249 Replacement Week 3 Problems

February 3, 2016

Problems from Stanley, EC Volume II.
5.66, 5.69, 5.72

Other problems (taken from previous courses by L. Williams and M. Haiman).

1. Let D be a digraph with p vertices $\{v_1, v_2, \ldots, v_p\}$, and let ℓ be a fixed positive integer. Suppose that for every pair u, v of (not necessarily distinct) vertices of D there is a unique (directed) walk of length ℓ from u to v.

   (a) Recall that the *adjacency matrix* $A(D)$ of D is the $p \times p$ matrix whose entry a_{ij} is equal to the number of edges from v_i to v_j. What are the eigenvalues of the (directed) adjacency matrix $A := A(D)$?

   (b) How many loops (v, v) does D have?

   (c) Show that every vertex has outdegree $p^{1/\ell}$.

   (d) Show that D is connected and balanced. Hint: You can solve explicitly for one of the eigenvectors of $A(D)$.

   (e) How many Eulerian tours does D have starting with a given edge e?

2. Stanley Vol II, Example 5.6.10. In this example we will compute the number of spanning trees of the n-dimensional cube. Let $\Gamma = (\mathbb{Z}/2\mathbb{Z})^n$ be the group of n-tuples of 0's and 1's under componentwise addition mod 2. Let C_n be the graph whose vertices are the elements of Γ, with 2 vertices α, β connected by an edge whenever $\alpha + \beta$ has exactly one component equal to 1. Thus, C_n is the graph of the n-dim. cube.

   Define a "scalar product" $\alpha \cdot \beta$ on Γ by

   $$(a_1, \ldots, a_n)\cdot(b_1, \ldots, b_n) = \sum a_i b_i \pmod 2 = \#\{i \mid a_i = b_i = 1\} \pmod 2$$

   Let V be the vector space of all functions $f : \Gamma \to \mathbb{Q}$.

   (a) Give a natural basis for V indexed by elements of Γ.

   (b) Define a linear transformation $\varphi : V \to V$ by

   $$(\varphi f)(\alpha) = n f(\alpha) - \sum_{\beta} f(\beta)$$

   where β ranges over all elements of γ adjacent to α in C_n. Show that the matrix of φ w.r.t the basis in (a) is exactly the Laplacian matrix $L(C_n)$.

(c) For $\gamma \in \Gamma$, define $\chi_\gamma \in V$ by

$$\chi_\gamma(\alpha) = (-1)^{\alpha \cdot \gamma}$$

Show that χ_γ is an eigenvector of φ.

(d) Using (c) and the Matrix-Tree theorem, show that the number of spanning trees on C_n is given by

$$c(C_n) = 2^{2^n - n - 1} \prod_{k=1}^{n} k^{\binom{n}{k}}$$

Remark: An open problem is to find a direct combinatorial proof of this formula.

3. In this problem we generalize Problem 2. The product $G \times H$ of two simple graphs (graphs without loops or multiple edges) is the graph on vertex set $V(G) \times V(H)$ with edges $\{(v, w), (v', w')\}$ for $v = v'$ and $\{w, w'\} \in E(H)$ or $w = w'$ and $\{v, v'\} \in E(G)$. The adjacency matrix A_G of a graph G on n vertices is the $n \times n$ matrix with rows and columns labelled by the vertices, and entries $(A_G)_{v,w} = 1$ if $\{v, w\} \in E(G)$, zero otherwise. Let D_G be the diagonal matrix whose (v, v) entry is the degree of v.

   (a) Let $f_G(r)$ be the number of rooted spanning forests of G with r roots, and let $F_G(z) = \sum_r f_G(r) z^r$ be the corresponding generating function. Show that $F_G(z) = \prod_i (z + \alpha_i)$, where the α_i's are the eigenvalues of $D_G - A_G$.

   (b) Show that $F_{G \times H}(z) = \prod_{i,j}(z + \alpha_i + \beta_j)$, where $F_G(z) = \prod_i (z + \alpha_i)$ and $F_H(z) = \prod_j (z + \beta_j)$. In particular, the numbers $f_G(r)$ and $f_H(r)$ for all r determine the corresponding numbers $f_{G \times H}(r)$.

   (c) Show that if Q_n is the graph formed by the vertices and edges of the n-cube, that is, the product of n copies of the complete graph on 2 vertices, then

$$F_{Q_n}(z) = \prod_{k=0}^{n} (z + 2k)^{\binom{n}{k}}.$$

   This reduces to Problem 2 by taking the coefficient of z.

4. (a) Let X be an $m \times n$ matrix and Y an $n \times m$ matrix. Given a subset $I \subseteq \{1, \ldots, n\}$, let X_I be the submatrix formed by the columns of X with indices $i \in I$ and let Y_I be the submatrix formed similarly by rows of Y. Prove or find a reference for the identity

$$\det XY = \sum_{|I|=m} \det(X_I) \det(Y_I)$$

(b) Let E be the $n \times \binom{n+1}{2}$ matrix constructed as follows: n of the columns are unit vectors e_i, and the remaining $\binom{n}{2}$ columns are differences $e_i - e_j$ for $i < j$. Show that an $n \times n$ square submatrix E_I of E is non-singular if and only if there is a rooted forest F on the vertex set $\{1, \ldots, n\}$ such that the columns of E_I which are unit vectors e_i correspond to the roots i of F and the and the columns which are difference vectors $e_i - e_j$ correspond to the edges $\{i, j\}$. Show in addition that in this case, $\det(E_I) = \pm 1$.

(c) Let $Y = E^t$ and let X be the matrix obtained from E by multiplying each unit vector column e_i by a scalar z_i, and each difference column $e_i - e_j$ by a scalar $-x_{ij}$. Use the formula in part (a) to deduce an alternate proof of the symmetric version of the Matrix-Tree Theorem, that is, its specialization with $x_{ji} = x_{ij}$.