249 Replacement: Week 2 Problems

January 27, 2016

Problems from Stanley, EC Volumes I (second edition), II.
1.5, 5.13ab, 5.20a

Other problems (taken from previous courses by L. Williams and M. Haiman).

1. Find the exponential generating function $D(x) = \sum_n D_n x^n/n!$, where D_n is the number of permutations $\sigma \in S_n$ with no fixed points. Deduce an explicit formula for D_n.

2. A binary tree is an ordered rooted tree in which every non-leaf node has exactly two children. Note that every binary tree has an odd number of vertices. Show that the number of binary trees with $2n+1$ vertices is equal to the Catalan number $C_n = \binom{2n}{n}/(n+1)$ by finding the ordinary generating function counting such trees.

3. An *unordered binary tree* is a rooted tree in which each non-leaf node has two children, but we do not order the children. Find the exponential generating function which counts unordered binary trees on n labelled nodes.

4. An *at most binary tree* is an unordered rooted tree in which each node has at most two children. Find the exponential generating function which counts at most binary trees on n labelled nodes.

5. (a) Fix $F(x)$ a formal power series with $F(0) = 0$. Show that map $G \mapsto G \circ F$ is a ring homomorphism.

   (b) Define a q-analog of functional composition by

   $$F \circ_q G(x) = \sum_k f_k\, G(x)G(qx)\cdots G(q^{k-1}x),$$

   where $G(x)$ is a formal series without constant term and q is an element of the ground ring. Show that the operator Ψ on formal power series defined by $\Psi(F) = F \circ_q G$ is continuous, linear over the ground ring and satisfies $\Psi(1) = 1$ and $\Psi(x\,F) = G(x)\,(\Psi(F)(qx))$.

   (c) Prove that the operator Ψ is determined by the properties in (b).

   (d) Assume now that both q and the coefficient of the linear term of $G(x)$ are invertible in the ground ring. Prove that the operator Ψ has an inverse.

   (e) Let $H = \Psi^{-1}(x)$, that is, $H \circ_q G = x$. Prove the identity $\Psi(HF) = x\Psi(F(qx))$, for all F.

   (f) Prove Garsia's Theorem, which states that Ψ^{-1} is given by $\Psi^{-1}(F) = F \circ_{1/q} H$. In particular, deduce that $G \circ_{1/q} H = x$.

6. Let $m_d(q)$ be the number of irreducible monic polynomials $f(x)$ of degree d, over the finite field $\mathbb{F}_q$ with q elements. Note that the number of *all* monic polynomials of degree d (irreducible or not) is just q^d.

   (a) Use unique factorization of polynomials to prove the generating function identity
   $$\prod_{d\geq 1} \frac{1}{(1-x^d)^{m_d(q)}} = \frac{1}{1-qx}.$$

   (b) By taking logarithms on both sides, derive the identity
   $$\sum_{d|n} d m_d(q) = q^n$$
   for all n, the sum ranging over the divisors of n. Equivalently,
   $$m_d(q) = \frac{1}{d}\sum_{m|d} \mu(d/m) q^m,$$
   where $\mu(n)$ is the Möbius function from number theory, *i.e.*, $\mu(n) = (-1)^k$ if n is a product of k distinct primes, and $\mu(n) = 0$ if n is divisible by a square.

   (c) Use (b) to prove that the product of all monic irreducible polynomials of degree dividing n is equal to $x^{q^n} - x$. [Hint: every element of $\mathbb{F}(q^n)$ is a root of $x^{q^n} - x$.]

   (d) A *necklace* is an equivalence class of words up to rotation. A necklace of length n is *primitive* if the corresponding rotation class consists of n distinct words, *i.e.*, it is not periodic with period d a proper divisor of n. (Example: 1122 is primitive; 1212 is not.) Note that every word of length n consists of n/d repetitions of a primitive necklace of length d dividing n. (Example: 1212 and 2121 both repeat the primitive necklace $12 = 21$.) Let $p_d(q)$ be the number of primitive necklaces of length d on an alphabet of q symbols. Prove that
   $$\sum_{d|n} d p_d(q) = q^n,$$
   and hence
   $$m_d(q) = p_d(q)$$
   when q is a power of a prime.

7. Fix some positive number $k \geq 2$. Show that the number of partitions of n in which every part appears at most $k-1$ times equals the number of partitions where every part is not divisible by k.

8. Prove that the number of partitions of n in which each part j is repeated less than j times is equal to the number of partitions of n in which no part is a square.