249 Replacement Week 4 Problems

February 17, 2016

1. Recall the species E of sets defined by $E[U] = \{U\}$ and the species X of characteristic of singletons defined by $X[U] = \{U\}$ if $|U| = 1$. Let $\mathcal{A}$, $\mathcal{A}_L$ denote the species of rooted and ordered rooted trees, respectively.

   (a) Show that $\mathcal{A} = X \cdot E(\mathcal{A})$.

   (b) Show that $\mathcal{A}_L = X \cdot L(\mathcal{A}_L)$, where L is the species of linear orderings.

2. Two species F and G are *equivalent* if they are naturally isomorphic as functors. More precisely, this means that one can give for each finite set S a bijection $\psi_S \colon F(S) \to G(S)$ such that for all $\sigma : S \to T$, one has $G(\sigma) \circ \psi_S = \psi_T \circ F(\sigma)$.

   (a) If L is the species of linear orderings, and P is the species of permutations, show that L and P are not equivalent.

   (b) The *Hadamard product* $F * G$ of two species is the species $(F * G)(S) = F(S) \times G(S)$. Show that the species $L * L$ and $P * L$ are equivalent. This can be understood as an explanation of the fact that the inequivalent species L and P have the same exponential generating function.

3. Given any species $\mathcal{F}$, we can define a new species $\mathcal{T}_\mathcal{F}$ such that an element of $\mathcal{T}_\mathcal{F}(X)$ consists of a rooted tree T with vertex set X, together with an $\mathcal{F}$-structure on the set of children of each vertex (these $\mathcal{F}$ structures to be chosen independently).

   (a) If $F(x), T_F(x)$ denote the exponential generating functions for these species, show that
   $$T_F(x) = xF(T_F(X))$$
   For exampe, when $\mathcal{F}$ is the trivial species, so $F(x) = e^x$, this gives the identity we saw previously for the rooted trees generating function $T(x)$.

   (b) Select the species $\mathcal{F}$ so that $\mathcal{T}_\mathcal{F}$ is rooted ordered trees, and use (a) to solve for the generating function for these trees. How is it related to the ordinary generating function for unlabelled ordered rooted trees, and why?

   (c) Select the species $\mathcal{F}$ so that $\mathcal{T}_\mathcal{F}$ is strictly binary trees (i.e., unordered labelled rooted trees in which every node has either zero or two children), and use (a) to solve for their exponential generating function.

4. A map $f \colon S \to S$ is *idempotent* if $f^2 = f$. Find the exponential generating function for the species of idempotent maps.

5. (a) Find an explicit formula for the cycle index Z_I of the species of involutions, $I(S) = \{\text{involutions } \sigma \colon S \to S\}$.

(b) Evaluate $Z_I[x]$ and verify that it agrees with the obvious ordinary generating function counting involutions up to conjugacy.

6. Recall (or show) that the cycle index of the trivial species is given by

$$Z_E = \exp \sum_{n=1}^{\infty} p_n/n.$$

Verify that the plethysm $Z_E * Z_C$ agrees with the formula we obtained by direct calculation for the cycle index of the species of permutations,

$$Z_P = \prod_{n=1}^{\infty} \frac{1}{1-p_n}.$$

Questions 7-9 deal with the species of perfect matchings.

7. A *perfect matching* on a set S of $2n$ elements is a partition of S into n blocks of two elements each. Perfect matchings form a species M, with $M(S) = \emptyset$ if $|S|$ is odd.

   (a) Find the exponential generating function for the species of perfect matchings.

   (b) Deduce algebraically that the number of perfect matchings on a set of $2n$ elements is $n!!$. Here and below the 'double factorial' notation $n!!$ stands for the product $(2n-1)(2n-3)\cdots 3\cdot 1$ of the first n odd numbers.

   (c) Give a direct counting argument for the result in (b).

8. Let $e(2n)$ be the number of permutations σ of a set of $2n$ elements with the property that every cycle of σ has even length.

   (a) Find the exponential generating function $\sum_n e(2n)x^{2n}/(2n)!$.

   (b) Deduce that $e(2n) = (n!!)^2$.

9. (a) From the preceding problems it follows that the number of permutations of a $2n$ element set S with only even cycles is equal to the number of pairs of perfect matchings on S. Construct a direct bijection between the two (to do this you will probably need to fix the set S to be $[2n]$ and use the numerical values of its elements to make some auxiliary choices).

   (b) Show that the species of permutations with even-length cycles and the species of pairs of perfect matchings are not equivalent. This can be understood as explaining the need for auxiliary choices in part (a).