join equal to $\hat{1}$ and meet equal to $\hat{0}$. Then

$$\sum_k (-1)^k n_k = -\mu(\hat{0}, \hat{1}) + \sum_{\substack{x \leq y \\ [x, y] \cap X = \emptyset}} \mu(\hat{0}, x)\mu(y, \hat{1}).$$

- (d) Prove the identity in Stanley, Exercise 3.92, known as *Crapo's complementation theorem*.